

# Explicit analysis of positive dimensional fibres of $\mathcal{P}_{g,r}$ and Xiao conjecture

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**ABSTRACT.** *We focus on the positive dimensional fibres of the Prym map  $\mathcal{P}_{g,r}$ . We present a direct procedure to investigate infinitely many examples of positive dimensional fibres. Such procedure uses families of Galois coverings of the line admitting a 2-sheeted Galois intermediate quotient. Then we generalize to families of Galois coverings of the line admitting a Galois intermediate quotient of higher degree and we show that the higher degree analogue of the aforementioned procedure gives all the known counterexamples to a conjecture by Xiao on the relative irregularity of a fibration.*

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## 1. Introduction

Let  $C$  be a curve of genus  $g \geq 1$  and let  $f : \tilde{C} \rightarrow C$  be a 2-sheeted covering of  $C$  ramified at  $r \geq 0$  points. The Prym variety  $P := P(\tilde{C}, C)$  is a polarized abelian variety of dimension  $g - 1 + \frac{r}{2}$  associated with  $f$ . It is defined as the identity component of the kernel of the Norm map  $\text{Nm}_f : J\tilde{C} \rightarrow JC$ . The theta divisor of  $J\tilde{C}$  induces a polarization on the Prym variety  $P$  of type  $\delta := (1, \dots, 1, 2, \dots, 2)$  with 2 repeated  $g$  times if  $r > 0$  and  $g - 1$  times if  $r = 0$ .

Let us denote by  $\mathcal{R}_{g,r}$  the coarse moduli space of isomorphism classes of coverings  $f$  and by  $\mathcal{A}_{g-1+\frac{r}{2}}^\delta$  the one of abelian varieties of dimension  $g - 1 + \frac{r}{2}$  with polarization of type  $\delta$ . The theory of double coverings provides an alternative description of  $\mathcal{R}_{g,r}$ . Indeed there is a 1-1 correspondence between double covers  $f : \tilde{C} \rightarrow C$  and triples  $(C, \eta, B)$  in

$$\mathcal{R}_{g,r} = \{(C, \eta, B) : C \in \mathcal{M}_g, \eta \in \text{Pic}^{\frac{r}{2}}(C), B \text{ reduced in } |\eta^{\otimes 2}|\} / \cong.$$

When  $r = 0$ , the branch divisor  $B$  is empty, hence we can identify  $f$  with a pair  $(C, \eta)$ , with  $\eta \in \text{Pic}^0(C) \setminus \{\mathcal{O}_C\}$  such that  $\eta^{\otimes 2} \cong \mathcal{O}_C$ .

The Prym map is the morphism

$$\mathcal{P}_{g,r} : \mathcal{R}_{g,r} \rightarrow \mathcal{A}_{g-1+\frac{r}{2}}^\delta$$

which sends triples  $[(C, \eta, B)]$  to the corresponding Prym variety  $P$ .

The case  $r = 0$  is very classical. Indeed unramified Prym varieties are principally polarized abelian varieties and they have been studied for over one hundred years. Our (algebraic) point of view has been presented for the first time by Mumford in [24] in 1974. Then many papers investigated this case and nowadays we have a lot of information on  $\mathcal{P}_{g,0}$ . Donagi very well discusses it in [6]. Good surveys are [8] and [29].

The case  $r > 0$  has become of interest only more recently. Indeed ramified Prym varieties are abelian varieties no longer principally polarized (except in the case of  $r = 2$ ). As such, they started to be studied quite late. Even if some cases with  $r = 4$  were already considered by [25] and by [2], the seminal paper is the one by Marucci and Pirola ([22]), which came out only in 2012. From this work, many other authors have investigated the ramified Prym maps  $\mathcal{P}_{g,r}$ . At this moment it is a very active area of research. For instance, very recently, it has been proved that if  $r \geq 6$  then  $\mathcal{P}_{g,r}$  is injective ([18, 28]).

When the genus  $g$  and the number  $r$  are low, more precisely when  $g < 6$  for  $r = 0$ ,  $g \leq 4$  for  $r = 2$  and  $g \leq 2$  for  $r = 4$ , the fibres of the Prym maps  $\mathcal{P}_{g,r}$  are positive dimensional and they carry plenty of geometry which is well-understood (see [6] and [13]). The structure of the generic positive dimensional fibre is so peculiar that one needs to find an ad-hoc procedure to describe each of them.

In [26], Naranjo stated that, for  $g$  large enough, the étale Prym map  $\mathcal{P}_{g,0}$  has positive dimensional fibres only on the locus of coverings of hyperelliptic curves and on some components of the locus of coverings of bielliptic curves. Naranjo and Ortega (see [28, Theorem 1.2]) showed that the ramified Prym map  $\mathcal{P}_{g,r}$ , with  $r = 2, 4$  and any value for  $g$ , has positive dimensional fibres when restricted to covers of hyperelliptic curves. In the same paper, the authors also proved that the Prym map  $\mathcal{P}_{5,2}$  carries positive dimensional fibres when restricted to the locus of trigonal curves. Finally, Casalaina-Martin and Zhang [4, Theorem 5.1] produced some positive dimensional fibres for  $\mathcal{P}_{3,4}$  under the assumption  $\eta$  effective.

Thus it turns out that, except for a few isolated cases, the hyperelliptic locus represents a good place to look for positive dimensional fibres for the Prym map. Nevertheless, when  $r > 0$  it is unknown if we should expect the existence of other examples. Indeed we only have the following:

**PROPOSITION 1.1** ([28]). *Assume  $r > 0$ . If the differential  $d\mathcal{P}_{g,r}$  is not injective then we are in one among these cases.*

$r = 2$  :  $C$  is hyperelliptic or trigonal or plane quintic or  $\eta = \mathcal{O}_C(x + y - z)$ , for  $x, y, z \in C$ .

$r = 4$ :  $C$  is hyperelliptic or  $h^0(C, \eta) > 0$ .

This Proposition doesn't conclude the classification. Apart from the hyperelliptic locus and the isolated examples mentioned in the previous paragraph (i.e. the trigonal locus in genus 5 and  $\eta$  effective in genus 3), we still do not know the behaviour of the differential  $d\mathcal{P}_{g,r}$  on the remaining cases.

The goal of this paper is to analyse infinitely many examples of positive dimensional fibres of the Prym maps, both in the étale or in the ramified case. We use positive dimensional families of Galois covers  $\tilde{C} \rightarrow \tilde{C}/\tilde{G} \cong \mathbb{P}^1$  of the line where the genus  $\tilde{g} := g(\tilde{C})$ , the number of ramification points  $s$  and the monodromy are fixed. Then we look for which among these families admit as intermediate quotient  $\tilde{C} \rightarrow C$  a 2:1 map ramified in  $r = 0, 2, 4$ . Hence we select the ones with associated  $d\mathcal{P}_{g,r}$  non-injective. This request corresponds to a simple numerical condition as follows.

**PROPOSITION 1.2.** *If  $\dim(S^2H^0(\omega_{\tilde{C}}))^{\tilde{G}} - \dim(S^2H^0(\omega_C))^{\tilde{G}} < \dim H^0(\omega_{\tilde{C}}^{\otimes 2})^{\tilde{G}}$  then the differential  $d\mathcal{P}_{g,r}$  along the family is not injective. Hence the Prym map has positive dimensional fibres along the family.*

In particular, when  $\dim(S^2H^0(\omega_{\tilde{C}}))^{\tilde{G}} = \dim(S^2H^0(\omega_C))^{\tilde{G}}$  the family is all contained in a fibre of  $\mathcal{P}_{g,r}$ . This is always the situation in case of  $H^0(\omega_{\tilde{C}}^{\otimes 2})^{\tilde{G}} = 1$ , i.e. of 1-dimensional families, under the assumption of Proposition 1.2.

We show the following:

**THEOREM 1.3.** *For any  $N \in \mathbb{N}$  there exist 1-dimensional families of Galois covers  $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^1$  contained in the fibres of  $\mathcal{P}_{2N,0}, \mathcal{P}_{2N,2}, \mathcal{P}_{2N-1,0}, \mathcal{P}_{2N-1,4}$ .*

Easily we show that (unfortunately) all such families arise as coverings of curves  $C$  lying in the hyperelliptic locus.

In general, starting from a family of curves, one can construct fibrations that have the curves of the family as fibres. In particular, we focus on those obtained from families under the assumption of Proposition 1.2. At the same time, we generalize to families of Galois coverings  $\tilde{C} \rightarrow \tilde{C}/\tilde{G} \cong \mathbb{P}^1$  (with fixed genus  $\tilde{g} := g(\tilde{C})$  and monodromy) admitting Galois intermediate quotient  $\tilde{C} \rightarrow C$  of degree  $d \geq 2$ . Accordingly, we give the definitions of Prym variety  $P(\tilde{C}, C)$  and of higher degree Prym map  $\mathcal{P}_{g,r}(d)$ .

First, we show that a higher-degree analogue of Proposition 1.2 holds. Indeed we prove the following:

**PROPOSITION 1.4.** *If  $\dim(S^2H^0(\omega_{\tilde{C}}))^{\tilde{G}} - \dim(S^2H^0(\omega_C))^{\tilde{G}} < \dim H^0(\omega_{\tilde{C}}^{\otimes 2})^{\tilde{G}}$  then the differential of the Prym map  $\mathcal{P}_{g,r}(d)$  along the family is not injective. Hence the Prym map has positive dimensional fibres along the family.*

It appears that fibrations  $h : S \rightarrow B$  obtained from families under the assumption of Proposition 1.4 are quite interesting. Indeed, using them, we produce all the known counterexamples to a conjecture of Xiao.

In [35], Xiao proved that a non trivial fibration with base curve  $B \cong \mathbb{P}^1$ , general fibre of genus  $\tilde{g}$  and irregularity  $q$ , satisfies  $q \leq \frac{\tilde{g}+1}{2}$ . Furthermore, for  $g(B) > 0$ , he conjectured that the relative irregularity of the fibration  $q_h$  satisfies  $q_h \leq \frac{\tilde{g}+1}{2}$ . The four known counterexamples have been constructed in [30] and in [1] as fibrations associated with families of cyclic prime odd étale covers of hyperelliptic curves (elliptic curves in case of [30]) carrying a non-injective differential  $d\mathcal{P}_{g,r}(d)$ . In particular, the three examples of [1] fit perfectly in the structure of our families: a Theorem by Ries ([31]) guarantees that they yield dihedral Galois cover of  $\mathbb{P}^1$ , i.e. they provide towers  $\tilde{C} \xrightarrow{d:1} C \rightarrow \tilde{C}/D_d \cong \mathbb{P}^1$ .

In light of Proposition 1.4, it seemed natural to us to check if there exist other families of Galois coverings  $\tilde{C} \xrightarrow{d:1} C \rightarrow \mathbb{P}^1$  with non-injective differential and which disprove the conjecture. Notice that we do not require  $C$  to be hyperelliptic and we consider any Galois (cyclic or not) intermediate quotient  $\tilde{C} \xrightarrow{d:1} C$  of any degree. By means of computer calculations (our MAGMA script is available at <http://mate.unipv.it/grosselli/publ/>), we show the following

**PROPOSITION 1.5.** *Up to  $\tilde{g} = 12, s = 14$  (i.e. dimension 11), the only positive dimensional families of Galois towers  $\tilde{C} \xrightarrow{d:1} C \rightarrow \mathbb{P}^1$  carrying non-injective differential and disproving Xiao's conjecture are the one of [30] and the ones of [1].*

The third example of [1, Theorem 1.2] is obtained via a degeneration argument. Therefore one of the four examples that we find with Proposition 1.5 is presented in a slightly different way from the original, although it is clearly the same. For this reason, we think it may be useful to give a very brief description of all the examples.

The paper is organized as follows: in Section 2 we give an overview on Prym maps while in Section 3 we explain our constructive method. Finally, in Section 4, we recall something on Xiao fibrations and we describe the counterexamples.

## 2. The state of the art

In this section we would like to overview the literature on positive dimensional fibres of the Prym maps  $\mathcal{P}_{g,r} : \mathcal{R}_{g,r} \rightarrow \mathcal{A}_{g-1+\frac{r}{2}}^\delta$ . We recall that

$$\dim \mathcal{R}_{g,r} = 3g - 3 + r \quad \text{and} \quad \dim \mathcal{A}_{g-1+\frac{r}{2}}^\delta = \frac{1}{2} \left( g - 1 + \frac{r}{2} \right) \left( g + \frac{r}{2} \right). \quad (2.1)$$

First, let us briefly recall the classical case, that is  $r = 0$  (the standard notation refers to  $\mathcal{R}_{g,0}$  as to  $\mathcal{R}_g$ , the same for  $\mathcal{P}_{g,0}$ ). Easily from (2.1) we see that the generic fibre of  $\mathcal{P}_g$  is positive dimensional when  $g \leq 5$ . A detailed study of its geometric structure is provided by the works of Verra for  $g = 3$  ([34]), Recillas for  $g = 4$  ([32]) and Donagi for  $g = 5$  ([6]). Cases with  $g = 1, g = 2$  are summarized in [6, Section 6]. When  $g = 6$  the fibre is generically finite of degree 27 ([7]). On the other hand, when  $g \geq 7$ , the Prym map is generically injective but never injective (see [6] and references therein).

The positive dimensional fibres of  $\mathcal{P}_g$  are characterized as follows:

**THEOREM 2.1** (Mumford [24], Naranjo [26]). *Assume  $g \geq 13$ . Then  $\mathcal{P}_g$  has positive dimensional fibres at  $(\tilde{C}, C)$  if and only if  $C$  is either hyperelliptic or it belongs to one among the components of the bielliptic locus where  $\tilde{C}$  carries  $\mathbb{Z}/2 \times \mathbb{Z}/2 \subseteq \text{Aut}(\tilde{C})$ .*

Now we focus on the ramified cases, i.e.  $r > 0$ . The inequality  $\dim \mathcal{R}_{g,r} > \dim \mathcal{A}_{g-1+\frac{r}{2}}^\delta$  is satisfied only in six cases, that is  $r = 2$  with  $1 \leq g \leq 4$  and  $r = 4$  with  $1 \leq g \leq 2$ . All of them are considered in [13]. Indeed it is shown the following:

**PROPOSITION 2.2.** *([13, Proposition 1.2 and Corollary 1.3]) Under the assumptions*

$$(g, r) \in \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (4, 2)\},$$

*the ramified Prym map  $\mathcal{P}_{g,r}$  is dominant. Therefore the generic fibre  $F_{g,r}$  of  $\mathcal{P}_{g,r}$  has  $\dim F_{1,2} = 1, \dim F_{2,2} = 2, \dim F_{3,2} = 2, \dim F_{4,2} = 1, \dim F_{1,4} = 1, \dim F_{2,4} = 1$ .*

Hence the paper gives a detailed description of the generic fibre for all the six cases (see [13, Theorem 0.1]).

Let us now focus on  $\dim \mathcal{R}_{g,r} \leq \dim \mathcal{A}_{g-1+\frac{r}{2}}^\delta$ . The first result is the following:

**THEOREM 2.3** (Marcucci-Pirola [22], Marcucci-Naranjo [21], Naranjo-Ortega [27]). *The ramified Prym map is generically injective as far as the dimension of  $\mathcal{R}_{g,r}$  is less than or equal to the dimension of  $\mathcal{A}_{g-1+\frac{r}{2}}^\delta$ .*

Actually, the equality between the dimensions is reached only in the case of  $g = 3$  and  $r = 4$  where more it is known:

**THEOREM 2.4** (Nagaraj-Ramanan [25]). *Let  $g \geq 3$ . The Prym map  $\mathcal{P}_{g,4}$  restricted to the locus of tetragonal curves has generically degree 3.*

Quite recently Theorem 2.3 has been improved:

**THEOREM 2.5** (Ikeda [18] for  $g = 1$ , Naranjo-Ortega [28] for all  $g$ ). *The Prym map  $\mathcal{P}_{g,r}$  is injective with injective differential for all  $r \geq 6$  and  $g > 0$ .*

Thus we will look for the positive dimensional fibres of  $\mathcal{P}_{g,2}$  and  $\mathcal{P}_{g,4}$ . Since  $\mathcal{P}_{1,2}$  and  $\mathcal{P}_{1,4}$  have positive dimensional generic fibre, we will assume  $g \geq 2$ .

The codifferential of  $\mathcal{P}_{g,r}$  at a point  $[(C, \eta, B)] \in \mathcal{R}_{g,r}$  is given by the multiplication map ([22])

$$d\mathcal{P}_{g,r}^*(C, \eta, B) : S^2H^0(C, \omega_C \otimes \eta) \rightarrow H^0(C, \omega_C^2 \otimes \mathcal{O}(B)). \quad (2.2)$$

We have the following:

PROPOSITION 2.6 ([28]). *Let  $L := \omega_C \otimes \eta$ . If  $d\mathcal{P}_{g,r}$  is not injective at  $[(C, \eta, B)]$  then one of the following holds:*

1.  $L$  is not very ample or
2.  $L$  very ample and
  - (a)  $r = 2$  and  $\text{Cliff}(C) \leq 1$  or
  - (b)  $r = 4$  and  $\text{Cliff}(C) = 0$ .

*Proof.* The proof is a straightforward application of Green-Lazarsfeld Theorem for the surjectivity of a multiplication map (see [16, Theorem 1]): the map  $d\mathcal{P}_{g,r}^* : S^2H^0(C, L) \rightarrow H^0(C, L^2)$  is surjective if  $L$  is very ample and  $\deg L \geq 2g + 1 - 2h^1(C, L) - \text{Cliff}(C)$ . Since

$$\deg L = 2g - 2 + \frac{r}{2} \quad \text{and} \quad h^1(C, L) = h^0(C, \omega_C \otimes L^{-1}) = 0$$

we conclude.  $\square$

As already observed in [28, Remark 2.2], the above Proposition can be rephrased as follows:

PROPOSITION 2.7. *If the differential  $d\mathcal{P}_{g,r}$  is not injective at  $[(C, \eta, B)]$  then:*

1.  $r = 2$  and  $\eta = \mathcal{O}(x + y - z)$  for  $x, y, z \in C$  or  $r = 4$  and  $h^0(C, \eta) > 0$ .  
Otherwise
2.  $r = 2$  and  $C$  is hyperelliptic, trigonal or a quintic plane curve or  $r = 4$  and  $C$  is hyperelliptic.

*Proof.* (1) is borrowed from [19, Lemma 2.1] while (2) follows from the definition of the Clifford Index.  $\square$

Now we list evidence of positive dimensional fibres that we find in the literature. For a proof of these results, we refer to the cited papers.

PROPOSITION 2.8. (Naranjo-Ortega, [28, Theorem 1.2]) *Let  $\mathcal{P}_{g,r}^h$  be the restriction of  $\mathcal{P}_{g,r}$  to the locus of coverings of hyperelliptic curves of genus  $g$  ramified in  $r$  points ( $r = 2, 4$ ). Then the generic fibre of  $\mathcal{P}_{g,2}^h$ , respectively of  $\mathcal{P}_{g,4}^h$ , is birational to a projective plane, respectively to an elliptic curve.*

REMARK 2.9. When  $g = 2$  the restriction  $\mathcal{P}_{2,r}^b$  coincides with  $\mathcal{P}_{2,r}$ . Indeed the paper [13] studies  $\mathcal{P}_{2,2}$ , respectively  $\mathcal{P}_{2,4}$ , and it shows that the generic fibre is isomorphic to a plane minus 15 lines, respectively to an elliptic curve minus 15 points.

PROPOSITION 2.10. (*Naranjo-Ortega, [28, Proposition 2.4]*) *Let  $\mathcal{P}_{5,2}^{tr}$  be the restriction of  $\mathcal{P}_{5,2}$  to the locus of coverings of trigonal curves of genus 5 ramified in 2 points. Then the fibres are all positive dimensional.*

PROPOSITION 2.11. (*Casalaina Martin-Zhang, [4, Theorem 5.1]*) *Let  $\mathcal{R}_{3,4}^{Eck} \subset \mathcal{R}_{3,4}$  be the subset of triples  $(C, \eta, B)$  such that  $C$  is a smooth quartic plane curve canonically embedded in a plane where  $B$  is reduced and cut by a line  $l$  and  $\eta$  is of degree 2 and cut by a bitangent. Then the generic fibre of  $\mathcal{P}_{3,4}$  restricted to  $\mathcal{R}_{3,4}^{Eck}$  is isomorphic to the elliptic curve described as the covering of  $l$  ramified on  $B$ .*

REMARK 2.12. Notice that here the positive dimensional fibres are realized under the assumption  $\eta$  effective.

### 3. The procedure

In this section, we describe our strategy to investigate infinitely many positive dimensional fibres of the Prym maps. In particular, we study families of towers of Galois covers  $\tilde{C} \xrightarrow{2:1} C \rightarrow \mathbb{P}^1$ . Our procedure does not bound the genus of the curves occurring in such towers. For this reason, we are able to describe infinitely many examples.

In order to do this, we introduce the Prym datum as given in [5, 11]. We recall the definition.

DEFINITION 3.1. *A Prym datum of type  $(s, r)$  is a triple  $\Xi = (\tilde{G}, \tilde{\theta}, \sigma)$ :  $\tilde{G}$  is a finite group,  $\tilde{\theta} : \Gamma_s \rightarrow \tilde{G}$  is an epimorphism,  $\sigma \in \tilde{G}$  is a central involution and*

$$r = \sum_{i: \sigma \in \langle \tilde{\theta}(\gamma_i) \rangle} \frac{|\tilde{G}|}{\text{ord}(\tilde{\theta}(\gamma_i))},$$

where  $\Gamma_s := \langle \gamma_1, \dots, \gamma_s : \gamma_1 \dots \gamma_s = 1 \rangle$ .

This datum corresponds to a family of Galois coverings and the generic point of the family fits in the following diagram

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{f} & C = \tilde{C}/\langle \sigma \rangle \\ & \searrow \pi & \swarrow \\ & \mathbb{P}^1 & \end{array} \quad (3.1)$$

where  $\tilde{C}$  and  $C$  are curves of genus  $\tilde{g}$  and  $g$  respectively. The cover  $\tilde{C} \rightarrow \mathbb{P}^1$  is branched on  $s$  points. The cover  $f$  is 2-sheeted and branched on a degree  $r$  divisor  $B$ . Let  $\eta \in \text{Pic}^{\frac{r}{2}}(C)$  be the corresponding line bundle such that  $\eta^2 = \mathcal{O}_C(B)$ .  $G := \tilde{G}/\langle \sigma \rangle$  is the quotient group acting on  $C$ , the composition of  $\tilde{\theta}$  with the projection to the quotient is an epimorphism  $\theta : \Gamma_s \rightarrow G$ . The maps  $\tilde{\theta}$  and  $\theta$  are respectively the monodromies of the two Galois covers  $\tilde{C} \rightarrow \mathbb{P}^1 = \tilde{C}/\tilde{G}$  and  $C \rightarrow \mathbb{P}^1 = C/G$ . Moreover, in all the examples we consider,  $s = 4$  and  $C \rightarrow \mathbb{P}^1$  is branched on the same points of  $\tilde{C} \rightarrow \mathbb{P}^1$ .

There is a natural identification between the tangent space to the family at the generic point and the space of the infinitesimal deformations of  $\tilde{C}$  that preserve the action of  $\tilde{G}$ . The latter is isomorphic to  $H^1(\tilde{C}, T_{\tilde{C}})^{\tilde{G}}$  ( $\cong H^0(\tilde{C}, \omega_{\tilde{C}}^2)^{\tilde{G}*}$ ). Thus  $\dim H^0(\tilde{C}, \omega_{\tilde{C}}^2)^{\tilde{G}} = s - 3$  equals the dimension of the family.

Recall that  $\sigma$  gives a decomposition of  $V := H^0(\tilde{C}, \omega_{\tilde{C}})$  in  $\pm 1$ -eigenspaces, resp.  $V_+$  and  $V_-$ , where  $V_+ \cong H^0(C, \omega_C)$  and  $V_- \cong H^0(C, \omega_C \otimes \eta)$ . Similarly we can define  $W := H^0(\tilde{C}, \omega_{\tilde{C}}^2)$  and get a decomposition  $W = W_+ \oplus W_-$  with  $W_+ \cong H^0(C, \omega_C^2 \otimes \eta^2) = H^0(C, \omega_C^2 \otimes \mathcal{O}(B))$ . Let us denote

$$\tilde{N} := \dim(S^2 H^0(\tilde{C}, \omega_{\tilde{C}}))^{\tilde{G}} \quad \text{and} \quad N := \dim(S^2 H^0(C, \omega_C))^G. \quad (3.2)$$

Immediately, we have  $\tilde{N} - N = \dim(S^2 V_-)^{\tilde{G}}$ .

We are interested in families that lie in positive dimensional fibres of the Prym map, so we look for a condition that makes the codifferential of the Prym map not surjective. We have the following

**PROPOSITION 3.2.** *If  $\tilde{N} - N < s - 3$  the differential of the Prym map  $d\mathcal{P}_{g,r}$  along the family is not injective, hence the Prym map has positive dimensional fibres along the family.*

*Proof.* As in (2.2), the codifferential of the Prym map at the generic element of the family is the multiplication map

$$m = (d\mathcal{P}_{g,r})^* : (S^2 V_-)^{\tilde{G}} \rightarrow W_+^{\tilde{G}}.$$

Since by assumption  $\dim(S^2 V_-)^{\tilde{G}} = \tilde{N} - N < s - 3 = \dim W_+^{\tilde{G}}$  the codifferential cannot be surjective hence its dual has a non-trivial kernel.  $\square$

**COROLLARY 3.3.** *If  $\tilde{N} = N$  the family is contained in a fibre of the Prym map.*

Now we present our construction. It produces infinitely many Prym data. They yield 1-dimensional families of Galois towers of type (3.1) that carry constant Prym variety. Thus they lie in positive dimensional fibres of the Prym map. We organize these data into 5 classes as follows.



**THEOREM 3.4.** *For any  $N \in \mathbb{N}$  there are 1-dimensional families of Galois covers  $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^1$  in the fibres of the Prym maps  $\mathcal{P}_{2N,2}$ ,  $\mathcal{P}_{2N-1,0}$  (2 families),  $\mathcal{P}_{2N,0}$  and  $\mathcal{P}_{2N-1,4}$ . All the families lie in their respective hyperelliptic locus.*

All the families carry an abelian Galois group, so we can refer to [12, Section 4]. We describe in detail the first case.

Fix a positive integer  $N$ , set the odd number  $k = 2N + 1$  and denote  $C_n \cong \mathbb{Z}/n\mathbb{Z}$ . The Prym datum is defined by the group  $\tilde{G} = C_2 \times C_{2k} \subset C_{2k}^2$  under the inclusion  $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} k \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , the monodromy  $\tilde{\theta} : \Gamma_4 \rightarrow \tilde{G}$  is represented by the matrix

$$A = \begin{pmatrix} k & 0 & 0 & k \\ 0 & k & 2 & k-2 \end{pmatrix}$$

where the  $i$ -th column is  $\tilde{\theta}(\gamma_i)$  under the inclusion in  $C_{2k}^2$  and the involution  $\sigma = (k, k)^t$ .

In case of abelian groups, the character group  $\tilde{G}^* = \text{Hom}(\tilde{G}, \mathbb{C})$  is isomorphic to  $\tilde{G}$ . In our situation, a character in  $\tilde{G}^*$  can be identified with an element  $n = (n_1, n_2)$  where  $n_1 \in C_2$  and  $n_2 \in C_{2k}$ . Set  $V = H^0(\tilde{C}, \omega_{\tilde{C}})$  and let  $V = V_+ \oplus V_-$  be the eigenspace decomposition induced by the action of  $\sigma$ . As before,  $V_+ \cong H^0(C, \omega_C)$ .

Our target is to compute the dimension of  $(S^2 H^0(\tilde{C}, \omega_{\tilde{C}})_-)^{\tilde{G}}$  and to show that it is zero. In this way, the codifferential of the Prym map would be trivial on the family and thus the Prym map would be constant.

Let  $H^0(\tilde{C}, \omega_{\tilde{C}})_n$  be the subspace of  $H^0(\tilde{C}, \omega_{\tilde{C}})$  where  $\tilde{G}$  acts via the character  $n$ , and denote by  $d_n$  its dimension. By [23, Prop. 2.8], the dimension for a non trivial character  $n$  is given by

$$d_n = -1 + \sum_{i=1}^4 \left\langle -\frac{\alpha_i}{2k} \right\rangle$$

where  $\langle x \rangle$  denotes the fractional part of a real number  $x$  and

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) := (n_1, n_2) \cdot A = (kn_1, kn_2, 2n_2, kn_1 + (k-2)n_2).$$

We remind that if  $x \in \mathbb{R} \setminus \mathbb{Z}$  then  $\langle x \rangle + \langle -x \rangle = 1$ . It is straightforward that for any  $n$  such that  $2n = 0$  we have  $d_n = 0$ . Indeed

$$d_n = -1 + \left\langle \frac{-n_1}{2} \right\rangle + \left\langle \frac{-n_2}{2} \right\rangle + \left\langle \frac{-n_2}{k} \right\rangle + \left\langle -\frac{kn_1 + (k-2)n_2}{2k} \right\rangle,$$

therefore when  $n_1 = 0, n_2 = k$

$$d_n = -1 + 0 + \left\langle -\frac{n_2}{2} \right\rangle + 0 + \left\langle -\frac{n_2}{2} \right\rangle = 0,$$

and when  $n_1 = 1$ ,  $n_2 = 0$ ,  $k$

$$d_n = -1 + \frac{1}{2} + \left\langle \frac{-n_2}{2} \right\rangle + 0 + \left\langle -\frac{1+n_2}{2} \right\rangle = 0.$$

So now we suppose  $-n \neq n$ , i.e.  $-n_2 \neq n_2$ , so  $n_2 \neq 0, k$ .

- If  $n_1 = 0$  and  $n_2$  is even, then  $d_n = -1 + \left\langle -\frac{n_2}{k} \right\rangle + \left\langle -\frac{(k-2)n_2}{2k} \right\rangle = 0$ .
- If  $n_1 = 0$  and  $n_2$  is odd, then  $d_n = -1 + \frac{1}{2} + \left\langle -\frac{n_2}{k} \right\rangle + \left\langle -\frac{(k-2)n_2}{2k} \right\rangle$ . Thus  $d_n + d_{-n} = 1$ , hence exactly one between  $d_n$  and  $d_{-n}$  is 1 and the other is 0. In both cases the product  $d_n d_{-n}$  is zero. Moreover the sum of all  $d_n$  of this kind is  $\frac{k-1}{2}$ .
- If  $n_1 = 1$  and  $n_2$  is even, then  $d_n = -1 + \frac{1}{2} + \left\langle -\frac{n_2}{k} \right\rangle + \left\langle -\frac{k+(k-2)n_2}{2k} \right\rangle$ . As in the previous case  $d_n + d_{-n} = 1$  and  $d_n d_{-n}$  is always 0. Therefore the sum of all  $d_n$  of this kind equals  $\frac{k-1}{2}$ .
- If  $n_1 = 1$  and  $n_2$  is odd, then  $d_n = -1 + \frac{1}{2} + \frac{1}{2} + \left\langle -\frac{n_2}{k} \right\rangle + \left\langle -\frac{k+(k-2)n_2}{2k} \right\rangle = 1$ . So all  $k-1$  terms of this form are 1, hence there are  $\frac{k-1}{2}$  couples such that  $d_n d_{-n} = 1$ .

The decomposition  $H^0(\tilde{C}, \omega_{\tilde{C}}) = \bigoplus_n H^0(\tilde{C}, \omega_{\tilde{C}})_n$  gives us

$$\tilde{g} = g(\tilde{C}) = \dim H^0(\tilde{C}, \omega_{\tilde{C}}) = \sum_n d_n = 2k - 2.$$

Moreover we have

$$\dim(S^2 H^0(\tilde{C}, \omega_{\tilde{C}}))^{\tilde{G}} = \frac{1}{2} \sum_{2n \neq 0} d_n d_{-n} = \frac{k-1}{2} = N.$$

As explained in [12, Lemma 4.3] the terms of  $H^0(\tilde{C}, \omega_{\tilde{C}})$  which are invariant for the action of  $\sigma$  are those such that  $n_1 + n_2$  is even. Hence the genus  $g$  of  $C$  is

$$g = \dim H^0(\tilde{C}, \omega_{\tilde{C}})_+ = \sum_{n_1+n_2 \text{ even}} d_n = k - 1$$

and by the Riemann-Hurwitz formula we obtain that the degree of the ramification divisor is

$$r = 2\tilde{g} - 2 - 2(2g - 2) = 2(2k - 2) - 4(k - 1) + 2 = 2.$$

Finally we compute the dimension of  $(S^2V_+)^{\tilde{G}}$ :

$$\dim(S^2H^0(\tilde{C}, \omega_{\tilde{C}})_+)^{\tilde{G}} = \frac{1}{2} \sum_{n_1+n_2 \text{ even}} d_n d_{-n} = \frac{k-1}{2} = N.$$

This gives  $(S^2H^0(\tilde{C}, \omega_{\tilde{C}})_-)^{\tilde{G}} = 0$  and thus it allows to conclude.

It only remains to observe that the unique element of  $G = \tilde{G}/\langle\sigma\rangle \cong C_{2k}$  of order 2 gives the hyperelliptic involution of  $C$ .

In the following table, we summarize all the examples outlined in Theorem 3.4. For any integer  $N$  we define  $k$  and consequently, we give the data. The first line corresponds to the example described above. As seen  $(S^2H^0(\tilde{C}, \omega_{\tilde{C}})_+)^{\tilde{G}} = N$ . The same holds for the remaining families. Since computations are almost identical, except case (3) which is slightly more tricky, we do not repeat them.

$n$	$k$	$\tilde{g}$	$g$	$r$	$\tilde{G}$	$A$	$\sigma$
(1)	$2N+1$	$2k-2$	$k-1$	2	$C_2 \times C_{2k}$	$\begin{pmatrix} k & 0 & 0 & k \\ 0 & k & 2 & k-2 \end{pmatrix}$	$\begin{pmatrix} k \\ k \end{pmatrix}$
(2)	$2N-1$	$2k-1$	$k$	0	$C_2 \times C_{2k}$	$\begin{pmatrix} k & 0 & 0 & k \\ 0 & k & 1 & k-1 \end{pmatrix}$	$\begin{pmatrix} k \\ k \end{pmatrix}$
(3)	$N$	$4k-3$	$2k-1$	0	$C_2 \times C_{2k}$	$\begin{pmatrix} k & k & 0 & 0 \\ k-1 & k-1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} k \\ k \end{pmatrix}$
(4)	$2N$	$2k$	$k$	0	$C_2 \times C_{2k}$	$\begin{pmatrix} k & 0 & 0 & k \\ 0 & k & 1 & k-1 \end{pmatrix}$	$\begin{pmatrix} k \\ k \end{pmatrix}$
(5)	$2N$	$2k-1$	$k-1$	4	$C_{2k}$	$(1 \ 1 \ k-1 \ k-1)$	$(k)$

In order to find other (possibly higher dimensional) families contained in the fibres of  $\mathcal{P}_{g,r}$  we use a MAGMA script similar to the one described in [11] and [12]. In these papers, the authors look for Shimura subvarieties generically contained in the Prym locus. For this reason, they require the differential of the Prym map to be an isomorphism. Here we want exactly the converse, so we look for data that satisfy Proposition 3.2. In this way, MAGMA produces various examples of families of different dimensions and genus. While in the higher dimensional cases we only find some sporadic examples, in the 1-dimensional case we realized that the examples behaved cyclically in the same way. This motivated our choice to organize them into five classes as in the above table.

#### 4. The Xiao conjecture

In this section we look at the higher-degree analogue of the Prym maps  $\mathcal{P}_{g,r}$ . Indeed nothing prevents us to consider Galois covers of curves  $C \in \mathcal{M}_g$  of

degree greater than 2. The theory of Prym varieties easily extends to these cases (see [19] for cyclic covers). In general, to any Galois covering  $f : \tilde{C} \xrightarrow{d:1} C$  one can associate an abelian variety  $P := P(\tilde{C}, C)$  defined as the connected component to the origin of the kernel of the Norm map  $\text{Nm} : J\tilde{C} \rightarrow JC$ . Letting  $\tilde{g}$ , resp.  $g$ , be the genus of the curves  $\tilde{C}$  and  $C$ , then  $P$  has dimension  $\tilde{g} - g$  and inherits a (non-principal) polarization  $L$  from the theta divisor associated with  $J\tilde{C}$ .

We denote by  $\mathcal{R}(K, g, r)$  the Hurwitz scheme parametrizing coverings  $f: K$  is the Galois group acting on  $\tilde{C}$ ,  $g$  is the genus of the quotient curve and  $r$  the number of branch points. Thus we can define the Prym map:

$$\begin{aligned} \mathcal{P}_{g,r}(d) : \mathcal{R}(K, g, r) &\longrightarrow \mathcal{A}_{\tilde{g}-g}^{\delta} \\ [f] &\longmapsto [P, L] \end{aligned} \quad (4.1)$$

Let us fix a group  $\tilde{G} \subseteq \text{Aut}(\tilde{C})$  with  $K \triangleleft \tilde{G}$  normal subgroup. For computational reasons, we assume that  $\tilde{C}$  has Galois quotient  $\tilde{C}/\tilde{G}$  isomorphic to  $\mathbb{P}^1$  and we let  $s$  be the number of branch points of  $\tilde{C} \rightarrow \tilde{C}/\tilde{G} \cong \mathbb{P}^1$ . This means that we focus on towers  $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^1$ . By moving the branch points in  $\mathbb{P}^1$ , we get a family of dimension  $s - 3$ . This family naturally gives rise to a family of the same dimension contained in  $\mathcal{R}(K, g, r)$ . For more details, we refer the reader to Section 3, where such construction is considered in case of covers  $f$  of degree 2.

If we set  $V := H^0(\tilde{C}, \omega_{\tilde{C}})$ , we decompose  $V = V_+ \oplus V_-$ , where we identify  $V_+ := H^0(\tilde{C}, \omega_{\tilde{C}})^K (\cong H^0(C, \omega_C))$  and  $V_- := H^{1,0}(P)$ . Similarly we define  $H_1(\tilde{C}, \mathbb{Z})_+$  and  $H_1(\tilde{C}, \mathbb{Z})_-$ . As already seen, the tangent space to the family at the generic point is  $H^1(\tilde{C}, T_{\tilde{C}})^{\tilde{G}}$ . The space of the infinitesimal deformations of  $(P, L)$  that preserve the action of  $\tilde{G}$  is  $S^2 H^{0,1}(P)^{\tilde{G}}$ . Indeed  $P = V_-^*/H_1(\tilde{C}, \mathbb{Z})_-$ , hence the tangent space of  $P$  at the origin is  $V_-^*$  and thus  $T_P \mathcal{A}_{\tilde{g}-g} = S^2 V_-^*$ . Since we have an inclusion of  $\tilde{G}$  in  $\text{Aut } P$ , we restrict to the  $\tilde{G}$ -invariant part. Thus the differential of the Prym map  $d\mathcal{P}_{g,r}(d)$  yields the map

$$d\mathcal{P}_{g,r}(d) : H^1(\tilde{C}, T_{\tilde{C}})^{\tilde{G}} \rightarrow S^2 H^{0,1}(P)^{\tilde{G}}.$$

Dualizing we get

$$d\mathcal{P}_{g,r}(d)^* = m|_{S^2 H^{1,0}(P)^{\tilde{G}}} : S^2 H^{1,0}(P)^{\tilde{G}} \rightarrow H^0(\tilde{C}, \omega_{\tilde{C}}^{\otimes 2})^{\tilde{G}},$$

where, as usual, the multiplication map  $m : S^2 H^0(\tilde{C}, \omega_{\tilde{C}}) \rightarrow H^0(\tilde{C}, \omega_{\tilde{C}}^{\otimes 2})$  is the codifferential of the Torelli map.

As in (3.2), we define  $\tilde{N}, N$ . We have a higher degree analogue of Proposition 3.2:

PROPOSITION 4.1. *If  $\tilde{N} - N < s - 3$ , the differential of the Prym map  $d\mathcal{P}_{g,r}(d)$  along the family is not injective, hence the Prym map has positive dimensional fibres along the family.*

*Proof.* The proof works exactly like the one of Proposition 3.2. □

Starting from our families of curves, we can construct fibrations  $h : S \rightarrow B$  that carry  $\tilde{C}$  as fibres. Indeed the closure of the image of the modular map  $t \mapsto [\tilde{C}_t]$  gives a curve in  $\mathcal{M}_{\tilde{g}}$ . Then, up to resolving singularities and taking pull-backs, we get a fibration  $h : S \rightarrow B$ , as claimed.

The *irregularity* of the surface  $S$  is  $q := \dim H^1(S, \mathcal{O}_S)$ , and  $q_h = q - g(B)$  is called *relative irregularity* of the fibration. It is quite famous that Xiao in [35] proved the following

THEOREM 4.2 (Xiao). *If  $h$  is not isotrivial and  $B = \mathbb{P}^1$  then*

$$q \leq \frac{\tilde{g} + 1}{2}.$$

Furthermore he conjectured that, for a base  $B$  of positive genus, the relative irregularity of the fibration should satisfy

$$q_h \leq \frac{\tilde{g} + 1}{2}. \tag{4.2}$$

It is known that the inequality (4.2) is false: Pirola in [30], resp. Albano and Pirola in [1], explicitly constructed 1 fibration, resp. 3 fibrations, that do not satisfy (4.2). Indeed, a modified version of the conjecture supposes  $q_h \leq \lceil \frac{\tilde{g} + 1}{2} \rceil$ .

All the counterexamples are constructed considering families of covers in  $\mathcal{R}(K, g, r)$ . They have data:

- $K = \mathbb{Z}/3\mathbb{Z}, g = 1, r = 3$ ;
- $K = \mathbb{Z}/5\mathbb{Z}, g = 2, r = 0$ ;
- $K = \mathbb{Z}/3\mathbb{Z}, g = 4, r = 0$ ;
- $K = \mathbb{Z}/3\mathbb{Z}, g = 3, r = 0$ ;

They all have constant Prym variety: indeed they have been found by considering families of coverings of hyperelliptic curves (elliptic curves in the example of Pirola [30]) which lie in positive dimensional fibres of Prym maps.

REMARK 4.3. The family studied by Pirola, i.e. the first one listed above, turns out to be interesting also from another point of view. Indeed, in [14], the authors show that the locus described by  $J\tilde{C}$ , for  $\tilde{C}$  varying in the family, yields a Shimura subvariety of  $\mathcal{A}_4$  generically contained in the Torelli locus. Indeed it satisfies the condition (\*) studied in the same paper which is sufficient to

produce Shimura subvarieties generically contained in the Torelli locus. Moreover, in [15] and also in [10], it is proven that, via its Prym map, it is fibred in totally geodesic curves, countably many of which are Shimura. One of these Shimura fibres is the family (12) of [9]. This is exactly the family we use to study the family of Pirola.

REMARK 4.4. The example with data  $K = \mathbb{Z}/5\mathbb{Z}, g = 2, r = 0$  is curious in the same spirit of the previous Remark. Indeed, by [31, Theorem 3.1], it involves curves whose Jacobians have a non-trivial endomorphism algebra and the endomorphisms are not induced by the automorphisms of the curves. In [33], the second author shows that the Jacobians of such curves yield a new explicit Shimura subvariety of  $\mathcal{A}_2$  generically contained in the Torelli locus.

In order to find new counterexamples to Xiao's conjecture, it seems quite natural to generalize the idea of [30] and [1] considering families of towers  $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^1$  whose Prym map is constant but without requiring  $C$  to be hyperelliptic and considering any Galois covering  $\tilde{C} \rightarrow C$  of any degree. We have the following:

PROPOSITION 4.5. *Up to  $\tilde{g} = 12, s = 14$  (i.e. dimension 11), the only positive dimensional families of Galois towers  $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^1$  that have  $\tilde{N} - N < s - 3$  and disprove (4.2) are the family (12) of [9] and the three examples of [1].*

*Proof.* Using Proposition 4.1, we know that when  $\tilde{N} - N < s - 3$  the differential of the Prym map associated with the family is not injective. Under this assumption, computer calculations in MAGMA that impose  $q_h > \frac{\tilde{g}+1}{2}$  find only the four examples of the statement.  $\square$

Now we would like to explicitly describe the four examples as families of Galois towers  $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^1$ .

The first example we treat is the family (12) of [9]. Indeed the family of Pirola ([30]) cannot be realized as a Galois cover of  $\mathbb{P}^1$ . For this reason, we will study the family (12) of [9] which is contained in the family of Pirola and which describes Galois towers  $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^1$ , as already mentioned in Remark 4.3.

EXAMPLE 1.

$d = 3, s = 4, \tilde{g} = 4, g = 1, r = 3.$

$\tilde{G} = G(6, 2) = \mathbb{Z}/6\mathbb{Z} = \langle g : g^6 = 1 \rangle.$

$(\tilde{\theta}(\gamma_1), \dots, \tilde{\theta}(\gamma_4)) = (g^3, g^5, g^5, g^5), \quad K = \langle \sigma = g^2 \rangle \cong \mathbb{Z}/3\mathbb{Z}.$  The monodromy matrix is  $A = (3, 5, 5, 5).$

The action of  $\tilde{G}$  gives the decomposition  $H^0(\tilde{C}, \omega_{\tilde{C}}) = W_3 \oplus W_4 \oplus W_5$ , where  $W_n$  is the subspace of  $H^0(\tilde{C}, \omega_{\tilde{C}})$  where  $\tilde{G}$  acts via the character  $n$ . We have  $\dim W_3 = \dim W_4 = 1$  and  $\dim W_5 = 2$ . Moreover  $H^0(\tilde{C}, \omega_{\tilde{C}})_+ = W_3$ . Therefore  $(S^2 H^0(\tilde{C}, \omega_{\tilde{C}}))^{\tilde{G}} = (S^2 H^0(\tilde{C}, \omega_{\tilde{C}})_+)^{\tilde{G}} = S^2 W_3$  and so the multiplication map  $m : (S^2 V_-)^{\tilde{G}} \rightarrow H^0(\omega_{\tilde{C}}^{\otimes 2})^{\tilde{G}}$  is trivial. Hence the family is a curve that

lies in a fibre of the Prym map  $\mathcal{P}_{1,3}(3)$ . The relative irregularity of the fibred surface is  $q_h = \tilde{g} - g = 3 > \frac{5}{2} = \frac{\tilde{g}+1}{2}$ , hence it violates the inequality (4.2).

Next example is the first one that appears in [1, Section 4], given by an étale 5:1 cover.

EXAMPLE 2.

$d = 5, s = 6, \tilde{g} = 6, g = 2, r = 0.$

$\tilde{G} = G(10, 1) = D_5 = \langle g_1, g_2 : g_1^2 = g_2^5 = 1, g_1 g_2 g_1 = g_2^{-1} \rangle.$

$(\tilde{\theta}(\gamma_1), \dots, \tilde{\theta}(\gamma_6)) = (g_1 g_2^2, g_1 g_2^4, g_1 g_2, g_1 g_2^4, g_1 g_2, g_1 g_2),$

$K = \langle \sigma = g_2 \rangle \cong \mathbb{Z}/5\mathbb{Z}.$

Using the notation of MAGMA, we get  $H^0(\tilde{C}, \omega_{\tilde{C}}) = 2V_2 \oplus V_3 \oplus V_4$ , where  $V_i$  are irreducible representations of  $\tilde{G}$  such that  $\dim V_2 = 1$  and  $\dim V_3 = \dim V_4 = 2$ . We have that  $(S^2 H^0(\tilde{C}, \omega_{\tilde{C}}))^{\tilde{G}} = 3S^2 V_2 \oplus (S^2 V_3)^{\tilde{G}} \oplus (S^2 V_4)^{\tilde{G}}$  has dimension 5 and that  $H^0(\tilde{C}, \omega_{\tilde{C}})_+ = 2V_2$ . Therefore  $\dim(S^2 H^0(\tilde{C}, \omega_{\tilde{C}})_+)^{\tilde{G}} = 3$ . Since  $\dim(S^2 H^0(\tilde{C}, \omega_{\tilde{C}})_-)^{\tilde{G}} = 2 < s - 3 = 3$ , the Prym map is not injective on the family. Again inequality (4.2) does not hold:  $\tilde{g} - g = 4 > \frac{7}{2} = \frac{\tilde{g}+1}{2}$ .

Here we have the example of [1, Section 5].

EXAMPLE 3.

$d = 3, s = 10, \tilde{g} = 10, g = 4, r = 0.$

$\tilde{G} = G(6, 1) = D_3 = \langle g_1, g_2 : g_1^2 = g_2^3 = 1, g_1 g_2 g_1 = g_2^{-1} \rangle.$

$(\tilde{\theta}(\gamma_1), \dots, \tilde{\theta}(\gamma_{10})) = (g_1, g_1, g_1 g_2^2, g_1 g_2^2, g_1 g_2, g_1, g_1 g_2^2, g_1 g_2, g_1 g_2, g_1),$

$K = \langle \sigma = g_2 \rangle \cong \mathbb{Z}/3\mathbb{Z}.$

In this case  $H^0(\tilde{C}, \omega_{\tilde{C}}) = 4V_2 \oplus 3V_3$ , where  $\dim V_2 = 1$  and  $\dim V_3 = 2$ ,  $H^0(\tilde{C}, \omega_{\tilde{C}})_+ = 4V_2$ . Then  $(S^2 H^0(\tilde{C}, \omega_{\tilde{C}}))^{\tilde{G}} = 10S^2 V_2 \oplus 6(S^2 V_3)^{\tilde{G}}$  has dimension 16 and  $(S^2 H^0(\tilde{C}, \omega_{\tilde{C}})_+)^{\tilde{G}} = 10S^2 V_2$  has dimension 10. Hence  $\tilde{N} - N = 6 < 7$  guarantees  $d\mathcal{P}_{4,0}(3)$  not injective. Moreover  $\tilde{g} - g = 6 > \frac{11}{2} = \frac{\tilde{g}+1}{2}$  violates (4.2).

Finally we have the example [1, Section 6].

EXAMPLE 4.

$d = 3, s = 7, \tilde{g} = 6, g = 2, r = 2.$

$\tilde{G} = G(6, 1) = D_3 = \langle g_1, g_2 : g_1^2 = g_2^3 = 1, g_1 g_2 g_1 = g_2^{-1} \rangle.$

$(\tilde{\theta}(\gamma_1), \dots, \tilde{\theta}(\gamma_7)) = (g_1 g_2, g_1 g_2, g_1, g_1 g_2^2, g_1, g_1, g_2),$

$K = \langle \sigma = g_2 \rangle \cong \mathbb{Z}/3\mathbb{Z}.$

MAGMA gives  $H^0(\tilde{C}, \omega_{\tilde{C}}) = 2V_2 \oplus 2V_3$ , where  $\dim V_2 = 1$  and  $\dim V_3 = 2$  and  $H^0(\tilde{C}, \omega_{\tilde{C}})_+ = 2V_2$ . We have that  $(S^2 H^0(\tilde{C}, \omega_{\tilde{C}}))^{\tilde{G}} = 3S^2 V_2 \oplus 3(S^2 V_3)^{\tilde{G}}$  has dimension 6 and that  $(S^2 H^0(\tilde{C}, \omega_{\tilde{C}})_+)^{\tilde{G}} = 3S^2 V_2$  has dimension 3. Once again the differential of the Prym map along the family is not injective and  $\tilde{g} - g = 4 > \frac{7}{2} = \frac{\tilde{g}+1}{2}$  violates (4.2).

As already said, our data give a slightly different presentation of the last example presented by [1]. We easily observe that they are the same. Indeed our family yields the following diagram:

$$\begin{array}{ccc}
 \tilde{C} & \xrightarrow{3:1} & C \\
 \pi \downarrow 2:1 & \searrow \psi & \downarrow 2:1 \\
 D & \xrightarrow{3:1} & \mathbb{P}^1 \cong \tilde{C}/D_3,
 \end{array} \tag{4.3}$$

the curve  $D$  is obtained as the quotient of  $\tilde{C}$  by a lift of the hyperelliptic involution of  $C$ . We have 7 branch points  $z_1, \dots, z_7$  in  $\mathbb{P}^1$  and the map  $\tilde{C} \rightarrow \mathbb{P}^1$  has three ramification points of order 2 over  $z_1, \dots, z_6$ . Let us call them  $p_{ij}$ ,  $i = 1, \dots, 6$ ,  $j = 1, 2, 3$ . Moreover  $\psi$  has 2 ramification points of order three over  $z_7$ . Let us call them  $q_1, q_2$ . The 2:1 map  $\tilde{C} \rightarrow D$  ramifies on one among the  $p_{ij}$  for every  $i$ , while it is étale over the remaining two and it is étale over  $q_1, q_2$ . If we denote  $p = \pi(q_1) = \pi(q_2)$ , then we get the map  $D \xrightarrow{3:1} \mathbb{P}^1$  associated with the linear system  $|3p|$ . This is the starting point of the construction of Albano and Pirola. Indeed the map  $\tilde{C} \rightarrow C$  is étale over all the  $p_{ij}$ 's while it is completely ramified over  $q_1$  and  $q_2$ . When these two points are glued we get the singular curve  $C_p$  of [1]. The fibration of [1, Section 6] is constructed desingularizing such curves  $C_p$ , i.e. considering the curves  $\tilde{C}$  provided by our example. Therefore this example is clearly another presentation of the one discussed in [1, Section 6].

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## REFERENCES

- [1] A. ALBANO AND G.P. PIROLA, *Dihedral monodromy and Xiao fibrations*, Ann. Mat. Pura App. **195** (2016), 1255–1268.
- [2] F. BARDELLI, C. CILIBERTO, AND A. VERRA, *Curves of minimal genus on a general abelian variety*, Compos. Math. **96** (1995), 115–147.
- [3] A. BEAUVILLE, *Prym varieties and the Schottky problem*, Invent. Math. **41** (1977), 149–196.



- [4] S. CASALAINA MARTIN AND Z. ZHANG, *The moduli space of cubic surface pairs via the intermediate Jacobians of Eckardt cubic threefolds*, J. Lond. Math. Soc. (2) **104** (2021), 1–34.
- [5] E. COLOMBO, P. FREDIANI, A. GHIGI, AND M. PENEGINI, *Shimura curves in the Prym locus*, Commun. Contemp. Math. **21** (2019), no. 2, 1850009.
- [6] R. DONAGI, *The fibers of the Prym map*, Curves, Jacobians, and abelian varieties, Contemp. Math., Vol. 136 (1992), Amer. Math. Soc., 55–125.
- [7] R. DONAGI AND R. SMITH, *The structure of the Prym map*, Acta Math. **146** (1981), 25–102.
- [8] G. FARKAS, *Prym varieties and their moduli*, Contributions to algebraic geometry, 2012, EMS, Zürich, 215–255.
- [9] P. FREDIANI, A. GHIGI, AND M. PENEGINI, *Shimura varieties in the Torelli locus via Galois coverings*, Int. Math. Res. Not. **20** (2015), 10595–10623.
- [10] P. FREDIANI, A. GHIGI, AND I. SPELTA, *Infinitely many Shimura varieties in the Jacobian locus for  $g \leq 4$* , Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **22** (2021), 1597–1619.
- [11] P. FREDIANI AND G.P. GROSSELLI, *Shimura curves in the Prym loci of ramified double covers*, Internat. J. Math. **32** (2021), No. 14, 2150110–2150133.
- [12] P. FREDIANI, G.P. GROSSELLI, AND A. MOHAJER, *Higher dimensional Shimura varieties in the Prym loci of ramified double covers*, to appear in Math. Nachr., arXiv:2101.09016.
- [13] P. FREDIANI, J. C. NARANJO, AND I. SPELTA, *The fibers of the ramified Prym map*, Commun. Contemp. Math., doi: 10.1142/S0219199721500309.
- [14] P. FREDIANI, M. PENEGINI, AND P. PORRU, *Shimura varieties in the Torelli locus via Galois coverings of elliptic curves*, Geom. Dedicata **181** (2016), 177–192.
- [15] S. GRUSHEVSKY AND M. MÖLLER, *Explicit formulas for infinitely many Shimura curves in genus 4*, Asian J. Math. **22** (2018), 381–390.
- [16] M. GREEN AND R. LAZARSFELD, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math. **83** (1986), 73–90.
- [17] G.P. GROSSELLI AND A. MOHAJER, *Shimura subvarieties in the Prym locus of ramified Galois coverings*, Collect. Math. (2021), doi: 10.1007/s13348-021-00342-5.
- [18] A. IKEDA, *Global Prym-Torelli theorem for double coverings of elliptic curves*, Algebr. Geom. **7** (2020), 544–560.
- [19] H. LANGE AND A. ORTEGA, *Prym varieties of cyclic coverings*, Geom. Dedicata **150** (2011), 391–403.
- [20] H. LANGE AND A. ORTEGA, *The trigonal construction in the ramified case*, arxiv:1902.00251.
- [21] V. MARCUCCI AND J.C. NARANJO, *Prym varieties of double coverings of elliptic curves*, Int. Math. Res. Not. **6** (2014), 1689–1698.
- [22] V. MARCUCCI AND G.P. PIROLA, *Generic Torelli for Prym varieties of ramified coverings*, Compos. Math. **148** (2012), 1147–1170.
- [23] A. MOHAJER AND K. ZUO, *On Shimura subvarieties generated by families of abelian covers of  $\mathbb{P}^1$* , J. Pure Appl. Algebra **222** (2018), 931–949.
- [24] D. MUMFORD, *Prym varieties, I*, Contributions to Analysis, Academic Press,

- New York, 1974, 325–350.
- [25] D.S. NAGARAJ AND S. RAMANAN, *Polarisations of type  $(1, 2, \dots, 2)$  on abelian varieties*, Duke Math. J. **80** (1995), 157–194.
  - [26] J. C. NARANJO, *The positive dimensional fibres of the Prym map*, Pacific J. Math. **172** (1996), 223–226.
  - [27] J.C. NARANJO AND A. ORTEGA, *Generic injectivity of the Prym map for double ramified coverings*, Trans. Amer. Math. Soc. **371** (2019), 3627–3646.
  - [28] J.C. NARANJO AND A. ORTEGA, *Global Prym-Torelli for double coverings ramified in at least 6 points*, to appear in J. Algebraic Geom., arxiv:2005.11108.
  - [29] A. ORTEGA, *The Prym map: from coverings to Abelian varieties*, Notices Amer. Math. Soc. **67** (2020), 1316–1323.
  - [30] G.P. PIROLA, *On a conjecture of Xiao*, J. Reine Angew. Math. **431** (1992), 75–89.
  - [31] J. RIES, *The Prym variety for a cyclic unramified cover of a hyperelliptic Riemann surface*, J. Reine Angew. Math. **340** (1983), 59–69.
  - [32] S. RECILLAS, *Jacobians of curves with  $g_4^1$  are Prym varieties of trigonal curves*, Bol. Soc. Mat. Mex. **19** (1974), 9–13.
  - [33] I. SPELTA, *Shimura subvarieties via endomorphisms*, arXiv:2109.07261
  - [34] A. VERRA, *The Fibre of the Prym Map in Genus Three*, Math. Ann. **276** (1987), 433–448.
  - [35] G. XIAO, *Irregularity of surfaces with a linear pencil*. Duke Math. J. **55** (1987), no. 3, 597–602.

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