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Squeezing multisets into real numbers

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to Eugenio.

Writing a paper for a special issue dedicated to our lifelong friend and colleague Eugenio Omodeo is, for the two of us, both a honour and a pleasure. We have known and worked with Eugenio for such a long time that our first thought, when we started thinking about the topic for this paper, was how to write something that not only gave him some pleasure reading it, but was also suitable to get him directly involved in subsequent developments. Given Eugenio's interest in decidability themes for Diophantine equations and set theory, we do hope that he will react positively. Actually, we are sure he will take this further chance to collaborate with us with the usual enthusiasm, the incredible breadth of knowledge, and the unique style that he has always been able to contribute in every single project we undertook together.

Abstract. *In this paper we study the encoding*

$$\mathbb{R}_A(x) \quad = \quad \sum_{y \in x} 2^{-\mathbb{R}_A(y)},$$

mapping hereditarily finite sets and hypersets *– hereditarily finite sets admitting circular chains of memberships – into real numbers. The map $\mathbb{R}_A$ somewhat generalizes the well-known Ackermann's encoding $\mathbb{N}_A(x) = \sum_{y \in x} 2^{\mathbb{N}_A(y)}$, whose co-domain is $\mathbb{N}$, to nonnegative real numbers.*
In this work we define and study the further natural extension of the map $\mathbb{R}_A$ to the so-called multisets*. Such an extension is simply obtained by multiplying by k the code of each element having multiplicity equal to k.*
We prove that, under a rather natural injectivity assumption of $\mathbb{R}_A$ on the universe of multisets, the map $\mathbb{R}_A$ sends almost all multisets into transcendental numbers.

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1. Introduction

In 1937, W. Ackermann proposed the following encoding of hereditarily finite sets by natural numbers:

$$\mathbb{N}_A(x) \quad = \quad \sum_{y \in x} 2^{\mathbb{N}_A(y)}$$

(see [1]).

In this paper we study a very simple variation of $\mathbb{N}_A$ that turns out to be much more powerful. Our proposed variant is obtained by simply adding a minus sign to each of the exponents of 2 in the definition of $\mathbb{N}_A(x)$. That is:

$$\mathbb{R}_A(x) \quad = \quad \sum_{y \in x} 2^{-\mathbb{R}_A(y)}.$$

If, on the one hand, the original encoding $\mathbb{N}_A$ was perfect to prove a strict correspondence between *hereditarily finite sets* – the cumulative hierarchy HF of finite sets whose elements are themselves hereditarily finite sets, see Definition 2.1 – and the collection $\mathbb{N}$ of natural numbers, on the other hand the function $\mathbb{R}_A$ can be used to establish a much more ductile link between sets and (real) numbers. The ductility of the link consists in the fact that $\mathbb{R}_A$ can be used to map much larger, albeit still finitary, collections of objects strictly related to sets.

Any object dubbed "set" we consider here is going to be a *hereditary* set. Quoting Halmos in his celebrated *Naive Set Theory* ([10]):

> *Sets, as they are usually conceived, have elements or members. An element of a set may be a wolf, a grape, or a pidgeon. It is important to know that a set itself may also be an element of some other set. [...] What may be surprising is not so much that sets may occur as elements, but that for mathematical purposes no other elements need ever be considered.*
>
> P. Halmos

The fact that it is not restrictive to play with *pure* sets (i.e., sets whose only elements are sets themselves) is important and often overlooked. Especially so when, as in our case, we are mainly interested in *encoding* sets by numbers.

Hence, given a set $h \in \mathsf{HF}$, we are entitled to apply both the mapping $\mathbb{N}_A$ and $\mathbb{R}_A$ to h, as well as to any of its members $h' \in h$.

As far as $\mathbb{N}_A$ is concerned, a number of elementary observations and, in our opinion, basic and natural questions arise. Consider, for example, the following simple facts – already proved in [5] and reported below in Section 2 for the sake of completeness:

- $\mathbb{N}_A$ is a bijection from HF to $\mathbb{N}$;
- the binary expansion of $\mathbb{N}_A(h)$, for $h \in \mathsf{HF}$, fully describes the membership relation of h with any other element of HF: $h' \in h$ if and only if there is a 1 at position $\mathbb{N}_A(h')$ of the binary expansion of $\mathbb{N}_A(h)$;
- the mapping $\mathbb{N}_A$ allows one to cast a (natural) total order on HF.

The first of the above points, namely the fact that $\mathbb{N}_A$ is bijective, is a strong limitation for an encoding. Unless significantly modified, $\mathbb{N}_A$ is unsuitable to deal with any extension of HF: $\mathbb{N}_A$ leaves simply "no space" in its range to map collections extending HF that can, in any reasonable sense, be called *sets*. This limitation was considered in [14] and was the main motivation for the introduction of $\mathbb{R}_A$, which can be used to map (the full class of) hereditarily finite *cyclic* sets – the so-called *hypersets* – to real numbers.[1] As a matter of fact, $\mathbb{R}_A$ is not the only possible encoding for hypersets. Indeed, in [6] a number of other alternatives has been considered and studied. However, the coding map $\mathbb{R}_A$ turns out to be the natural variant of the Ackermann encoding $\mathbb{N}_A$, and it is definitely more elegant than its alternatives.

The fact that $\mathbb{R}_A$ can be used to encode a richer cumulative hierarchy of *sets* stems from the simple fact that its range is $\mathbb{R}$. Consider, for example, the set-theoretic equation $\zeta = \{\zeta\}$, which is satisfied by the sole hyperset having itself as its only member. The corresponding code is the unique real number satisfying the equation $x = 2^{-x}$, a fact easily seen to be true and briefly discussed in Section 2.3.

The "extra space" provided by having $\mathbb{R}_A$ ranging over $\mathbb{R}$ suggests other simple and natural questions. For instance, we can easily observe that the codes of the first four sets in the HF-hierarchy (namely $\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$, and $\{\{\varnothing\},\varnothing\}$) are rational numbers, the codes of the twelve sets of rank 3 in HF are quadratic irrationals, and – as is easily seen in some specific cases – some codes are even non-algebraic:

$$\mathbb{R}_A(\varnothing) = 0, \qquad \mathbb{R}_A(\{\varnothing\}) = 1, \qquad \mathbb{R}_A(\{\varnothing\}^2) = \frac{1}{2}, \qquad \mathbb{R}_A(\{\varnothing\}^3) = \frac{1}{\sqrt{2}},$$

$$\mathbb{R}_A(\{\varnothing\}^4) = 2^{-\frac{1}{\sqrt{2}}}, \qquad \mathbb{R}_A\big(\{\{\varnothing\}^3, \varnothing\}\big) = 2^{-\frac{1}{\sqrt{2}}} + 1, \qquad \text{etc.},$$

where, for the n-th iterated singleton $\underbrace{\{\cdots\{}_{n}\ \varnothing\ \underbrace{\}\cdots\}}_{n}$ of $\varnothing$, we are using the shorthand $\{\varnothing\}^n$. So, $\{\varnothing\}^2 = \{\{\varnothing\}\}$, $\{\varnothing\}^3 = \{\{\{\varnothing\}\}\}$, etc.

The main result of this paper is that, apart from the codes of the first sixteen sets of HF in the Ackermann encoding $\mathbb{N}_A$, the codes $\mathbb{R}_A(h)$ for $h \in \mathsf{HF}$

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[1] To the best of the authors' knowledge, the term "hyperset" has been first introduced by Jon Barwise and John Etchemendy in [3, page 38].

are all *non*-algebraic real numbers, under a conjecture on the injectivity of the extension of $\mathbb{R}_A$ to the *multi*-sets, to be discussed in Section 4, over a large portion of the multisets.

Multisets live between sets and numbers, as they are genuine sets but do use multiplicities to provide the possibility to express the fact that membership can be numerically qualified.

This "intermediate nature" is an example of a situation in which useful and syntactically well-defined objects take naturally the scene, while their semantics still poses some problem. The following quotation from [15] clearly illustrate the situation:

> *The case of mathematical logic, where the syntax was developed before the semantics, is exceptional because of the unusual history of the subject. An instance of a semantics without syntax is the theory of multisets. A multiset of a set S is a generalization of a subset of S, where elements are allowed to occur with multiplicities. Multisets can be added and multiplied; however, a characterization by algebraic operations of the family of multisets of a set S – an analog of what Boolean algebra is for sets – is not known at present.*
>
> G.C. Rota

We hope that our suggested usage of $\mathbb{R}_A$ on multisets – together with the consequences we will be able to prove here – will contribute to a clarification and a deeper understanding of their semantics.

2. Basics and Ackermann encoding $\mathbb{N}_A$.

In the following, we will extensively use the collection of hereditarily finite sets as domain of our encodings. Such collection is built using the classic *powerset* operator $\mathscr{P}(\cdot)$, which, in due time, will be adapted to our needs to build the cumulative hierarchy of multisets.

Let us begin by fixing some standard notation and notions.

Throughout the paper, we denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{N}^*$ the set of positive integers.

By recursion, the collection HF of the hereditarily finite sets is defined as follows.

Definition 2.1. *Let*

$$\begin{array}{rcl} \mathsf{HF}_0 & = & \emptyset, \\ \mathsf{HF}_{n+1} & = & \mathscr{P}(\mathsf{HF}_n), \quad \textit{for } n \in \mathbb{N}. \end{array}$$

Then

$$\mathsf{HF} = \bigcup_{n \in \mathbb{N}} \mathsf{HF}_n$$

is the cumulative hierarchy of the hereditarily finite sets.

For every $h \in \mathsf{HF}$, the rank of h – denoted $\mathsf{rk}(h)$ – is the least integer r such that $h \in \mathsf{HF}_{r+1}$.

Remark 2.2. The first four layers of the hierarchy HF are the following ones:

$$\begin{aligned}
\mathsf{HF}_0 &= \varnothing \\
\mathsf{HF}_1 &= \{\varnothing\} \\
\mathsf{HF}_2 &= \{\varnothing, \{\varnothing\}\} \\
\mathsf{HF}_3 &= \Big\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\Big\}.
\end{aligned}$$

Hence,

$$\mathsf{rk}(\varnothing) = 0, \quad \mathsf{rk}(\{\varnothing\}) = 1, \quad \mathsf{rk}(\{\{\varnothing\}\}) = \mathsf{rk}(\{\varnothing, \{\varnothing\}\}) = 2, \quad \text{etc.}$$

The Ackermann encoding $\mathbb{N}_A$, introduced in [1], is so defined.

Definition 2.3. *For $h \in \mathsf{HF}$, by recursion on $\mathsf{rk}(h)$ we put:*

$$\mathbb{N}_A(h) \quad = \quad \sum_{x \in h} 2^{\mathbb{N}_A(x)},$$

where, conventionally, the empty sum is evalutated to 0.

A number of very natural and elegant properties of $\mathbb{N}_A$ can be proved. To begin with, it is easy to see that the map $\mathbb{N}_A$ is a bijection between HF and $\mathbb{N}$, inducing an order over HF that we will call *Ackermann order*. Thus, we will henceforth denote by h_i the i-th element of HF in the Ackermann ordering; that is,

$$\mathbb{N}_A(h_i) = i, \quad \text{for } i \in \mathbb{N}.$$

We will denote by $\prec$ the Ackermann order on HF induced by $\mathbb{N}_A$, namely

$$h_i \prec h_j \quad \text{iff} \quad i < j, \quad \text{for } i, j \in \mathbb{N}.$$

Plainly, for all hereditarily finite sets h_i and h_j, we have:

$$h_i \in h_j \quad \Longrightarrow \quad i < j. \tag{1}$$

Indeed, if $h_i \in h_j$ then

$$i < 2^i = 2^{\mathbb{N}_A(h_i)} \leqslant \sum_{x \in h_j} 2^{\mathbb{N}_A(x)} = \mathbb{N}_A(h_j) = j.$$

The following proposition can be read as a restating of the bitwise comparison among natural numbers in set-theoretic terms.

Proposition 2.4. *For $h_i, h_j \in \mathsf{HF}$, we have:*

$$h_i \prec h_j \quad \textit{iff} \quad \max_{\prec}(h_i \backslash h_j) \prec \max_{\prec}(h_j \backslash h_i),$$

where we agree that $\max_{\prec} \varnothing \prec \max_{\prec} h$*, for any nonnull hereditarily finite set* h.

Proof. Since $h_i \prec h_j$ is equivalent to saying that $i < j$, it is sufficient to compare the base-two expansions of i and j in light of Definition 2.3 and the definition of Ackermann order. □

An immediate consequence of Proposition 2.4 is stated next.

Corollary 2.5. *For $h_i, h_j \in \mathsf{HF}$, if $h_i \subsetneq h_j$ then $i < j$.*

2.1. The real-valued map $\mathbb{R}_A$.

The map $\mathbb{R}_A$ is defined by recursion on rank as follows.

Definition 2.6. *For $h \in \mathsf{HF}$, we put:*

$$\mathbb{R}_A(h) = \sum_{x \in h} 2^{-\mathbb{R}_A(x)}.$$

The above definition bears a strong formal similarity with $\mathbb{N}_A$, as it is obtained from $\mathbb{N}_A$ by simply prefixing a minus sign to each of the exponents of 2 in $\mathbb{N}_A(x)$, but calls into play *real* numbers.

Plainly, $\mathbb{R}_A(h) \geqslant 0$, for every $h \in \mathsf{HF}$. In addition, we immediately have:

Lemma 2.7. *For $h \in \mathsf{HF}$,*

$$\mathbb{R}_A(h) = 0 \quad \textit{iff} \quad h = \varnothing.$$

From the principle of inclusion and exclusion, the following additional property for the map $\mathbb{R}_A$ can also be easily proved.

Lemma 2.8. *For $h, h' \in \mathsf{HF}$,*

$$\mathbb{R}_A(h \cup h') = \mathbb{R}_A(h) + \mathbb{R}_A(h') - \mathbb{R}_A(h \cap h').$$

Hence, if $h \cap h' = \varnothing$, we have

$$\mathbb{R}_A(h \cup h') = \mathbb{R}_A(h) + \mathbb{R}_A(h').$$

Table 1 reports the first 16 hereditarily finite sets and their $\mathbb{R}_A$-codes.

Table 1: The first 16 hereditarily finite sets and their $\mathbb{R}_A$-codes

i	h_i	$\mathbb{R}_A(h_i)$
0	$\varnothing$	0
1	$\{\varnothing\}$	1
2	$\{\varnothing\}^2$	$\frac{1}{2}$
3	$\{\{\varnothing\},\varnothing\}$	$\frac{3}{2}$
4	$\{\varnothing\}^3$	$\frac{1}{\sqrt{2}}$
5	$\{\{\varnothing\}^2,\varnothing\}$	$\frac{1}{\sqrt{2}}+1$
6	$\{\{\varnothing\}^2,\{\varnothing\}\}$	$\frac{1}{\sqrt{2}}+\frac{1}{2}$
7	$\{\{\varnothing\}^2,\{\varnothing\},\varnothing\}$	$\frac{1}{\sqrt{2}}+\frac{3}{2}$
8	$\{\{\{\varnothing\},\varnothing\}\}$	$\frac{1}{2\sqrt{2}}$
9	$\{\{\{\varnothing\},\varnothing\},\varnothing\}$	$\frac{1}{2\sqrt{2}}+1$
10	$\{\{\{\varnothing\},\varnothing\},\{\varnothing\}\}$	$\frac{1}{2\sqrt{2}}+\frac{1}{2}$
11	$\{\{\{\varnothing\},\varnothing\},\{\varnothing\},\varnothing\}$	$\frac{1}{2\sqrt{2}}+\frac{3}{2}$
12	$\{\{\{\varnothing\},\varnothing\},\{\varnothing\}^2\}$	$\frac{3}{2\sqrt{2}}$
13	$\{\{\{\varnothing\},\varnothing\},\{\varnothing\}^2,\varnothing\}$	$\frac{3}{2\sqrt{2}}+1$
14	$\{\{\{\varnothing\},\varnothing\},\{\varnothing\}^2,\{\varnothing\}\}$	$\frac{3}{2\sqrt{2}}+\frac{1}{2}$
15	$\{\{\{\varnothing\},\varnothing\},\{\varnothing\}^2,\{\varnothing\},\varnothing\}$	$\frac{3}{2\sqrt{2}}+\frac{3}{2}$

2.2. Algebraic and transcendental $\mathbb{R}_A$-codes

We recall that a real number is *algebraic* if it is a root of a non-zero polynomial in one variable with integer coefficients. Algebraic real numbers are closed under sum, difference, product, and quotient (with a non-zero denominator).

A real number that is not algebraic is *transcendental*.

By inspecting Table 1, it turns out that the $\mathbb{R}_A$-codes of the first sixteen h.f. sets have the form

$$\frac{1}{2}\Big(a + \frac{b}{\sqrt{2}}\Big), \quad \text{for } 0 \leqslant a, b \leqslant 3.$$

In particular, the first four h.f. sets have rational $\mathbb{R}_A$-codes of the form $\frac{a}{2}$ (for $0 \leqslant a \leqslant 3$), while the subsequent twelve sets in HF have quadratic irrational $\mathbb{R}_A$-codes of the form $\frac{1}{2}\Big(a + \frac{b}{\sqrt{2}}\Big)$ (for $0 \leqslant a \leqslant 3$ and $1 \leqslant b \leqslant 3$).

As an application of the Gelfond-Schneider-theorem (whose variant for real numbers is recalled below), we can prove that all the sets in HF of the form $\{ h_i \} \cup h_j$, where $4 \leqslant i \leqslant 15$ and $0 \leqslant j \leqslant 15$, have a transcendental $\mathbb{R}_A$-code.

Gelfond-Schneider theorem (for real numbers) ([9, 16]). *If a and b are algebraic real numbers (where $a > 0$, $a \neq 1$, and b is irrational), then a^b is transcendental.*[2]

Lemma 2.9. *For all i and j such that $4 \leqslant i \leqslant 15$ and $0 \leqslant j \leqslant 15$, the code $\mathbb{R}_A(\{h_i\} \cup h_j)$ is transcendental.*

Proof. It is enough to observe that $\mathbb{R}_A(h_i)$ is an irrational algebraic number for every i such that $4 \leqslant i \leqslant 15$, so that $\mathbb{R}_A(\{h_i\}) = 2^{-\mathbb{R}_A(h_i)}$ is transcendental by the Gelfond-Schneider theorem, and that $\mathbb{R}_A(h_j)$ is algebraic for every j such that $0 \leqslant j \leqslant 15$ (see Table 1). Thus, $\mathbb{R}_A(\{h_i\} \cup h_j) = \mathbb{R}_A(\{h_i\}) + \mathbb{R}_A(h_j)$ is transcendental for $4 \leqslant i \leqslant 15$ and $0 \leqslant j \leqslant 15$. $\square$

In fact, we conjecture that all h_i's in HF such that $i \geqslant 16$ have a transcendental $\mathbb{R}_A$ code. This will be shown to be a consequence of an injectivity conjecture for the extension $\mathbb{R}_A^\mu$ of $\mathbb{R}_A$ to multisets, to be stated in Section 4.2.

2.3. $\mathbb{R}_A$ on Hereditarily Finite Hypersets

Following [5], it can be seen that the domain of $\mathbb{R}_A$ can be expanded so as to include also the *non-well-founded* hereditarily finite sets, namely, the sets

[2] We recall that the Gelfond-Schneider theorem, obtained independently in 1934 by A. O. Gelfond and Th. Schneider, solves completely the seventh in a well-celebrated list of twenty-three problems posed by David Hilbert at the *International Congress of Mathematicians* held in Paris, 1900 (see [11]).

defined by (finite) systems of equations of the following form

$$\begin{cases} \varsigma_1 = \{\varsigma_{1,1}, \dots, \varsigma_{1,m_1}\} \\ \quad \vdots \\ \varsigma_n = \{\varsigma_{n,1}, \dots, \varsigma_{n,m_n}\}, \end{cases} \tag{2}$$

with *bisimilarity* as equality criterion (see [2] and [4], where the term *hyperset* is also used). For instance, the special case of the single set equation $\varsigma = \{\varsigma\}$ results into the equation (in real numbers)

$$x = 2^{-x}, \tag{3}$$

which provides the code of the unique (under bisimilarity) hyperset $\Omega = \{\Omega\}$.

Systems of set-theoretic equations of the form (2) are intended to fully specify the entire *transitive closure* of a collection of sets. To express this fact it is customary to require that any $\varsigma_{i,j}$ is chosen among $\varsigma_1, \dots, \varsigma_n$, a choice that also underlines the hereditary finiteness of the objects under study.

Going back to Ω, it is easy to see that the equation (3) has a unique solution in $\mathbb{R}$, since the functions x and 2^{-x} are, respectively, strictly increasing and strictly decreasing, so that the function $x - 2^{-x}$ is strictly increasing. In addition, we have:

$$x - 2^{-x}|_{x=\frac{1}{2}} = \frac{1}{2} - \frac{1}{\sqrt{2}} < 0 < 1 - \frac{1}{2} = x - 2^{-x}|_{x=1}.$$

Thus, the solution $\boldsymbol{\Omega}$ of (3) over $\mathbb{R}$, namely, the code of the hyperset defined by the set equation $\zeta = \{\zeta\}$, satisfies $\frac{1}{2} < \boldsymbol{\Omega} < 1$. Furthermore, much by the same argument used by the Pythagoreans to prove the irrationality of $\sqrt{2}$, it can easily be shown that $\boldsymbol{\Omega}$ is irrational. In fact, $\boldsymbol{\Omega}$ is transcendental. Indeed, if $\boldsymbol{\Omega}$ were algebraic, so would be $-\boldsymbol{\Omega}$ and therefore, by the Gelfond-Schneider theorem, $2^{-\boldsymbol{\Omega}} = \boldsymbol{\Omega}$ would be transcendental, contradicting the assumed algebraicity of $\boldsymbol{\Omega}$. Thus, $\boldsymbol{\Omega}$ must be transcendental after all.

It is interesting to notice that the solution to the equation $x = e^{-x}$ is the so-called *omega* constant, introduced by Lambert in [12] and studied also by Euler in [8].

While the encoding $\mathbb{N}_A$ is defined inductively (and this is perfectly in line with our intuition of the very basic properties of the collections of natural numbers $\mathbb{N}$ and of hereditarily finite sets HF – called HF^0 in [4]), the definition of $\mathbb{R}_A$, instead, is not inductive when extended to non-well-founded sets, and thus it requires a more careful analysis, as it must be *proved* that it univocally (and possibly injectively) associates (real) numbers to sets.

The injectivity of $\mathbb{R}_A$ on the collection of well-founded and non-well-founded hereditarily finite sets – henceforth, to be referred to as $\mathsf{HF}^{1/2}$, see [4] – was conjectured in [14] and is still an open problem. In [5] we proved that, given any finite collection $\hbar_1, \dots, \hbar_n$ of pairwise distinct sets in $\mathsf{HF}^{1/2}$ satisfying a system of set-theoretic equations of the form (2) in n unknowns, one can univocally determine real numbers $\mathbb{R}_A(\hbar_1), \dots, \mathbb{R}_A(\hbar_n)$ satisfying the following system of equations:

$$\begin{cases} \mathbb{R}_A(\hbar_1) = \sum_{k=1}^{m_1} 2^{-\mathbb{R}_A(\hbar_{1,k})} \\ \qquad\vdots \\ \mathbb{R}_A(\hbar_n) = \sum_{k=1}^{m_n} 2^{-\mathbb{R}_A(\hbar_{n,k})}. \end{cases}$$

A consequence of the results in [5] is the fact that the definition of $\mathbb{R}_A$ is well-given, as it associates a unique (real) number to each hereditarily finite hyperset in $\mathsf{HF}^{1/2}$. This extends to $\mathsf{HF}^{1/2}$ the first of the properties that the encoding $\mathbb{N}_A$ enjoys with respect to HF. Should $\mathbb{R}_A$ also enjoy the injectivity property, the proposed adaptation of $\mathbb{N}_A$ would be completely satisfactory, and $\mathbb{R}_A$ could be coherently dubbed an *encoding* for $\mathsf{HF}^{1/2}$.

The proof of the above result was given by defining a procedure operating by computing successive approximations of the final $\mathbb{R}_A$-codes. These approximations – infinitely many of them were necessary when the code involved hypersets – turned out to be real values that were naturally interpreted as the encodings of particular multisets. Elements with multiplicities greater than one resulted as (equal) approximations of different codes that were not yet being separated by the ongoing precision refinement of their final $\mathbb{R}_A$-value.

In [5] a full-fledged extension of $\mathbb{R}_A$ to the realm of multisets was not undertaken. This is done here, in the following section.

3. A gentle introduction to multisets

Given n *distinct* objects $O_1, O_2, \dots, O_n$ and n positive integers $m_1, m_2, \dots, m_n$, we denote by

$$[\,\underbrace{O_1, \dots, O_1}_{m_1},\ \underbrace{O_2, \dots, O_2}_{m_2},\ \dots,\ \underbrace{O_n, \dots, O_n}_{m_n}\,], \tag{4}$$

or any permutation of it, the multiset containing extacly m_i copies of O_i, for $i = 1, \dots, n$, and no additional members. Hence, the cardinality of (4) is $m_1 + m_2 + \cdots + m_n$. The integer m_i, for $i = 1, \dots, m$, is the *multiplicity* of O_i in (4).

We refer to the list notation (4) as the *multiset-builder square-bracket notation*. Thus, with the square-bracket notation element repetitions do count, whereas the order of elements is still irrelevant.

The multiset (4) is also denoted by the expression

$$\{^{m_1}O_1, {}^{m_2}O_2, \ldots, {}^{m_n}O_n\},$$

where the m_i's are usually omitted if they are equal to 1. the *multiset* M whose elements are $O_1, \ldots, O_n$ with multiplicities $m_1, \ldots, m_n$, respectively.[3]

Given a multiset M, an object O, and a multiplicity $m \in \mathbb{N}^*$, as a handy notation we write

$$O \in^m M \tag{5}$$

to mean that the object O belongs to M with a multiplicity exactly equal to m. In the context of multisets, we will also write $O \in M$, when $O \in^m M$ for some $m \in \mathbb{N}^*$. We conveniently extend the notation (5) also to the case when $m = 0$. Hence, we write $O \in^0 M$ to mean that $O \notin M$, namely $O \notin^m M$ for any $m \in \mathbb{N}^*$.

Following [13, p. 128], in this paper we define the multiset subset relation $M_1 \subseteq M_2$ by

$$(\forall O)(O \in M_1 \Longrightarrow O \in M_2),$$

namely $(\forall O)\big((\exists m \geqslant 1)O \in^m M_1 \Longrightarrow (\exists n \geqslant 1)O \in^n M_2\big)$. This is to be contrasted with the more common semantics

$$(\forall O)\big((\exists m \geqslant 1)O \in^m M_1 \Longrightarrow (\exists n \geqslant m)O \in^n M_2\big),$$

which enjoys antisymmetry.

Every multiset M can be conveniently described by its *multiplicity map* μ_M, where $\mu_M(O)$ is the multiplicity, possibly 0, of O in M. Hence, we have

$$O \in^m M \Longleftrightarrow \mu_M(O) = m,$$

for all M, O, and $m \in \mathbb{N}$.

Multisets can be handily described by means of the following square-brackets comprehension schema:

$$M = [e(\vec{x}) \mid \varphi(\vec{x})], \tag{6}$$

where $e(\vec{x})$ is a multiset expression over the variables $\vec{x}$ and $\varphi(\vec{x})$ is a predicate over $\vec{x}$. Specifically, for the multiset M defined by (6), $J \in^m M$ if and only if there are *exactly* m distinct ways to instantiate the variables $\vec{x}$ in (6) in such a way that $e(\vec{x}) = J$ and $\varphi(\vec{x})$ is true.

We will also make use of the following curly-brackets comprehension schema

$$M = \{e(\vec{x}) \mid \varphi(\vec{x})\}$$

to define a multiset M such that:

[3]We have chosen to depart from the more conventional notation for multisets in which multiplicities are indicated as right-upper indices, since right-upper indices will be used later as exponents to denote powers of multisets.

- $J \in M$ if and only if $\vec{x}$ can be instantiated in such a way that $e(\vec{x}) = J$ and $\varphi(\vec{x})$ is true, and

- all the members of M have multiplicity equal to 1.

We refer the reader also to [7] for formal theory of multisets.

3.1. Hereditarily finite multisets

When all the members of a given finite multiset M are finite multisets themselves, and this is true at any nesting depth, namely for members of members, for members of members of members, and so on, we say that M is a *hereditarily finite multiset*.

To formally define the collection HF^{μ} of hereditarily finite (h.f., for short) multisets, we will make use of the finitary μ-power-set operator $\mathscr{P}^{\mu}$ (a variant of the ordinary power-set operator $\mathscr{P}$). Specifically, for each multiset X, $\mathscr{P}^{\mu}(X)$ is the collection of all the multisets of the form $\left\{{}^{m_1}x_1, \ldots, {}^{m_n}x_n\right\}$, each with multiplicity equal to 1, where $x_1, \ldots, x_n$ stand for distinct members of X and $m_1, \ldots, m_n \in \mathbb{N}^*$, with $n \in \mathbb{N}$. In symbols:

$$\mathscr{P}^{\mu}(X) = \big\{\{{}^{m_1}x_1, \ldots, {}^{m_n}x_n\} \mid x_1, \ldots, x_n \in X \text{ (pairwise distinct) },\\ m_1, \ldots, m_n \in \mathbb{N}^*, n \in \mathbb{N}\big\}.$$

We are now ready to define the cumulative hierarchy HF^{μ} of the *hereditarily finite multisets.*

DEFINITION 3.1. *Let*

$$\begin{array}{rcl} \mathsf{HF}^{\mu}_0 & = & \varnothing, \\ \mathsf{HF}^{\mu}_{n+1} & = & \mathscr{P}^{\mu}(\mathsf{HF}^{\mu}_n), \quad \text{for } n \in \mathbb{N}. \end{array}$$

Then

$$\mathsf{HF}^{\mu} = \bigcup_{n \in \mathbb{N}} \mathsf{HF}^{\mu}_n$$

is the cumulative hierarchy of the hereditarily finite multisets.

$\mathsf{HF}^{\mu} = \bigcup_{n \in \mathbb{N}} \mathsf{HF}^{\mu}_n$ *is the collection of the* hereditarily finite multi-*sets, where:*

$$\begin{array}{rcl} \mathsf{HF}^{\mu}_0 & = & \varnothing, \\ \mathsf{HF}^{\mu}_{n+1} & = & \mathscr{P}^{\mu}(\mathsf{HF}^{\mu}_n). \end{array}$$

REMARK 3.2. The first four layers of the hierarchy HF^{μ} are:

$$\begin{aligned}
&\mathsf{HF}^{\mu}_0 = \varnothing\\
&\mathsf{HF}^{\mu}_1 = \{\,\varnothing\,\}\\
&\mathsf{HF}^{\mu}_2 = \Big\{\{\,{}^{0}\varnothing\,\},\{\,{}^{1}\varnothing\,\},\{\,{}^{2}\varnothing\,\},\dots\Big\} \qquad \text{(where } \{\,{}^{0}\varnothing\,\} \text{ stands for } \varnothing)\\
&\mathsf{HF}^{\mu}_3 = \Big\{\{\,{}^{m_1}\{\,{}^{n_1}\varnothing\,\},{}^{m_2}\{\,{}^{n_2}\varnothing\,\},\dots,{}^{m_k}\{\,{}^{n_k}\varnothing\,\}\,\} \mid n_1 < n_2 < \dots < n_k,\\
&\qquad\qquad m_1, m_2, \dots, m_k \in \mathbb{N}^*, k \in \mathbb{N}\Big\}.
\end{aligned}$$

On the grounds of the above definition, we can introduce a natural extension of the *rank* function to h.f. multisets.

DEFINITION 3.3. *The* rank $\mathsf{rk}(H)$ *of a multiset* $H \in \mathsf{HF}^{\mu}$ *is the least integer* r *such that* $H \in \mathsf{HF}^{\mu}_{r+1}$.

Thus, for instance,

$$\begin{aligned}
&\mathsf{rk}(\varnothing) = 0\\
&\mathsf{rk}\big(\{\,\varnothing\,\}\big) = \mathsf{rk}\big(\{\,{}^{2}\varnothing\,\}\big) = \mathsf{rk}\big(\{\,{}^{3}\varnothing\,\}\big) = \dots = 1\\
&\mathsf{rk}\Big(\{\,{}^{m_1}\{\,{}^{n_1}\varnothing\,\},{}^{m_2}\{\,{}^{n_2}\varnothing\,\},\dots,{}^{m_k}\{\,{}^{n_k}\varnothing\,\}\,\}\Big) = 2,
\end{aligned}$$

for all pairwise distinct $n_1, n_2, \dots, n_k \in \mathbb{N}$ such that $n_1 + \dots + n_k \geqslant 1$ and all $m_1, \dots, m_k \in \mathbb{N}^*$, with $k \geqslant 1$.

As is the case for sets in HF, for all multisets $H, H' \in \mathsf{HF}^{\mu}$ such that $H \in H'$, we plainly have $\mathsf{rk}(H) < \mathsf{rk}(H')$.

However, contrary to the case of ordinary sets, there are infinitely many multisets for any given rank $k \geqslant 1$, that is HF^{μ}_{k+1} is infinite for all $k \geqslant 1$.

The cardinality operator can be extended to multisets in the most natural way, by putting

$$|H| = \sum_{J \in H} \mu_H(J),$$

for all $H \in \mathsf{HF}^{\mu}$.

3.2. Set-theoretic operations on multisets

Given $H, K \in \mathsf{HF}^{\mu}$, the Boolean combinations $H \cup K$, $H \cap K$, and $H \backslash K$ of H and K, and the multiset $\bigcup H$ are defined in such a way that their multiplicity maps $\mu_{H \cup K}$, $\mu_{H \cap K}$, $\mu_{H \backslash K}$, and $\mu_{\bigcup H}$ satisfy pointwise the following identities:

$$\begin{aligned}
\mu_{H \cup K} &= \max(\mu_H, \mu_K),\\
\mu_{H \cap K} &= \min(\mu_H, \mu_K),\\
\mu_{H \backslash K} &= \mu_H - \mu_{H \cap K},\\
\mu_{\bigcup H} &= \max\{\mu_J \mid J \in H\}.
\end{aligned}$$

Plainly, when H and K are ordinary sets, the above definitions just yield the standard operators of union, intersection, set difference, and unary union, respectively.

From the very definition of $\bigcup H$, for every multiset H we have

$$\bigcup H = \bigcup\{K \mid K \in H\}.$$

In addition, much as with ordinary sets, for all $H, J \in \mathsf{HF}^{\mu}$, we have:

$$J \in \bigcup H \quad \Longleftrightarrow \quad J \in H', \text{ for some } H' \in H.$$

Indeed,

$$\begin{aligned} J \in \bigcup H \quad &\Longleftrightarrow \quad \mu_{\bigcup H}(J) \geqslant 1 \\ &\Longleftrightarrow \quad \mu_{H'}(J) \geqslant 1, \text{ for some } H' \in H \\ &\Longleftrightarrow \quad J \in H', \text{ for some } H' \in H. \end{aligned}$$

The operators of sum and product, which we define next, are specific to multisets

For $H, K \in \mathsf{HF}^{\mu}$, we denote by $H + K$ the multiset whose multiplicity map is $\mu_{H+K} = \mu_H + \mu_K$, so that

$$H + K = \left\{{}^{\mu_H(J)+\mu_K(J)}J \mid J \in H \cup K\right\}.$$

Accordingly, for $H_1, \ldots, H_n \in \mathsf{HF}^{\mu}$, we denote by $\sum_{i=1}^{n} H_i$ the multiset whose multiplicity map is $\sum_{i=1}^{n} \mu_{H_i}$. Thus, we have:

$$\sum_{i=1}^{n} H_i = \left\{{}^{\sum_{i=1}^{n} \mu_{H_i}(J)}J \;\middle|\; J \in \bigcup_{i=1}^{n} H_i\right\},$$

and therefore for all J we have

$$J \in \sum_{i=1}^{n} H_i \quad \Longleftrightarrow \quad J \in \bigcup_{i=1}^{n} H_i, \tag{7}$$

though in general $\sum_{i=1}^{n} H_i \neq \bigcup_{i=1}^{n} H_i$.[4]

Plainly, for all $H_1, \ldots, H_n \in \mathsf{HF}^{\mu}$ we have

$$\Big|\sum_{i=1}^{n} H_i\Big| = \sum_{i=1}^{n} |H_i|.$$

[4] In fact, $\sum_{i=1}^{n} H_i = \bigcup_{i=1}^{n} H_i$ holds true if and only if the multisets $H_1, \ldots, H_n$ are pairwise disjoint.

Indeed,

$$\Big|\sum_{i=1}^{n} H_i\Big| = \sum_{J \in \bigcup_{i=1}^{n} H_i} \sum_{i=1}^{n} \mu_{H_i}(J) = \sum_{i=1}^{n} \sum_{J \in \bigcup_{i=1}^{n} H_i} \mu_{H_i}(J)$$

$$= \sum_{i=1}^{n} \sum_{J \in H_i} \mu_{H_i}(J) = \sum_{i=1}^{n} |H_i|.$$

Next, for $H \in \mathsf{HF}^{\mu}$ and $a \in \mathbb{N}$, we denote by aH the multiset $\underbrace{H + \cdots + H}_{a \text{ times}}$, namely the multiset whose multiplicity map is $a\mu_H$. Thus, we have:

$$aH = \left\{{}^{a\mu_H(J)}J \mid J \in H\right\},$$

and when $a = 0$ we have $aH = \varnothing$.

Notice that, for $H, J \in \mathsf{HF}^{\mu}$ and $a \in \mathbb{N}^*$, we have:

$$J \in aH \iff J \in H, \tag{8}$$

and $|aH| = a|H|$.

Finally, the product $H \cdot K$ of two multisets H and K is defined as

$$H \cdot K = \left[{}^{\mu_H(H')\mu_K(K')}\big(H' + K'\big) \;\Big|\; H' \in H,\ K' \in K\right]. \tag{9}$$

Plainly, we have:

$$|H \cdot K| = |H| \cdot |K|.$$

Indeed,

$$|H \cdot K| = \sum_{\substack{H' \in H \\ K' \in K}} \mu_H(H')\mu_K(K') = \sum_{K' \in K} \Big(\mu_K(K') \cdot \sum_{H' \in H} \mu_H(H')\Big)$$

$$= \sum_{K' \in K} \big(\mu_K(K') \cdot |H|\big) = |H| \cdot \sum_{K' \in K} \mu_K(K') = |H| \cdot |K|.$$

By iterating the operation of multisets product, we can define powers of multisets with positive integer exponents. Specifically, for $H \in \mathsf{HF}^{\mu}$, we put $(H)^1 = H$, and recursively, for $n \in \mathbb{N}^*$, we put $(H)^{n+1} = (H)^n \cdot H$. Plainly, by induction it can be proved that $|(H)^n| = |H|^n$. Moreover, we have:

$$(H)^n = \left[{}^{\prod_{i=1}^{n} \mu_H(H_i)}\Big(\sum_{i=1}^{n} H_i\Big) \;\Big|\; H_1, \ldots, H_n \in H\right], \tag{10}$$

for all $n \in \mathbb{N}^*$.

Indeed, (10) clearly holds for $n = 1$. In addition, by assuming that (10) holds for a given $n \in \mathbb{N}^*$, we have:

$$
\begin{aligned}
(H)^{n+1} &= (H)^n \cdot H \\
&= \Big[{}^{\prod_{i=1}^n \mu_H(H_i)}\Big(\sum_{i=1}^n H_i\Big) \mid H_1, \dots, H_n \in H \Big] \cdot H \\
&= \Big[{}^{\left(\prod_{i=1}^n \mu_H(H_i)\right)\cdot \mu_H(H_{n+1})}\Big(\Big(\sum_{i=1}^n H_i\Big) + H_{n+1}\Big) \\
&\qquad\qquad \mid H_1, \dots, H_n \in H \wedge H_{n+1} \in H \Big] \\
&= \Big[{}^{\prod_{i=1}^{n+1} \mu_H(H_i)}\Big(\sum_{i=1}^{n+1} H_i\Big) \mid H_1, \dots, H_n, H_{n+1} \in H \Big],
\end{aligned}
$$

proving that (10) holds for $n + 1$ too. Thus, by induction, we get (10) for all $n \in \mathbb{N}^*$.

From (10), it follows immediately that

$$
J \in (H)^n \quad \Longleftrightarrow \quad J = \sum_{i=1}^n H_i, \quad \text{for some } H_1, \dots, H_n \in H, \tag{11}
$$

for all $H, J \in \mathsf{HF}^\mu$ and $n \in \mathbb{N}^*$.

Finally, we mention that the direct image of a multiset H under a map $f \colon \mathsf{HF}^\mu \to \mathsf{HF}^\mu$ can be expressed as follows

$$
f[H] = \sum_{J \in H} \left\{ {}^{\mu_H(J)} f(J) \right\}.
$$

Direct images distribute over multiset sums, namely for all $H, K \in \mathsf{HF}^\mu$ we have:

$$
f[H + K] = f[H] + f[K]. \tag{12}
$$

Indeed,

$$
\begin{aligned}
f[H + K] &= \sum_{J \in H+K} \left\{ {}^{\mu_{H+K}(J)} f(J) \right\} \\
&= \sum_{J \in H+K} \left\{ {}^{\mu_H(J)+\mu_K(J)} f(J) \right\} \\
&= \sum_{J \in H+K} \left(\left\{ {}^{\mu_H(J)} f(J) \right\} + \left\{ {}^{\mu_K(J)} f(J) \right\} \right) \\
&= \sum_{J \in H+K} \left\{ {}^{\mu_H(J)} f(J) \right\} + \sum_{J \in H+K} \left\{ {}^{\mu_K(J)} f(J) \right\} \\
&= f[H] + f[K].
\end{aligned}
$$

Then, by induction, we have

$$f[nH] = nf[H], \tag{13}$$

for all $H \in \mathsf{HF}^{\mu}$ and $n \in \mathbb{N}$.

Remark 3.4. It is easy to check that the multiset operators of union, intersection, sum, and product defined above are commutative and associative.

3.3. Embedding HF into HF^{μ}

Ordinary sets can just be regarded as multisets in which all multiplicities are equal to 1, at any nesting depth. More precisely, the collection HF can be embedded into HF^{μ} in a very simple and natural manner, via the following recursively defined *canonical embedding* $\mathcal{E}\colon \mathsf{HF} \to \mathsf{HF}^{\mu}$, where

$$\mathcal{E}(h) = \big\{\mathcal{E}(h') \mid h' \in h\big\}, \quad \text{for every } h \in \mathsf{HF}.$$

It is not hard to prove that the canonical embedding $\mathcal{E}$ is an injective homomorphism from HF into HF^{μ}, which, among many others, satisfies the following basic properties, for all $h, h' \in \mathsf{HF}$:

(a) $\mathsf{rk}(\mathcal{E}(h)) = \mathsf{rk}(h)$;

(b) $|\mathcal{E}(h)| = |h|$;

(c) $\mathcal{E}(h \cup h') = \mathcal{E}(h) \cup \mathcal{E}(h')$;

(d) $\mathcal{E}(h \cap h') = \mathcal{E}(h) \cap \mathcal{E}(h')$;

(e) $\mathcal{E}(\bigcup h) = \bigcup \mathcal{E}(h)$;

(f) $h' \in h \iff \mathcal{E}(h') \in \mathcal{E}(h)$;

(g) $h' \subseteq h \iff \mathcal{E}(h') \subseteq \mathcal{E}(h)$.

In the multiset context, in what follows we will freely identify a h.f. set h with its multiset image $\mathcal{E}(h)$, but this should generate no confusion.

3.4. Multiset polynomial expressions

Let $\mathbb{N}[x]$ be the collection of all polynomials in a single indeterminate x with coefficients in $\mathbb{N}$.

For every multiset $H \in \mathsf{HF}^{\mu}$ and every polynomial $P = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ in $\mathbb{N}[x]$ with $a_n > 0$, we can define the multiset $P(H)$ by recursion on the degree $n = \deg(P)$ of P, where we put $\deg(P_0) = 0$ for the null polynomial $P_0 = 0$.

For the base case, if $\deg(P) = 0$ and $P = a_0 > 0$, we put $P(H) = \{^{a_0}\varnothing\}$, whereas if $P = 0$ we put $P(H) = \varnothing$. Then, for $n \geqslant 1$, by assuming that $P = a_n x^n + Q$ (with $a_n \geqslant 1$ and $\deg(Q) < n$), we recursively set $P(H) = a_n(H)^n + Q(H)$.

The following property holds.

LEMMA 3.5. *Let $P \in \mathbb{N}[x]$ be any polynomial of positive degree with a* null *constant term. Then, for all $H \in \mathsf{HF}^{\mu}$ and $J \in P(H)$, we have $J = \sum_{i=1}^{\ell} H_i$, for some $H_1, \ldots, H_\ell \in H$ and $1 \leqslant \ell \leqslant \deg(P)$.*

Proof. Let $P = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x$, with $n, a_n \geqslant 1$, and let $H \in \mathsf{HF}^{\mu}$. Then we have:

$$
\begin{aligned}
J \in P(H) &\iff J \in \sum_{\ell=1}^{n} a_\ell (H)^\ell \\
&\iff J \in \bigcup_{\ell=1}^{n} (H)^\ell && \text{(by (7) and (8))} \\
&\iff \bigvee_{\ell=1}^{n} J \in (H)^\ell \\
&\iff \bigvee_{\ell=1}^{n} \Big(J = \sum_{i=1}^{\ell} H_i\Big), \text{ for some } H_1, \ldots, H_n \in H && \text{(by (11))} \\
&\iff J = \sum_{i=1}^{\ell} H_i, \text{ for some } H_1, \ldots, H_\ell \in H \\
&\qquad\qquad\qquad \text{and } 1 \leqslant \ell \leqslant n = \deg(P),
\end{aligned}
$$

proving the lemma. □

Next, we intend to prove that for any two polynomials $P, Q \in \mathbb{N}[x]$ with distinct degrees and any set $h \in \mathsf{HF}$ of rank at least 2, the multisets $P(h)$ and $Q(h)$ are distinct. Later, we will strengthen this fact to prove the main result of the paper, namely the transcendentality of the $\mathbb{R}_A$-codes of all the h.f. sets of rank at least 4 under the conjectured hypothesis that the coding map $\mathbb{R}_A^{\mu}$ is injective over a certain portion of HF^{μ}.

For every nonempty multiset $H \in \mathsf{HF}^{\mu}$, we denote by $H^{\top}$ the multiset comprising the members of H of maximum rank with the same multiplicities as in H, namely

$$H^{\top} = [J \mid J \in H \wedge \mathsf{rk}(J) + 1 = \mathsf{rk}(H)],$$

and we write $J \mathrel{\dot\in} H$ as a shorthand for $J \in H^{\top}$.

We now define a basic operator over multisets, which will play a key role in our proof of transcendentality.

Definition 3.6. *For every nonempty multiset $H \in \mathsf{HF}^{\mu}$, we define $\mathsf{M}(H)$ as the largest multiplicity in H of any member of maximum rank in H, namely*

$$\mathsf{M}(H) = \max \mu_H[H^{\top}].$$

It is convenient to extend the map M also to $\varnothing$, by conventionally setting $\mathsf{M}(\varnothing) = 0$.

Notice that $\mathsf{M}(H) \geqslant 1$, for every nonempty multiset H, and that $\mathsf{M}(h) = 1$, for every nonempty hereditarily finite set h.

The following lemma collects some basic properties of the operator M.

Lemma 3.7. *For all $H, K \in \mathsf{HF}^{\mu}$ and $h \in \mathsf{HF}$ such that $\mathsf{rk}(h) \geqslant 4$, the following equalities hold true:*

(a) $\mathsf{M}(H + K) \leqslant \mathsf{M}(H) + \mathsf{M}(K)$, *and therefore* $\mathsf{M}\big(\sum_{i=1}^{n} H_i\big) \leqslant \sum_{i=1}^{n} \mathsf{M}(H_i)$ *for all* $H_1, \ldots, H_n \in \mathsf{HF}^{\mu}$*;*

(b) *if* $\bigcup H \neq \varnothing$, *then* $\mathsf{M}\big(\bigcup H\big) = \max\{\mathsf{M}(J) \mid J \mathrel{\dot{\in}} H\}$*;*

(c) $\mathsf{M}\big(\bigcup(H + K)\big) = \max\big(\mathsf{M}(\bigcup H), \mathsf{M}(\bigcup K)\big)$*;*

(d) $\mathsf{M}\big(\bigcup(aH)\big) = \mathsf{M}(\bigcup H)$, *for every* $a \in \mathbb{N}^*$*;*

(e) $M\big(\bigcup(h)^n\big) = n$*;*

(f) $M\big(\bigcup P(h)\big) = \deg(P)$*.*

Proof. Concerning (a), if either one of H and K is empty, say H, then

$$\mathsf{M}(H + K) = \mathsf{M}(K) = \mathsf{M}(H) + \mathsf{M}(K).$$

On the other hand, if $H \neq \varnothing$ and $K \neq \varnothing$, then we have

$$\begin{aligned} \mathsf{M}(H + K) &= \max \mu_{H+K}[H + K] \\ &= \max\{\mu_H(J) + \mu_K(J) \mid J \in H + K\} \\ &= \max\{\mu_H(J) + \mu_K(J) \mid J \in H \cup K\} \\ &\leqslant \mathsf{M}(H) + \mathsf{M}(K), \end{aligned}$$

since $\mu_H(J) \leqslant \mathsf{M}(H)$ and $\mu_K(J) \leqslant \mathsf{M}(K)$ for all $J \in H \cup K$. In addition, by induction on $n \in \mathbb{N}^*$, it is immediate to prove that $\mathsf{M}\big(\sum_{i=1}^{n} H_i\big) \leqslant \sum_{i=1}^{n} \mathsf{M}(H_i)$, for all $H_1, \ldots, H_n \in \mathsf{HF}^{\mu}$.

Next, as for (b), if $\bigcup H \neq \varnothing$ we have:

$$\begin{aligned}
\mathsf{M}\big(\bigcup H\big) &= \max\ \mu_{\bigcup H}[\bigcup H] \\
&= \max\{\mu_{\bigcup H}(H'') \mid H'' \in \bigcup H\} \\
&= \max\big\{\max\{\mu_{H'}(H'') \mid H' \in H\} \bigm| H'' \in \bigcup H\big\} \\
&= \max\big\{\max\big(\{\mu_{H'}(H'') \mid H'' \in H' \in H\} \cup \{0\}\big) \bigm| H'' \in \bigcup H\big\} \\
&= \max\big\{\max\{\mu_{H'}(H'') \mid H'' \in H'\} \bigm| \varnothing \neq H' \in H\big\} \\
&= \max\{\max\ \mu_{H'}[H'] \bigm| \varnothing \neq H' \in H\} \\
&= \max\{\mathsf{M}(H') \bigm| \varnothing \neq H' \in H\}.
\end{aligned}$$

Regarding (c), if $\bigcup H = \varnothing$, then $\bigcup(H+K) = \bigcup K$ and $\mathsf{M}(\bigcup H) = 0$. Therefore $\mathsf{M}\big(\bigcup(H+K)\big) = \mathsf{M}\big(\bigcup K\big) = \max\big(\mathsf{M}(\bigcup H), \mathsf{M}(\bigcup K)\big)$. Likewise, if $\bigcup K = \varnothing$, then $\mathsf{M}\big(\bigcup(H+K)\big) = \max\big(\mathsf{M}(\bigcup H), \mathsf{M}(\bigcup K)\big)$.

On the other hand, if $\bigcup H \neq \varnothing$ and $\bigcup K \neq \varnothing$, then we have:

$$\begin{aligned}
\mathsf{M}\big(\bigcup(H+K)\big) &= \max\{\mathsf{M}(J) \mid J \mathrel{\dot\in} H+K\} && \text{(by (b))} \\
&= \max\big(\{\mathsf{M}(J) \mid J \mathrel{\dot\in} H\} \cup \{\mathsf{M}(J) \mid J \mathrel{\dot\in} K\}\big) \\
&= \max\big(\max\{\mathsf{M}(J) \mid J \mathrel{\dot\in} H\}, \max\{\mathsf{M}(J) \mid J \mathrel{\dot\in} K\}\big) \\
&= \max\big(\mathsf{M}(\bigcup H), \mathsf{M}(\bigcup K)\big) && \text{(by (b)).}
\end{aligned}$$

Next, concerning (d), let $a \in \mathbb{N}^*$. If $\bigcup H = \varnothing$ then $\bigcup(aH) = \varnothing$, and therefore $\mathsf{M}\big(\bigcup(aH)\big) = 0 = \mathsf{M}(\bigcup H)$. On the other hand, if $\bigcup H \neq \varnothing$, then by two applications of (b)) we have:

$$\begin{aligned}
\mathsf{M}\big(\bigcup(aH)\big) &= \max\{\mathsf{M}(J) \mid J \mathrel{\dot\in} aH\}\} \\
&= \max\{\mathsf{M}(J) \mid \varnothing \neq J \mathrel{\dot\in} H\}\} \\
&= \mathsf{M}(\bigcup H).
\end{aligned}$$

Concerning (e), we plainly have

$$M(\bigcup(h)^n) = \max\ \Big\{M\Big(\sum_{i=1}^{n} h'_i\Big) \Bigm| \varnothing \neq \sum_{i=1}^{n} h'_i \wedge h'_1, \ldots, h'_n \in h\Big\}. \tag{14}$$

Indeed,

$$\begin{aligned}
M(\bigcup(h)^n) &= \max\{M(H') \bigm| \varnothing \neq H' \in (h)^n\} && \text{(by (b))} \\
&= \max\ \Big\{M\Big(\sum_{i=1}^{n} h'_i\Big) \Bigm| \varnothing \neq \sum_{i=1}^{n} h'_i \wedge h'_1, \ldots, h'_n \in h\Big\} && \text{(by (10)).}
\end{aligned}$$

Hence, we have

$$M(\bigcup(h)^n) \leqslant n, \tag{15}$$

since, by (a),

$$M\Big(\sum_{i=1}^{n} h'_i\Big) \leqslant \sum_{i=1}^{n} M(h'_i) \leqslant n,$$

for all $h'_1, \ldots, h'_n \in h$.

In addition, letting $\bar{\bar{h}} \in \bar{h} \in h$ (since $\bigcup h \neq \varnothing$), then by (14) and (15) we have

$$n = \mu_{n\bar{h}}\big(\bar{\bar{h}}\big) \leqslant M(n\bar{h}) \leqslant M(\bigcup(h)^n) \leqslant n,$$

and therefore $M(\bigcup(h)^n) = n$.

Finally, as for (f), we proceed by induction on $n = \deg(P)$. If $\deg(P) = 0$, then $\bigcup P(h) = \varnothing$, and therefore $M(\bigcup P(h)) = 0 = \deg(P)$. For the inductive step, if $n = \deg(P) \geqslant 1$, then $P = a_n \cdot x^n + Q$, for some $a_n \geqslant 1$ and $Q \in \mathbb{N}[x]$ such that $\deg(Q) < n$, so that

$$\begin{aligned} M(\bigcup P(h)) &= M\big(\bigcup(a_n \cdot (h)^n + Q(h))\big) \\ &= \max\Big(M\big(\bigcup(a_n \cdot (h)^n)\big), M\big(\bigcup Q(h)\big)\Big) && \text{(by (c))} \\ &= \max\Big(M\big(\bigcup(h)^n\big), M\big(\bigcup Q(h)\big)\Big) && \text{(by (d))} \\ &= \max\Big(n, M\big(\bigcup Q(h)\big)\Big) && \text{(by (e))}. \end{aligned}$$

Since, by inductive hypothesis, $M\big(\bigcup Q(h)\big) = \deg(Q) < n$, then we have $\max(n, M(\bigcup Q(h))) = n$, so that $M(\bigcup P(h)) = n = \deg(P)$. Thus, by induction, property (f) follows. □

Later we will prove that properties (e) and (f) of the preceding lemma can be generalized with a certain reduction map to be defined in due course (see Definition 4.6 and Lemma 4.10(b),(c)).

The preceding lemma yields readily the following interesting injectivity result.

Corollary 3.8. *Let $h \in \mathsf{HF}$ be such that $\mathsf{rk}(h) \geqslant 2$, and let $P, Q \in \mathbb{N}[x]$ be polynomials with distinct degrees. Then $P(h) \neq Q(h)$.*

Proof. From Lemma 3.7(f), we have

$$M\big(\bigcup P(h)\big) = \deg(P) \neq \deg(Q) = M\big(\bigcup Q(h)\big).$$

Hence, it follows immediately that $P(h) \neq Q(h)$. □

4. $\mathbb{R}_A^\mu$ on Hereditarily Finite Multisets

The map $\mathbb{R}_A$ can be extended in a very natural manner to a map $\mathbb{R}_A^\mu$ over the collection HF^μ of the h.f. multisets.

Definition 4.1. *For every multiset $H \in \mathsf{HF}^\mu$, we put recursively:*

$$\mathbb{R}_A^\mu(H) = \sum_{K \in H} \mu_H(K) \cdot 2^{-\mathbb{R}_A^\mu(K)}.$$

It can easily be proved that for each $h \in \mathsf{HF}$, we have $\mathbb{R}_A^\mu\big(\mathcal{E}(h)\big) = \mathbb{R}_A(h)$, where $\mathcal{E}$ is the canonical embedding of HF into HF^μ defined in Section 3.3. Often we will freely write $\mathbb{R}_A^\mu(h)$ in place of $\mathbb{R}_A^\mu\big(\mathcal{E}(h)\big)$, for $h \in \mathsf{HF}$, but this will cause no confusion.

Remark 4.2. It should be readily observed that there are cases in which the injectivity of $\mathbb{R}_A^\mu$ does not hold, e.g.,

$$\mathbb{R}_A^\mu([\{\varnothing\},\{\varnothing\}]) = 2 \cdot 2^{-1} = 1 = 2^{-0} = \mathbb{R}_A^\mu([\varnothing]).$$

An analogous – rather annoying – feature would occur even if we modified ($\mathbb{R}_A$ and) $\mathbb{R}_A^\mu$ replacing the base 2 with any integer. Moreover, this fact will force us to state our conjecture (Conjecture 4.11) on the injectivity of $\mathbb{R}_A^\mu$ over just *a portion* of the universe of hereditarily finite multisets. Under Conjecture 4.11 we will prove the transcendence of $\mathbb{R}_A$-codes (Theorem 4.12).

We suspect that by replacing the base 2 of the exponent in the definition of ($\mathbb{R}_A$ and) $\mathbb{R}_A^\mu$ with non integral positive numbers greater than 1, say with e, we can solve this *local* problem and obtain the transcendence of the (modified) $\mathbb{R}_A$-codes under the assumption of the injectivity of $\mathbb{R}_A^\mu$ over the *full* HF^μ. However, we did not pursue this path here since the current definition of $\mathbb{R}_A^\mu$ is perfectly in line with $\mathbb{R}_A$ (and, ultimately, with $\mathbb{N}_A$) and, moreover, since we believe the base 2 could ease the proof of injectivity of $\mathbb{R}_A$.

Finally, we observe that, had we used a transcendental base for $\mathbb{R}_A$, the application of Gelfond-Schneider theorem, giving us some initial insight on the status of the collection of codes, would not have been possible.

4.1. Some basic identities of $\mathbb{R}_A^\mu$

We prove some basic identities related to the coding map $\mathbb{R}_A^\mu$.

Lemma 4.3. *For every $n \in \mathbb{N}^*$ and $H, H_1, \ldots, H_n \in \mathsf{HF}^\mu$, the following identities hold true:*

(a) $\mathbb{R}_A^\mu\Big(\sum_{i=1}^n H_i\Big) = \sum_{i=1}^n \mathbb{R}_A^\mu(H_i)$,

(b) $\mathbb{R}_A^\mu(nH) = n \cdot \mathbb{R}_A^\mu(H)$, *and*

(c) $\mathbb{R}^{\mu}_{A}(H_1 \cdot H_2) = \mathbb{R}^{\mu}_{A}(H_1) \cdot \mathbb{R}^{\mu}_{A}(H_2)$.

Proof. As for (a), letting $\bar{H} = \sum_{i=1}^{n} H_i$ we have:

$$\begin{aligned}
\mathbb{R}^{\mu}_{A}\Big(\sum_{i=1}^{n} H_i\Big) &= \sum_{K \in \bar{H}} \mu_{\bar{H}}(K) \cdot 2^{-\mathbb{R}^{\mu}_{A}(K)} = \sum_{K \in \bar{H}} \Big(\sum_{i=1}^{n} \mu_{H_i}(K)\Big) \cdot 2^{-\mathbb{R}^{\mu}_{A}(K)} \\
&= \sum_{i=1}^{n} \sum_{K \in \bar{H}} \mu_{H_i}(K) \cdot 2^{-\mathbb{R}^{\mu}_{A}(K)} \\
&= \sum_{i=1}^{n} \sum_{K \in H_i} \mu_{H_i}(K) \cdot 2^{-\mathbb{R}^{\mu}_{A}(K)} = \sum_{i=1}^{n} \mathbb{R}^{\mu}_{A}(H_i).
\end{aligned}$$

Next, concerning (b) we have:

$$\mathbb{R}^{\mu}_{A}(nH) = \sum_{K \in nH} \mu_{nH}(K) \cdot 2^{-\mathbb{R}^{\mu}_{A}(K)} = n \sum_{K \in H} \mu_{H}(K) \cdot 2^{-\mathbb{R}^{\mu}_{A}(K)} = n \cdot \mathbb{R}^{\mu}_{A}(h).$$

Finally, as regards (c) we have:

$$\begin{aligned}
\mathbb{R}^{\mu}_{A}(H_1 \cdot H_2) &= \sum_{K \in H_1 \cdot H_2} \mu_{H_1 \cdot H_2}(K) \cdot 2^{-\mathbb{R}^{\mu}_{A}(K)} \\
&= \sum_{\substack{K_1 \in H_1 \\ K_2 \in H_2}} \mu_{H_1}(K_1) \cdot \mu_{H_2}(K_2) \cdot 2^{-\mathbb{R}^{\mu}_{A}(K_1 + K_2)} \\
&= \sum_{\substack{K_1 \in H_1 \\ K_2 \in H_2}} \mu_{H_1}(K_1) \cdot \mu_{H_2}(K_2) \cdot 2^{-(\mathbb{R}^{\mu}_{A}(K_1) + \mathbb{R}^{\mu}_{A}(K_2))} && \text{(by (a))} \\
&= \sum_{K_1 \in H_1} \mu_{H_1}(K_1) 2^{-\mathbb{R}^{\mu}_{A}(K_1)} \cdot \sum_{K_2 \in H_2} \mu_{H_2}(K_2) 2^{-\mathbb{R}^{\mu}_{A}(K_2)} \\
&= \mathbb{R}^{\mu}_{A}(H_1) \cdot \mathbb{R}^{\mu}_{A}(H_2).
\end{aligned}$$

□

LEMMA 4.4. *For $h \in \mathsf{HF}$ and $P \in \mathbb{N}[X]$, we have*

$$\mathbb{R}^{\mu}_{A}\big(P(h)\big) = P\big(\mathbb{R}_{A}(h)\big).$$

Proof. If $P = 0$, then $P(h) = \varnothing$ and $P(\mathbb{R}_A(h)) = 0$. Hence, $\mathbb{R}^{\mu}_{A}(P(h)) = \mathbb{R}^{\mu}_{A}(\varnothing) = 0 = P(\mathbb{R}_A(h))$.

If $P = a_0 > 0$, then $P(h) = \{{}^{a_0}\varnothing\}$ and $P(\mathbb{R}_A(h)) = a_0$. Hence,

$$\mathbb{R}^{\mu}_{A}\big(\{{}^{a_0}\varnothing\}\big) = \mu_{\{{}^{a_0}\varnothing\}}(\varnothing) \cdot 2^{-\mathbb{R}^{\mu}_{A}(\varnothing)} = a_0 = P(\mathbb{R}_A(h)).$$

For $n = \deg(P) > 0$, we proceed inductively. In this case, we have $P(h) = a_n(h)^n + Q(h)$, for some $a_n \in \mathbb{N}^*$ and some polynomial $Q \in \mathbb{N}[x]$ of degree strictly less than n. Hence, we have:

$$\begin{aligned}
\mathbb{R}^{\mu}_{A}(P(h)) &= \mathbb{R}^{\mu}_{A}(a_n(h)^n + Q(h)) \\
&= \mathbb{R}^{\mu}_{A}(a_n(h)^n) + \mathbb{R}^{\mu}_{A}(Q(h)) && \text{(by Lemma 4.3(a))} \\
&= a_n \cdot \big(\mathbb{R}_{A}(h)\big)^n + Q(\mathbb{R}_{A}(h)) && \text{(by Lemma 4.3(b) and} \\
& && \quad \text{by inductive hypothesis)} \\
&= P(\mathbb{R}_{A}(h)). && \square
\end{aligned}$$

4.2. A conjecture and some consequences

We say that a multiset J occurs in (the transitive closure of) a multiset K *at nesting depth* $n \geqslant 0$, when there exist multisets $H_1, H_2, \ldots, H_n$ such that

$$J \in H_1 \in H_2 \in \cdots \in H_n \in K$$

(hence, J occurs in a multiset K at nesting depth 0 just when $J \in K$ holds).

The following construction will allow us to single out a maximal portion of HF^{μ} on which it is reasonable to expect $\mathbb{R}^{\mu}_{A}$ to be injective, in view of Remark 4.2. The steps we take below are based on the observation that, in order to obtain injectivity, it is necessary to enforce a restriction that allow us to uniquely retrieve m and n from any equation of the form $\mathbb{R}^{\mu}_{A}(\{{}^{m}\varnothing, {}^{n}\{\varnothing\}\}) = \frac{a}{2}$, where $a \in \mathbb{N}^*$. The simplest restriction takes the form $n \leqslant 1$, namely $n \in \{0,1\}$, which amounts to forbidding multiple occurrences of $\{\varnothing\}$ *at any level of depth* in multisets.

For $n \in \mathbb{N}$, we let $\mathfrak{H}^{n}_{1}$ be the collection of h.f. multisets that, at any nesting depth *at most* n, contain no occurrence of the set $\{\varnothing\}$ with a multiplicity larger than 1. Similarly, we let $\mathfrak{H}^{\infty}_{1}$ be the collection of h.f. multisets that, at *any* nesting depth, contain no occurrence of the set $\{\varnothing\}$ with a multiplicity larger than 1. More formally, we put:

$$\mathfrak{H}^{0}_{1} = \{H \in \mathsf{HF}^{\mu} \mid \mu_H(\{\varnothing\}) \leqslant 1\},$$

and recursively, for $n \in \mathbb{N}$,

$$\mathfrak{H}^{n+1}_{1} = \big\{H \in \mathfrak{H}^{0}_{1} \mid H \subseteq \mathfrak{H}^{n}_{1}\big\}. \tag{16}$$

Then we set

$$\mathfrak{H}^{\infty}_{1} = \bigcap_{n=0}^{\infty} \mathfrak{H}^{n}_{1}.$$

Lemma 4.5. *The following properties hold:*

(a) *if* $H \in \mathfrak{H}^{0}_{1}$ *and* $H \subseteq \mathfrak{H}^{\infty}_{1}$, *then* $H \in \mathfrak{H}^{\infty}_{1}$;

(b) *if* $H \in \mathfrak{H}_1^1$ *and* $\bigcup H \subseteq \mathfrak{H}_1^\infty$, *then* $H \in \mathfrak{H}_1^\infty$*;*

(c) $\mathsf{HF} \subseteq \mathfrak{H}_1^\infty$.[5]

Proof. Concerning (a), let H be such that $H \in \mathfrak{H}_1^0$ and $H \subseteq \mathfrak{H}_1^\infty$. Hence, by (16), we have $H \in \mathfrak{H}_1^{n+1}$ for every $n \in \mathbb{N}$, as $H \subseteq \mathfrak{H}_1^\infty \subseteq \mathfrak{H}_1^n$. Thus, by induction, $H \in \bigcap_{n=0}^\infty \mathfrak{H}_1^n = \mathfrak{H}_1^\infty$.

Next, as regards (b), let H be such that $H \in \mathfrak{H}_1^1$ and $\bigcup H \subseteq \mathfrak{H}_1^\infty$, and assume that $H \in \mathfrak{H}_1^n$ for some $n \geqslant 1$. We prove that $H \subseteq \mathfrak{H}_1^n$. Thus, let $J \in H$, so that

$$J \subseteq \bigcup H \subseteq \mathfrak{H}_1^\infty. \tag{17}$$

Since $H \in \mathfrak{H}_1^1$, then $H \subseteq \mathfrak{H}_1^0$, and therefore $J \in \mathfrak{H}_1^0$. From (a), the latter membership relation, together with (17), implies $J \in \mathfrak{H}_1^\infty$. By the arbitrariness of $J \in H$, we have $H \subseteq \mathfrak{H}_1^\infty \subseteq \mathfrak{H}_1^n$. Hence, $H \in \mathfrak{H}_1^{n+1}$, and so by induction $H \in \bigcap_{n=0}^\infty \mathfrak{H}_1^n = \mathfrak{H}_1^\infty$.

Finally, as for (c), we preliminarily observe that we plainly have $\mathsf{HF} \subseteq \mathfrak{H}_1^0$. Next, assuming that $\mathsf{HF} \subseteq \mathfrak{H}_1^n$ for some $n \in \mathbb{N}$, for every $h \in \mathsf{HF}$ we have $h \subseteq \mathfrak{H}_1^n$ and $h \in \mathfrak{H}_1^0$, so that (16) yields $h \in \mathfrak{H}_1^{n+1}$. Hence, $\mathsf{HF} \subseteq \mathfrak{H}_1^{n+1}$ holds. Thus, by induction, we have:

$$\mathsf{HF} \subseteq \bigcap_{n=0}^{\infty} \mathfrak{H}_1^n = \mathfrak{H}_1^\infty. \qquad \square$$

We define a reduction operator, which will be of basic relevance in the proof of transcendentality in Section 4.2.

Definition 4.6 (Reduction operator). *The reduction operator* $\boldsymbol{\rho}\colon \mathsf{HF}^\mu \to \mathsf{HF}^\mu$ *is defined by putting*

$$\boldsymbol{\rho}(H) = \left(H \backslash \left\{ {}^{2\lfloor \frac{k}{2} \rfloor} \{\varnothing\} \right\}\right) + \left\{ {}^{\lfloor \frac{k}{2} \rfloor} \varnothing \right\},$$

for $H \in \mathsf{HF}^\mu$*, where* $k = \mu_H(\{\varnothing\})$.

In plain terms, the reduction operator $\boldsymbol{\rho}$ replaces each *pair* of occurrences of $\{\varnothing\}$ in its argument by a single occurrence of $\varnothing$. Thus, $\mu_{\boldsymbol{\rho}(H)}(\{\varnothing\}) \leqslant 1$, for every $H \in \mathsf{HF}^\mu$, and therefore $\boldsymbol{\rho}(H) \in \mathfrak{H}_1^0$.

Lemma 4.7. *For every* $H \in \mathsf{HF}^\mu$*, we have*

$$\mathbb{R}_A^\mu(\boldsymbol{\rho}(H)) = \mathbb{R}_A^\mu(H).$$

[5]Here and throughout the lemma, we are freely identifying each h.f. set h with its image $\mathcal{E}(h)$ under the canonical embedding $\mathcal{E}$ of HF into HF^μ.

Proof. Notice that $\mathbb{R}^{\mu}_{A}(\boldsymbol{\rho}(H)) = \mathbb{R}^{\mu}_{A}(H)$. Indeed, putting again $k = \mu_H(\{\varnothing\})$, we have:

$$\begin{aligned}
\mathbb{R}^{\mu}_{A}(\boldsymbol{\rho}(H)) &= \mathbb{R}^{\mu}_{A}\Big(\Big(H \setminus \big\{^{2\lfloor\frac{k}{2}\rfloor}\{\varnothing\}\big\}\Big) + \big\{^{\lfloor\frac{k}{2}\rfloor}\varnothing\big\}\Big)\\
&= \mathbb{R}^{\mu}_{A}(H) - \mathbb{R}^{\mu}_{A}\Big(\big\{^{2\lfloor\frac{k}{2}\rfloor}\{\varnothing\}\big\}\Big) + \mathbb{R}^{\mu}_{A}\Big(\big\{^{\lfloor\frac{k}{2}\rfloor}\varnothing\big\}\Big)\\
&= \mathbb{R}^{\mu}_{A}(H) - 2\left\lfloor\frac{k}{2}\right\rfloor \cdot \mathbb{R}^{\mu}_{A}(\{\{\varnothing\}\}) + \left\lfloor\frac{k}{2}\right\rfloor \cdot \mathbb{R}^{\mu}_{A}(\{\varnothing\})\\
&= \mathbb{R}^{\mu}_{A}(H) - 2\left\lfloor\frac{k}{2}\right\rfloor \cdot \frac{1}{2} + \left\lfloor\frac{k}{2}\right\rfloor\\
&= \mathbb{R}^{\mu}_{A}(H). \qquad \square
\end{aligned}$$

The preceding lemma yields readily the following result.

COROLLARY 4.8. *For each $H \in \mathsf{HF}^{\mu}$, we have $\mathbb{R}^{\mu}_{A}(\boldsymbol{\rho}[H]) = \mathbb{R}^{\mu}_{A}(H)$.*

Proof. Indeed, by Lemma 4.7:

$$\begin{aligned}
\mathbb{R}^{\mu}_{A}(\boldsymbol{\rho}[H]) &= \mathbb{R}^{\mu}_{A}\Big(\sum_{J\in H}\big\{^{\mu_H(J)}\boldsymbol{\rho}(J)\big\}\Big)\\
&= \sum_{J\in H}\mathbb{R}^{\mu}_{A}\Big(\big\{^{\mu_H(J)}\boldsymbol{\rho}(J)\big\}\Big)\\
&= \sum_{J\in H}\mu_H(J)\cdot 2^{-\mathbb{R}^{\mu}_{A}(\boldsymbol{\rho}(J))}\\
&= \sum_{J\in H}\mu_H(J)\cdot 2^{-\mathbb{R}^{\mu}_{A}(J)}\\
&= \mathbb{R}^{\mu}_{A}(H). \qquad \square
\end{aligned}$$

LEMMA 4.9. *Let $h \in \mathsf{HF}$ be such that $\{\varnothing\} \notin h$ and let $P \in \mathbb{N}[x]$ be any polynomial with a null constant term. Then $\boldsymbol{\rho}[P(h)] \in \mathfrak{H}^{\infty}_{1}$.*

Proof. Since P has a null constant term, by Lemma 3.5 all members of $P(h)$ have the form

$$h'_1 + \cdots + h'_\ell, \tag{18}$$

for some $h'_1, \ldots, h'_\ell \in h$ (with $1 \leqslant \ell \leqslant \deg(P)$).

In consideration that $\{\varnothing\} \notin h$, no element of the form (18) can equal $\{\varnothing\}$. Thus, $\{\varnothing\} \notin P(h)$ and therefore $\{\varnothing\} \notin \boldsymbol{\rho}[P(h)]$ as well, so that $\boldsymbol{\rho}[P(h)] \in \mathfrak{H}^{0}_{1}$. In addition, for every $K \in \boldsymbol{\rho}[P(h)]$, we have $K = \boldsymbol{\rho}(H)$, for some $H \in P(h)$, and therefore $K \in \mathfrak{H}^{0}_{1}$. Hence, $\boldsymbol{\rho}[P(h)] \subseteq \mathfrak{H}^{0}_{1}$ and so $\boldsymbol{\rho}[P(h)] \in \mathfrak{H}^{1}_{1}$.

Plainly, $\bigcup\boldsymbol{\rho}[P(h)] \subseteq \mathsf{HF}$. Indeed, if $J \in \bigcup\boldsymbol{\rho}[P(h)]$, then J is a member of some member of $\boldsymbol{\rho}[P(h)]$ and so $J \in \boldsymbol{\rho}(H)$, for some $H \in P(h)$. Hence, either $J = \varnothing$ or J is a member of a multiset of the form (18). In either case, $J \in \mathsf{HF}$, and therefore $\bigcup \boldsymbol{\rho}[P(h)] \subseteq \mathsf{HF}$. Thus, from Lemma 4.5(c), we have $\bigcup \boldsymbol{\rho}[P(h)] \subseteq \mathfrak{H}_1^{\infty}$, and therefore Lemma 4.5(b) yields $\bigcup \boldsymbol{\rho}[P(h)] \in \mathfrak{H}_1^{\infty}$, as $\bigcup \boldsymbol{\rho}[P(h)] \in \mathfrak{H}_1^{1}$. □

Lemma 4.10. *For every $H \in \mathsf{HF}^{\mu}$ such that $\mathsf{rk}(H) \geqslant 3$ and every $h \in \mathsf{HF}$ such that $\mathsf{rk}(h) \geqslant 4$, the following equalities hold true:*

(a) $\mathsf{M}(\boldsymbol{\rho}(H)) = \mathsf{M}(H)$;

(b) $\mathsf{M}\big(\bigcup \boldsymbol{\rho}[(h)^n]\big) = n$, *for every* $n \in \mathbb{N}$;

(c) $\mathsf{M}\big(\bigcup \boldsymbol{\rho}[P(h)]\big) = \deg(P)$, *for every* $P \in \mathbb{N}[x]$.

Proof. Concerning (a), let $H \in \mathsf{HF}^{\mu}$ be such that $\mathsf{rk}(H) \geqslant 3$. Hence, $\boldsymbol{\rho}(H)^{\top} = H^{\top}$, so that

$$\mathsf{M}(\boldsymbol{\rho}(H)) = \max \mu_{\boldsymbol{\rho}(H)}[\boldsymbol{\rho}(H)^{\top}] = \max \mu_H[H^{\top}] = \mathsf{M}(H).$$

Next, as for (b), let $n \in \mathbb{N}$. Then we have:

$$\begin{aligned} \mathsf{M}(\textstyle\bigcup \boldsymbol{\rho}[(h)^n]) &= \max\{\mathsf{M}(H) \mid H \mathrel{\dot\in} \boldsymbol{\rho}[(h)^n]\} && \text{(by (10))} \\ &= \max \Big\{\mathsf{M}\Big(\boldsymbol{\rho}\Big(\sum_{i=1}^{n} h'_i\Big)\Big) \;\Big|\; h'_1 \mathrel{\dot\in} h, h'_2, \ldots, h'_n \in h\Big\}. \end{aligned}$$

But,

$$\begin{aligned} \mathsf{M}\Big(\boldsymbol{\rho}\Big(\sum_{i=1}^{n} h'_i\Big)\Big) &= \mathsf{M}\Big(\sum_{i=1}^{n} h'_i\Big) && \text{(by (a))} \\ &\leqslant \sum_{i=1}^{n} \mathsf{M}(h'_i) && \text{(by Lemma 3.7(a))} \\ &\leqslant n, \end{aligned}$$

for all $h'_1 \mathrel{\dot\in} h, h'_2, \ldots, h'_n \in h$. Hence,

$$\mathsf{M}(\textstyle\bigcup \boldsymbol{\rho}[(h)^n]) \leqslant n \tag{19}$$

holds.

In addition, letting $\bar{h} \mathrel{\dot\in} h$ and $\bar{\bar{h}} \mathrel{\dot\in} \bar{h}$, we have

$$\begin{aligned} n = \mu_{n\bar{h}}(\bar{\bar{h}}) &\leqslant \mathsf{M}(n\bar{h}) \\ &\leqslant \mathsf{M}(\textstyle\bigcup (h)^n) && (\text{since } n\bar{h} \subseteq \textstyle\bigcup (h)^n) \\ &\leqslant \mathsf{M}(\boldsymbol{\rho}[\textstyle\bigcup (h)^n]) && \text{(by (a))}. \end{aligned}$$

Thus, in view of (19), we get $\mathsf{M}(\boldsymbol{\rho}[\bigcup (h)^n]) = n$, proving (b).

Finally, concerning (c), we proceed by induction on $\deg(P)$. If $\deg(P) = 0$, then $\bigcup P(h) = \emptyset$, and so $\bigcup \boldsymbol{\rho}[P(h)] = \emptyset$. Hence, $\mathsf{M}(\bigcup \boldsymbol{\rho}[P(h)]) = 0 = \deg(P)$.

For the inductive step, if $n = \deg(P) \geqslant 1$, then $P = a_n x^n + Q$, for some $a_n \geqslant 1$ and some polynomial $Q \in \mathbb{N}[x]$ of degree less than n. Hence, we have:

$$\begin{aligned}
\mathsf{M}(\textstyle\bigcup \boldsymbol{\rho}[P(h)]) &= \mathsf{M}\big(\textstyle\bigcup \boldsymbol{\rho}[a_n (h)^n + Q(h)]\big) \\
&= \mathsf{M}\Big(\textstyle\bigcup \big(\boldsymbol{\rho}[a_n (h)^n] + \boldsymbol{\rho}[Q(h)]\big)\Big) && \text{(by (12))} \\
&= \max\Big(M\big(\textstyle\bigcup \boldsymbol{\rho}[a_n (h)^n]\big), M\big(\textstyle\bigcup \boldsymbol{\rho}[Q(h)]\big)\Big) && \text{(by Lemma 3.7(c))} \\
&= \max\Big(M\big(\textstyle\bigcup (a_n \boldsymbol{\rho}[(h)^n])\big), M\big(\textstyle\bigcup \boldsymbol{\rho}[Q(h)]\big)\Big) && \text{(by (13))} \\
&= \max\Big(M\big(\textstyle\bigcup \boldsymbol{\rho}[(h)^n]\big), M\big(\textstyle\bigcup \boldsymbol{\rho}[Q(h)]\big)\Big) && \text{(by Lemma 3.7(d))} \\
&= \max\Big(n, M\big(\textstyle\bigcup \boldsymbol{\rho}[Q(h)]\big)\Big) && \text{(by (b)).}
\end{aligned}$$

Since, by inductive hypothesis,

$$M\big(\textstyle\bigcup \boldsymbol{\rho}[Q(h)]\big) = \deg(Q) < n\,, \quad \max(n, \mathsf{M}(\textstyle\bigcup \boldsymbol{\rho}[Q(h)])) = n\,,$$

and therefore $\mathsf{M}(\bigcup \boldsymbol{\rho}[P(h)]) = n = \deg(P)$. Thus, by induction, property (c) follows. □

CONJECTURE 4.11. *The coding map $\mathbb{R}^{\mu}_{A}$ is injective over the collection $\mathfrak{H}^{\infty}_{1}$.*

THEOREM 4.12. *Under Conjecture 4.11, every hereditarily finite set of rank at least* 4 *has a transcendental $\mathbb{R}_A$-code.*

Proof. Let us assume that Conjecture 4.11 holds and that, for contradiction, there exists a set $h \in \mathsf{HF}$ of rank at least 4 and such that its code $\mathbb{R}_A(h)$ is algebraic.

Without loss of generality, we may assume that $h \cap \mathsf{HF}_3 = \emptyset$, namely that all members of h have rank at least 3. Indeed, letting $\bar{h} = h \setminus \mathsf{HF}_3$, then $h = \bar{h} \cup (h \cap \mathsf{HF}_3)$ and therefore, by Lemma 2.8,

$$\mathbb{R}_A(\bar{h}) = \mathbb{R}_A(h) - \mathbb{R}_A(h \cap \mathsf{HF}_3),$$

since $\bar{h} \cap (h \cap \mathsf{HF}_3) = \emptyset$. Thus the code $\mathbb{R}_A(\bar{h})$ is algebraic, as it is the difference of two algebraic numbers.[6] In addition, just by construction we have

$$\mathsf{rk}(\bar{h}) = \mathsf{rk}(h) \geqslant 4 \quad \text{and} \quad \bar{h} \cap \mathsf{HF}_3 = \emptyset.$$

Let $P \in \mathbb{Z}[x]$ be any non-null polynomial with a *null constant term* for which $P(\mathbb{R}_A(h)) = 0$ holds. Let us gather all the positive terms of P in the polynomial

[6] Indeed, $\mathbb{R}_A(h \cap \mathsf{HF}_3) \in \left\{ \frac{1}{2}\left(a + \frac{b}{\sqrt{2}}\right) \;\middle|\; 0 \leqslant a, b \leqslant 3 \right\}$; see Table 1.

$P^+ \in \mathbb{N}[x]$ and set $P^- = P^+ - P$, so that $P^- \in \mathbb{N}[x]$ and $P = P^+ - P^-$. Given that $P(\mathbb{R}_A(h)) = 0$, then $P^+(\mathbb{R}^\mu_A(h)) = P^-(\mathbb{R}^\mu_A(h))$ and therefore, by Lemma 4.4, we have

$$\mathbb{R}^\mu_A(P^+(h)) = \mathbb{R}^\mu_A(P^-(h)). \tag{20}$$

Since both P^+ and P^- have a null constant term, Lemma 3.5 implies that both $P^+(h)$ and $P^-(h)$ are multisets whose members all have the form $h'_1 + h'_2 + \ldots + h'_\ell$, with $h'_1, h'_2, \ldots, h'_\ell \in h$ (for some $\ell \in \{1, \ldots, \deg(P)\}$). Thus, recalling that $\varnothing, \{\varnothing\} \notin h$, we get $\varnothing, \{\varnothing\} \notin P^+(h), P^-(h)$, so that $P^+(h), P^-(h) \in \mathfrak{H}^0_1$. Given that $\mathsf{rk}(h) \geqslant 4$ and that, just by construction, $\deg(P^+) \neq \deg(P^-)$, Corollary 3.8 yields $P^+(h) \neq P^-(h)$.

If $P^+(h), P^-(h) \in \mathfrak{H}^\infty_1$, in view of Conjecture 4.11, we would readily have

$$\mathbb{R}^\mu_A(P^+(h)) \neq \mathbb{R}^\mu_A(P^-(h)),$$

contradicting (20).

However, $P^+(h)$ and/or $P^-(h)$ may fail to belong to $\mathfrak{H}^\infty_1$, since – as already observed – all the members of $P^+(h)$ and $P^-(h)$ have the form $h'_1 + \cdots + h'_\ell$, with $h'_1, \ldots, h'_\ell \in h$, and therefore some of them may contain the set $\{\varnothing\}$ with a multiplicity strictly greater than 1. If this were the case, we will need to follow an alternate route, which makes use of the reduction operator $\boldsymbol{\rho}$.

Specifically, in place of $P^+(h)$ and $P^-(h)$, we will consider their direct images $\boldsymbol{\rho}[P^+(h)]$ and $\boldsymbol{\rho}[P^-(h)]$ under the reduction operator $\boldsymbol{\rho}$.

In view of Corollary 4.8, by (20) we have

$$\mathbb{R}^\mu_A(\boldsymbol{\rho}[P^+(h)]) = \mathbb{R}^\mu_A(\boldsymbol{\rho}[P^-(h)]). \tag{21}$$

On the other hand, by two applications of Lemma 4.10(c), we have

$$\mathsf{M}\big(\bigcup \boldsymbol{\rho}[P^+(h)]\big) = \deg(P^+) \neq \deg(P^-) = \mathsf{M}\big(\bigcup \boldsymbol{\rho}[P^-(h)]\big).$$

Hence,

$$\boldsymbol{\rho}[P^+(h)] \neq \boldsymbol{\rho}[P^-(h)]. \tag{22}$$

Since $\boldsymbol{\rho}[P^+(h)], \boldsymbol{\rho}[P^-(h)] \in \mathfrak{H}^\infty_1$ (by Lemma 4.9), in the light of Conjecture 4.11, (22) yields

$$\mathbb{R}^\mu_A(\boldsymbol{\rho}[P^+(h)]) \neq \mathbb{R}^\mu_A(\boldsymbol{\rho}[P^-(h)]).$$

However, the latter inequality is in blatant contradiction with (21). Thus, our initial assumption that (under Conjecture 4.11) there may exist a set $h \in \mathsf{HF}$ of rank at least 4 and such that $\mathbb{R}_A(h)$ is algebraic is untenable, proving the theorem. $\square$

5. Conclusions and Open Problems

The theme we explored in this paper concerns the interplay between sets and numbers, more specifically, the interplay between *hereditarily finite multi*-sets and *real* numbers. Our initial motivation was the study of a numerical map $\mathbb{R}_A(h) = \sum_{h' \in h} 2^{-\mathbb{R}_A(h')}$ sending well-founded hereditarily finite sets into real numbers, a map that is structured in complete analogy to the celebrated Ackermann bijection $\mathbb{N}_A(h) = \sum_{h' \in h} 2^{\mathbb{N}_A(h')}$.

The main characteristic of $\mathbb{N}_A$ is the fact that it is a recursively defined bijection. The range of $\mathbb{N}_A$ (and its surjectivity) impose strict limitations to $\mathbb{N}_A$: it cannot be directly adapted to more extended notions of sets (e.g., non well-founded structures such *hypersets*) and it cannot be used to map the "middle-earth" of multisets into numbers. Both these limitations are overcome when the range of the map is extended. This move "gives space" for mapping objects defined by modifying the very notion of *set*.

The basic feature of a set is that it is fully understood when its elements are listed. While sets do not change when their elements are listed in a different order or when some of their elements are listed more than once, multisets are defined by dropping the latter assumption and keeping into account *multiplicities* of elements, intended to *count* the number of times any single element occurs in them. In this sense, multisets represent a *middle-earth* between pure sets and numbers.

Mapping multisets into *real* numbers is therefore possible and doing it with $\mathbb{R}^{\mu}_A$ – defined in complete analogy to $\mathbb{R}_A, \mathbb{N}_A$ – allowed us to use multiplicities to express the product of codes by integers as an iterated sum of them (see Section 3.2). Here we proved that the injectivity of $\mathbb{R}^{\mu}_A$ on a large portion of the family of hereditarily finite multisets implies that its range is almost everywhere non-algebraic. That is, mapping multisets into numbers – in a manner coherent with $\mathbb{N}_A$ – calls into play a rather complex numerical field.

The main problem we leave open is, clearly, the injectivity of $\mathbb{R}^{\mu}_A$ (on $\mathfrak{H}^{\infty}_1$). We conjecture that $\mathbb{R}^{\mu}_A$ is injective – with the limitations discussed above – and this is a reinforcement of our previous conjecture on the injectivity of $\mathbb{R}_A$ (see [14, 5]).

Under the validity of our previous conjecture(s), codes are irrational numbers and do not necessarily admit finite representations. Hence, any finite computation of $\mathbb{R}^{\mu}_A$ (respectively, $\mathbb{R}_A$) on a given finite collection of multisets (respectively, sets), must be stopped at some predetermined level of approximation. A further direction we consider worth exploring is the level of approximation (e.g., the number of digits) that is sufficient to compute in order to distinguish the codes of an input collection of multisets (respectively, sets).

Finally, we hope that the tools we have developed for the proof of our main result here will represent a simple and concrete operational link between the fields of (multi-)set theories and Diophantine equations. Computational problems and results written in one of the two mathematical languages can now be easily rewritten and reconsidered in the other. We hope that studying and exploring the potential of this transfer can be a source of new ideas and techniques.

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