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# Discrete dualities for groupoids

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Abstract. *We present discrete dualities for groupoid-based algebras and their associated frames starting from a plain groupoid and then expanding step by step its signature and/or axioms assuming existence of a compatible partial ordering, commutativity, associativity, idempotency, existence of the identity element, and existence of left and right residuals of the groupoid operator. In the final section we present a brief overview of lattice-ordered groupoids and provide hints how to obtain discrete dualities for these from the dualities established in the preceding sections and relevant results in the literature.*

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## 1. Introduction

A *discrete duality* is a duality between a class of algebras and a class of frames (relational systems). To establish a discrete duality for a class Alg of algebras and a class Frm of frames we proceed according to the following steps:

- $\mathsf{S}_1$ For every algebra $A$ in *Alg* we define its *canonical frame* $X_A \subseteq 2^A$ and prove that it belongs to the class Frm. If $f : A \to A^n$ is an $n$-ary operator in the signature of $A$, then there is an $n+1$-ary relation $R_A$ on $X_A$ such that the properties which qualify it as a member of Frm are provable from the axioms of $f$ in Alg.
- $\mathsf{S}_2$ For every frame $X$ in Frm we define its *complex algebra* $A_X \subseteq 2^X$ and prove that it belongs to the class Alg. If $R$ is an $n+1$-ary relation in the signature of $X$, then there is an $n$-ary operator on $A_X$ such that the properties which qualify it as a member in Alg are provable from the axioms for $R$ in $X$.
- $\mathsf{S}_3$ We prove two representation theorems:
  - (a) Every algebra $A \in$ Alg is embeddable into the complex algebra $A_{X_A}$ of its canonical frame $X_A$.
  - (b) Every frame $X \in$ Frm is embeddable into the canonical frame $X_{A_X}$

of its complex algebra $A_X$. The embeddings $h : A \hookrightarrow A_{X_A}$ and $k : X \hookrightarrow X_{A_X}$ are defined by

$$h(a) \stackrel{\mathrm{df}}{=} \{x \in X_A : a \in x\} \subseteq 2^{2^A}, \tag{1}$$

$$k(x) \stackrel{\mathrm{df}}{=} \{B \in A_X : x \in B\} \subseteq 2^{2^X}. \tag{2}$$

The tradition to connect $n$-ary operations on $A$ with $n+1$-ary relations on $X_A$ and vice versa goes back to [15] in the context of Boolean algebras with operators and their ultrafilter frames, a process which led to canonical extensions of such algebras, see [14] for an overview.

We say that a *discrete duality* (DD) holds for a class $\mathsf{Alg}$ of algebras and a class $\mathsf{Frm}$ of frames whenever the representation theorems are proved.

A discrete duality leads, among others, to what is called *duality via truth* (DvT) for a logic whose semantics is determined by the structures for which a DD holds: For every formula $\alpha$ of the logic, $\alpha$ is true in the algebraic semantics of the logic if and only if it is true in its frame semantics. To prove DvT we need a result which says that for every frame $X$ in $\mathsf{Frm}$ a formula $\alpha$ is true in $X$ if and only if it is true in its complex algebra $A_X$. Thus, a discrete duality contributes to the completeness theorem, once a deduction system for the logic is given. In this context, a class of algebras and a class of relational structures (frames) are considered dual, if both provide semantics for the same logic; in this sense, algebras and frames are considered equally. Details on the process of DvT and applications can be found in [21, 22].

The algebras for which discrete dualities were established are in most cases signature extensions of three basic types with their respective axioms: Boolean algebras, bounded distributive lattices, bounded general lattices. The universes of the canonical frames are sets of ultrafilters, respectively, prime filters or filter-ideal pairs. Each of these has $\subseteq$ as a built-in partial order; this order is discrete in the case of ultrafilters, and thus, it is usually not mentioned. The complex algebras of the respective frames $X$ are families of subsets of $X$. Thus, for the representation of an ordered algebra the order $\leq$ must be mapped into the $\subseteq$ relation on the powerset of the universe of its canonical frame. If the order of the algebra is induced by a lattice operation, this will always be the case since the order is definable by the operators. In the present paper we shall explore how far the representation theorems can be carried on groupoids $\langle G, \circ, \leq \rangle$ where the order is compatible with $\circ$, but not necessarily induced by a lattice operation.

Groupoid–related structures play an important role in the studies of a large variety of non-classical logics involving their semantics, proof theory, and applications. One important family of groupoid-based logics are substructural logics; for details of such logics and their hierarchy see [24] or [10] and the references therein. In the present work we restrict ourselves to order based fragments of such logics without lattice operations, such as entailment structures and their relatives.

In studies of substructural logics the frame semantics is usually defined in terms of ternary relations. It was presented and developed in several papers [25, 26, 27], see also [7]. There were also attempts to provide a semantics in terms of binary relations such as [1, 5, 6, 17, 31].

The paper is structured as follows: In Section 3 we discuss if and how a discrete duality for groupoids without order can be established. In Section 4 a discrete duality for groupoids with a compatible order is presented; this discrete duality serves as a basis for discrete dualities established in the sequel. We also discuss the possible choices for the universes of complex algebras or canonical frames. Section 5 investigates axiomatic extensions of ordered groupoids such as commutativity, associativity, and idempotency. In the next two sections we enrich the signature of ordered groupoids: In Section 6 we augment the signature of ordered groupoids with a left or a right identity, and in Section 7 we consider ordered groupoids with the left and right residuals of its operator. Finally, in Section 8 we present a brief overview of existing discrete dualities for algebras which have a groupoid operator in the signature and are Boolean-ordered, distributive-lattice-ordered, or general-lattice-ordered.

## 2. Notation and first definitions

Suppose that $\langle P, \leq \rangle$ is a partially ordered set. If $Q \subseteq P$, then $\uparrow Q \stackrel{\text{df}}{=} \{y \in P : (\exists x)[x \in Q \text{ and } x \leq y\}$; if $Q = \{x\}$, we just write $\uparrow x$. $Q$ is called *down directed* if each $\{p, q\} \subseteq Q$ has a lower bound in $Q$. $Q$ is called an *order filter*, if $Q \neq \emptyset$ and $Q = \uparrow Q$. The set of all order filters of $P$ is denoted by $\mathcal{F}_P^{\mathsf{O}}$, the set of down directed order filters by $\mathcal{F}_P^{\mathsf{D}}$, and the set of principal filters by $\mathcal{F}_P^{\mathsf{P}}$; clearly, $\mathcal{F}_P^{\mathsf{P}} \subseteq \mathcal{F}_P^{\mathsf{D}} \subseteq \mathcal{F}_P^{\mathsf{O}}$. It may be worth to remark that $\mathcal{F}_P^{\mathsf{O}}$ is closed under $\cup$ and $\cap$, $\mathcal{F}_P^{\mathsf{D}}$ is closed under $\cap$, and $\mathcal{F}_P^{\mathsf{P}}$ is closed under neither.

With some abuse of notation we shall identify an algebra or a frame with its respective universe, if no confusion can arise. As we shall work on different levels of sets, we will generally adhere to the following convention: Elements of an algebra are denoted by letters from the beginning of the alphabet such as $a, b, c, \ldots$, elements of a frame by letters towards the end of the alphabet such as $\ldots, x, y, z$, and elements of the complex algebra of a frame by capital letters from the front of the alphabet such as $A, B, C, \ldots$ Quantifier free axioms are assumed to be universally quantified.

## 3. Groupoids

In this section we will take a look at unordered groupoids and unordered frames. A *Brandt–groupoid* [4], or simply, *groupoid* – also called *binar* or *magma* – is an algebra $\langle G, \circ\rangle$ where $\circ : G \times G \to G$ is a binary operator. Since $\circ$ is binary, the frame relation we consider will need to be ternary [15].

We shall start with frames. A *G-frame* is a relational structure $\langle X, R\rangle$ where $X$ is a nonempty set and $R$ is a nonempty ternary relation on $X$. Each G-frame leads to a binary operation $\circ_X$ on $2^X$ defined by

$$A \circ_X B \stackrel{\mathrm{df}}{=} \{z \in X : (\exists x, y)[x \in A \text{ and } y \in B \text{ and } R(x, y, z)]\}. \tag{3}$$

This is the definition given in [15, Theorem 3.3] which paved the way to connect algebraic and relational semantics of classical modal logic. It was later used in [19] and [30] in the context of relevant logics and their algebras, see also [9]. In such a way, we proceed from a ternary frame $X$ to a groupoid on $2^X$. We shall fix this definition of $\circ_X$ for all structures which we consider.

The other ingredient of a complex algebra of $X$ is its universe $G_X \subseteq 2^X$. The choice of $G_X$ may depend on signature extensions of G-frames: If we have no explicit ordering on $X$, the most general choice of $G_X$ is whole power set of $X$ without explicitly including $\subseteq$ in the signature. If more resources are available, such as a partial order on $X$ or a topology, then one may choose the collection of order filters or open or clopen sets in the topology. A structure $\langle G_X, \circ_X\rangle$ is called a *complex algebra of* $X$ if $G_X \subseteq 2^X$ and $G_X$ is closed under $\circ_X$; note that the largest complex algebra of $X$ is $\langle 2^X, \circ_X\rangle$.

Various first order properties of $R$ induce algebraic properties of $G_X$. Consider the following:

$F_1$ $R(x, y, z) \Rightarrow R(y, x, z)$.
$F_2$ $R(x, y, z)$ and $R(z, y', z') \Rightarrow (\exists u)[R(y, y', u)$ and $R(x, u, z')$,
$F_3$ $R(x, y, z)$ and $R(x', z, z') \Rightarrow (\exists u)[R(x', x, u)$ and $R(u, y, z')]$.

THEOREM 3.1. *1. If $R$ satisfies $F_1$, then $G_X$ is commutative.*
*2. If $R$ satisfies $F_2$ and $F_3$, then $G_X$ is associative.*

*Proof.* This can be shown by an easy adaptation of [20, Proposition 12.4.1] for 1, and [20, Proposition 5.2.3]. for 2. These proofs have $2^X$ as a universe, and do not use any order theoretic properties of $2^X$. Since commutativity and associativity are universal statements, the claim is true for each subalgebra of $\langle 2^X, \circ_X\rangle$. □

Starting from a groupoid $\langle G, \circ\rangle$, we will define a canonical frame $\langle X_G, R_G\rangle$ of $G$ as a family of subsets of $G$ and a suitable ternary relation $R_G$ on $X_G$. As an auxiliary tool we shall use the complex extension $\circ^{\mathrm{c}}$ of $\circ$ over $2^G$, that is, for all $x, y \subseteq G$

$$x \circ^{\mathrm{c}} y = \{c \in G : (\exists a, b)[a \in x, b \in y \text{ and } a \circ b = c\}. \tag{4}$$

The structure $\langle 2^G, \circ^{\mathrm{c}}\rangle$ is called the *full algebra of complexes* of $G$. Each groupoid can be embedded into its algebra of complexes by the assignment $a \mapsto \{a\}$. For the related topic of Boolean groupoids, we refer the reader to [2].

In most, if not all, signature extensions of groupoids, the universe of the canonical frame $X_G$ is not the whole powerset of $G$; for example, it is the set of ultrafilters of an underlying Boolean algebra, or the set of prime filters of an underlying distributive lattice. In both cases, the structures are explicitly or implicitly ordered by $\subseteq$; more examples can be found in [20]. But even in an explicitly order free setup, we cannot escape the implicit ordering of subsets when choosing $R_G$:

LEMMA 3.2. *Let $\langle G, \circ\rangle$ be a groupoid, $X_G \subseteq 2^G$ and $h : G \hookrightarrow G_{X_G}$ be the canonical embedding, where $\circ_{X_G}$ is defined by* (3)*. Suppose that $x, y, z \in X_G$ and $R_G(x, y, z)$. Then, $x \circ^{\mathrm{c}} y \subseteq z$.*

*Proof.* Since $h$ is a homomorphism and by (3) we have for all $a, b \in G$,

$$h(a) \circ_{X_G} h(b) \overset{(3)}{=} \{z \in X_G : (\exists x, y)[a \in x, b \in y \text{ and } R_G(x, y, z)]\} \\ = h(a \circ b), \quad (5)$$

Suppose that $x, y, z \in X_G$ as well as $R_G(x, y, z)$, and assume that $x \circ^{\mathrm{c}} y \not\subseteq z$. Then, there are $a \in x, b \in y$ such that $a \circ b \notin z$, that is, $z \notin h(a \circ b)$. By (5), $a \in x$ and $b \in y$ imply that $\neg R_G(x, y, z)$, contradicting the hypothesis. □

Thus,

$$R_G(x, y, z) \text{ implies } x \circ^{\mathrm{c}} y \subseteq z \tag{6}$$

is a necessary condition for the definition of $R_G$ in the setup so far. Since we do not have $\subseteq$ in the signature of the canonical frame,we are left with

$$R_G(x, y, z) \overset{\mathrm{df}}{\iff} x \circ^{\mathrm{c}} y = z. \tag{7}$$

This is the definition given in [15, Theorem 3.3] for Boolean algebras with operators. However, it will not work in our context. Suppose that $R_G$ is defined by (7). While it is easily shown that $h(a) \circ_{X_G} h(b) \subseteq h(a \circ b)$, the reverse inclusion does not always hold:

EXAMPLE 3.3. Suppose that $|G| \geq 2$ and fix some $c \in G$; for all $a, b \in G$ let $a \circ b \overset{\mathrm{df}}{=} c$. Then, $x \circ^{\mathrm{c}} y = \{c\}$ for all $x, y \in X_G$, and, therefore, $R_G = \{\langle x, y, \{c\}\rangle : x, y \in X_G\}$. Let $a \neq c$, and consider $h(a \circ c)$; then, $\{a, c\} \in h(a \circ c) = h(c)$. On the other hand, $\neg R_G(x, y, \{a, c\})$, and therefore, $\{a, c\} \notin h(a) \circ_{x_G} h(c)$.

We conclude that in our setup with $\circ_X$, respectively $h$, defined by (3), respectively, by (1), groupoids without ordering cannot be handled well, if at all.

## 4. Ordered groupoids

Thus, we turn to ordered groupoids. An *O-groupoid* $\langle G, \circ, \leq \rangle$ is a groupoid endowed with a compatible partial order $\leq$, i.e for all $a, b, c \in G$,

$$a \leq b \Rightarrow a \circ c \leq b \circ c \text{ and } c \circ a \leq c \circ b. \tag{8}$$

According to [19], the ordering may be interpreted as a relation of entailment which partially orders propositions. The embedding $h$ now need not only preserve $\circ$, but also the ordering $\leq$. This restricts the choice of $X_G$:

LEMMA 4.1. *1. $h$ preserves $\leq$ if and only if every element of $X_G$ is an order filter of $G$.*
*2. Suppose that $X_G$ contains all principal order filters of $G$. Then, $\uparrow a$ is the smallest element of $h(a)$ in $\langle X_G, \subseteq \rangle$.*

*Proof.* 1. "$\Rightarrow$": Let $x \in X_G$, $a \in x$, that is, $x \in h(a)$. If $a \leq b$, then $h(a) \subseteq h(b)$ by the hypothesis, which implies $b \in x$.

"$\Leftarrow$": Suppose that $a, b \in G$, $a \leq b$. and $x \in h(a)$. Then, $a \in x$, and $a \leq b$ together with the hypothesis implies that $b \in x$, i.e. $x \in h(b)$.

2. Let $x \in h(a)$. Then, $a \in x$, and the fact that $x$ is an order filter implies that $\uparrow a \subseteq x$. □

Therefore, we suppose in the sequel that $X_G \subseteq \mathcal{F}_G^{\mathsf{O}}$. In view of Lemma 3.2, we fix $R_G \subseteq X_G^3$ as the smallest relation compatible with both (1) and (3), namely,

$$R_G(x, y, z) \stackrel{\text{df}}{\Longleftrightarrow} x \circ^{\text{c}} y \subseteq z. \tag{9}$$

The relation $R_G$ in (9) originates with [27] in the semantic analysis of relevant logic,[1] and was later used, among others, in [30] and [20].

Based on these considerations we define a *canonical frame* of $G$ as a triple $\langle X_G, R_G, \subseteq \rangle$ where $X_G \subseteq \mathcal{F}_G^{\mathsf{O}}$, and $R_G \subseteq X_G^3$ is defined by (9)

In [30] and [20] the operation $\odot$ defined on $2^G$ by

$$x \odot y \stackrel{\text{df}}{=} \{c \in G : (\exists a, b)[a \in x, b \in y, a \circ b \leq c]\}, \tag{10}$$

is used for the definition of $R_G$. It is easy to see that $x \odot y = \bigcup \{\uparrow(a \circ b) : a \in x, b \in y\}$, and, unlike $x \circ^{\text{c}} y$, it is always an order filter of $G$. Clearly, $\odot$ is compatible with $\subseteq$ and commutative, respectively, associative if and only if $\circ$ is commutative, respectively, associative. Observe that $\langle \mathcal{F}_G^{\mathsf{O}}, \odot, \subseteq \rangle$ is an ordered groupoid itself. For the definition of $R_G$ we may use $\circ^{\text{c}}$ or $\odot$:

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[1] We thank A. Urquhart for pointing this out to us.

LEMMA 4.2. *Suppose that $x, y, z \in X_G$.*
*Then, $R_G(x, y, z)$ if and only if $x \odot y \subseteq z$.*

*Proof.* "$\Rightarrow$": Suppose that $a \in x, b \in y$ and $a \circ b \leq c$. Then, $a \circ b \in z$, and therefore, $c \in z$ since $z$ is increasing.
"$\Leftarrow$": This follows immediately from $x \circ^{\mathsf{c}} y \subseteq x \odot y$. □

The following observation which leads to a frame condition follows directly from Lemma 4.2:

LEMMA 4.3. *Suppose that $X_G$ is closed under $\odot$. If $x, y \in X_G$, there is a smallest $z \in X_G$ with respect to $\subseteq$ such that $R_G(x, y, z)$.*

*Proof.* Suppose that $x, y \in X_G$. Since $X_G$ is closed under $\odot$, we have $x \odot y \in X_G$, and clearly, $x \odot y$ is the smallest $z \in X_G$ with $R_G(x, y, z)$. □

We consider several choices for $X_G$: The set $\mathcal{F}_G^{\mathsf{P}}$ of all principal order filters, the set $\mathcal{F}_G^{\mathsf{D}}$ of down directed order filters, and the set $\mathcal{F}_G^{\mathsf{O}}$ of all order filters. The next lemma generalizes [30, Lemma 2.1] and shows that each of these is a valid choice for $X_G$ to apply Lemma 4.3:

LEMMA 4.4. *$\mathcal{F}_G^{\mathsf{P}}, \mathcal{F}_G^{\mathsf{D}}$, and $\mathcal{F}_G^{\mathsf{O}}$ are closed under $\odot$.*

*Proof.* $\mathcal{F}_G^{\mathsf{P}}$: Let $a, b \in G$; then

$$\begin{aligned} c \in {\uparrow} a \odot {\uparrow} b &\iff (\exists a', b')[a \leq a', b \leq b', a' \circ b' \leq c], \\ &\iff a \circ b \leq c, && \text{by (8)}, \\ &\iff c \in {\uparrow}(a \circ b). \end{aligned}$$

$\mathcal{F}_G^{\mathsf{D}}$: Let $x, y \in \mathcal{F}_G^{\mathsf{D}}$, and $c_1, c_2 \in x \odot y$; then there are $a_1, a_2 \in x, b_1, b_2 \in y$ such that $a_1 \circ b_1 \leq c_1$ and $a_2 \circ b_2 \leq c_2$. Since $x$ and $y$ are down directed, there are $d_1 \in x, d_2 \in y$ such that $d_1 \leq a_1, a_2$ and $d_2 \leq b_1, b_2$. By compatibility, $d_1 \circ d_2 \leq a_1 \circ b_1 \leq c_1$, and $d_1 \circ d_2 \leq a_2 \circ b_2 \leq c_2$, and thus, $d_1 \circ d_2$ is a lower bound of $\{c_1, c_2\}$ in $\mathcal{F}_G^{\mathsf{D}}$.
$\mathcal{F}_G^{\mathsf{O}}$: This follows immediately from the definitions. □

The choice of $X_G$ has some connection to the completion of ordered algebras; for example, if $X_G = \mathcal{F}_G^{\mathsf{P}}$, the order of the embedding algebra $G_{X_G}$ of $G$ is isomorphic to its order ideal completion of $\langle G, \leq \rangle$. We shall not pursue this further, and invite the reader to consult [12] for an overview.

Conversely, an *O-frame* is a frame $\langle X, R, \leq \rangle$ where $\leq$ is a partial order, and $R$ is a ternary relation on $X$ which satisfies for all $x, y, z, x', y', z'$ the conditions

$F_4$ $R(x, y, z), x' \leq x, y' \leq y, z \leq z' \Rightarrow R(x', y', z')$, (Monotonicity).
$F_5$ For all $x, y \in X$ there is a smallest $z \in X$ such that $R(x, y, z)$, (From Lemma 4.3).

The monotonicity property $F_4$ ensures that $\subseteq$ is compatible with $\circ_X$. As in Lemma 4.1 it can be easily shown that

LEMMA 4.5. *$k$ preserves $\leq$ if and only if every element of $G_X$ is an order filter of $\langle 2^X, \subseteq\rangle$.*

This restricts the choice of $G_X$, and we define a *complex algebra of $X$* as an ordered groupoid $\langle G_X, \circ_X, \subseteq\rangle$, where

1. $G_X \subseteq \mathcal{F}_X^{\mathsf{O}}$,
2. $\circ_X$ is defined by (3),
3. $G_X$ is closed under $\circ_X$.

In analogy to Lemma 4.4 we exhibit several choices for $G_X$:

LEMMA 4.6. *$\mathcal{F}_X^{\mathsf{P}}$, $\mathcal{F}_X^{\mathsf{D}}$, and $\mathcal{F}_X^{\mathsf{O}}$ are closed under $\circ_X$.*

*Proof.* $\mathcal{F}_X^{\mathsf{P}}$: Let $x, y \in X$. By $F_5$ there is smallest $z$ with $R(x, y, z)$. Then, $\uparrow x \circ_X \uparrow y = \uparrow z$.

$\mathcal{F}_X^{\mathsf{D}}$: Suppose that $A, B \in \mathcal{F}_X^{\mathsf{D}}$, $x, x' \in A$, $y, y' \in B$, $R(x, y, z)$, and $R(x', y', z')$. Since $A, B$ are down directed, there are $u \in A, v \in B$ such that $u \leq x, x'$ and $v \leq y, y'$. From $F_4$ we obtain $R(u, v, z)$ and $R(u, v, z')$, and by $F_5$ there is some $w$ such that $R(u, v, w)$ and $w \leq z, z'$. Thus, $w \in A \circ_G B$, and $w$ is a lower bound of $\{z, z'\}$.

$\mathcal{F}_X^{\mathsf{O}}$: This follows immediately from $F_4$. $\square$

This shows that the largest complex algebra of $X$ is the ordered groupoid $\langle \mathcal{F}_X^{\mathsf{O}}, \circ_X, \subseteq\rangle$, and that $\mathcal{F}_X^{\mathsf{D}}$ and $\mathcal{F}_X^{\mathsf{P}}$ are subalgebras of $\mathcal{F}_G^{\mathsf{O}}$.

In the rest of the paper, we suppose that $\langle G, \circ, \leq\rangle$ is an O-groupoid, $X_G$ a set of order filters of $\langle G, \leq\rangle$ containing all principal order filters and closed under $\odot$, and that $\langle X, R, \leq\rangle$ is an O-frame and $G_X$ is a set of order filters of $\langle X, \leq\rangle$ containing all principal order filters and closed under $\circ_X$.

We now have the tools necessary to prove a discrete duality for O-groupoids and O-frames. Such theorems have been known for some time for lattice ordered structures that have a groupoid reduct, for example, relation algebras or distributive residuated lattices [20]. These situations differ from ours since we have only a binary operation and a compatible ordering and no lattice operation.

THEOREM 4.7. *1. $h$ is an embedding of O-groupoids.*
*2. $k$ is an embedding of O-frames.*

*Proof.* 1. Because $X_G$ contains all principal filters, $h(a) \neq \emptyset$ for all $a \in G$, and $h$ is injective since $\leq$ is antisymmetric. By Lemma 4.1, $h$ preserves the ordering $\leq$. Let $a, b \in G$; we need to show that $h(a) \circ_{X_G} h(b) = h(a \circ b)$.

"$\subseteq$": We have

$$\begin{aligned}
&z \in h(a) \circ_{X_G} h(b) \\
&\quad\Rightarrow (\exists x, y)[x \in h(a), y \in h(b), R_G(x, y, z)], && \text{definition of } \circ_X. \\
&\quad\Rightarrow (\exists x, y)[a \in x, b \in y, x \circ^{\mathrm{c}} y \subseteq z], && \text{definition of } R_G, \\
&\quad\Rightarrow a \circ b \in z, && \text{as } a \in x, b \in y, \\
&\quad\Rightarrow z \in h(a \circ b). && \text{definition of } h.
\end{aligned}$$

"$\supseteq$": Let $z \in h(a \circ b)$, i.e. $a \circ b \in z$. Set $x \overset{\mathrm{df}}{=} {\uparrow} a, y \overset{\mathrm{df}}{=} {\uparrow} b$; then, $x \in h(a), y \in h(b)$, and all that is left to show is $R_G(x, y, z)$, i.e. that ${\uparrow} a \circ^{\mathrm{c}} {\uparrow} b \subseteq z$. Suppose that $c \in {\uparrow} a \circ^{\mathrm{c}} {\uparrow} b$; then, there are $a', b'$ such that $a \leq a', b \leq b'$ and $a' \circ b' = c$. By (8), $a \circ b \leq c$, and $a \circ b \in z$ as well as the fact that $z$ is an order filter imply that $c \in z$.

2. Let $x, y, z \in X$. We need to show that

$$R(x, y, z) \text{ if and only if } R_{G_X}(k(x), k(y), k(z)).$$

Observe that

$$\begin{aligned}
R_{G_X}(k(x), k(y), k(z)) &\iff k(x) \circ_X^{\mathrm{c}} k(y) \subseteq k(z), \quad \text{by (9)} \\
&\iff (\forall A, B \in G_X)[x \in A \text{ and } y \in B \Rightarrow z \in A \circ_X B], \\
&\iff (\forall A, B \in G_X)[x \in A \text{ and } y \in B \Rightarrow \\
&\qquad (\exists u, v \in X)(u \in A, v \in B \text{ and } R(u, v, z))].
\end{aligned}$$

If $R(x, y, z)$, then we set $u \overset{\mathrm{df}}{=} x$ and $v \overset{\mathrm{df}}{=} y$ showing that $R_{G_X}(k(x), k(y), k(z))$. Conversely, if $R_{G_X}(k(x), k(y), k(z))$, then setting $A \overset{\mathrm{df}}{=} {\uparrow} x$ and $B \overset{\mathrm{df}}{=} {\uparrow} y$ shows that $R(x, y, z)$. By Lemma 4.5, $k$ preserves the ordering as well. $\square$

In the sequel we shall use this result to establish discrete dualities for various axiomatic and signature extension of $G$, respectively, of $X$.

## 5. Some axiomatic extensions of O-groupoids

In Theorem 3.1 we have shown that certain frame properties imply algebraic properties of $G_X$ . We will use this to establish discrete dualities for these:

Theorem 5.1. *1. There is a discrete duality between commutative O-groupoids and O-frames that satisfy $F_1$.*

*2. There is a discrete duality between associative O-groupoids and O-frames that satisfy $F_2$ and $F_3$.*

*Proof.* By Theorem 3.1 it is sufficient that $X_G$ has the frame property corresponding to the algebraic property.

1. This can be shown by adapting the proof of [20, Proposition 12.4.2] for commutative residuated lattices, which uses only the definition of $R_G$ and holds for $X_G \subseteq \mathcal{F}^{\mathsf{P}}_G$.

2. Here, we cannot directly use the corresponding result for composition of relation algebras given in [20, Proposition 5.2.4], since the universe of the canonical frame of a relation algebra is the set of the ultrafilters of its Boolean reduct which we do not have at our disposal. Nevertheless, the proofs for groupoids follow a similar strategy. For $F_2$, let $x, y, y', z, z' \in X_G$, and suppose that $R_G(x, y, z)$ and $R_G(z, y', z')$; then, $x \circ^{\mathrm{c}} y \subseteq z$ and $z \circ^{\mathrm{c}} y' \subseteq z'$ by definition of $R_G$. Set $u \overset{\mathrm{df}}{=} y \circ^{\mathrm{c}} y'$; then, $R(y, y', u)$ by definition of $R_G$, and

$$\begin{aligned}
R_G(x,y,z) \ \text{ and } \ R_G(z,y',z')& \\
&\Rightarrow x \circ^{\mathrm{c}} y \subseteq z \text{ and } z \circ^{\mathrm{c}} y' \subseteq z', && \text{by definition of } R_G \\
&\Rightarrow (x \circ^{\mathrm{c}} y) \circ^{\mathrm{c}} y' \subseteq z', \\
&\Rightarrow x \circ^{\mathrm{c}} (y \circ^{\mathrm{c}} y') \subseteq z', && \text{by associativity} \\
&\Rightarrow x \circ^{\mathrm{c}} u \subseteq z', && \text{by definition of } u \\
&\Rightarrow R_G(x,u,z'), && \text{by definition of } R_G.
\end{aligned}$$

$F_3$ is shown analogously. Note that the ordering was only used for the definition of $R_G$. □

Next, we turn to idempotency. The operation $\circ$ is called *expanding*, if $a \leq a \circ a$, and *contracting*, if $a \circ a \leq a$ for all $a \in G$. It is called *idempotent*, if it is both expanding and contracting. We shall start with an expanding operator. The relevant frame condition

$F_6$ $R(x, x, x)$

was introduced in [19] as part of the semantics for basic relevant logics, and they state that it is related to modus ponens. The following lemma shows that it works for all order filters on the frame side:

LEMMA 5.2. *$\circ_X$ is expanding if and only if $F_6$ holds in $X$.*

*Proof.* "$\Rightarrow$": Let $z \in X$; then $\uparrow z \subseteq \uparrow z \circ_X \uparrow z$ by the hypothesis, in particular, $z \in \uparrow z \circ_X \uparrow z$. By (3) there are $x, y \in X$ such that $z \leq x, y$ and $R(x, y, z)$. Now, $F_4$ implies $R(z, z, z)$.

"$\Leftarrow$": Suppose that $A \in G_X$, and $x \in A$. Then, $R(x, x, x)$ and (3) imply that $x \in A \circ_X A$. □

The proof which uses only principal filters implies that the result continues to hold if $G_X = \mathcal{F}^{\mathsf{P}}_X$ or $G_X = \mathcal{F}^{\mathsf{D}}_X$. On the other hand, the following example shows that there is some $G$ such that $\circ$ is expanding, and $F_6$ does not hold in $X_G$, if $X_G$ is the set of all order filters of $G$.

Example 5.3. Let $G = \{a, b, c\}$, and define $\circ$ and $\leq$ as below:

| $\circ$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $c$ | $c$ |
| $b$ | $c$ | $b$ | $c$ |
| $c$ | $c$ | $c$ | $c$ |

$$\begin{tikzcd} a & & b \\ & c \arrow[ul, "\leq"] \arrow[ur, "\leq"'] & \end{tikzcd}$$

Then, $\circ$ is idempotent and compatible with $\leq$; indeed, $\langle G, \circ, \leq \rangle$ is a meet semilattice. Let $X_G$ be the set of all order filters of $G$, and set $x \overset{\text{df}}{=} \{a, b\}$; then, $x \in X_G$, and $x \circ^{\mathrm{c}} x = G$. Since $c \in x \circ^{\mathrm{c}} x$, but $c \notin x$, it follows that $\neg R_G(x, x, x)$.

The reason for the failure of the condition seems to be that by allowing all order filters in $X_G$, we catch too many elements outside $h[G]$, for which $A \subseteq A \circ_X A$ does not hold, in our example the union of two incomparable principal filters. Indeed, a necessary and sufficient condition for $R_G(x, x, x)$ is that each $x \in X_G$ is an increasing subgroupoid of $G$. If we restrict $X_G$ to $\mathcal{F}_G^{\mathsf{D}}$, then $F_6$ holds in $X_G$:

Lemma 5.4. *Suppose that $X_G$ is the set of down directed order filters of $G$. Then, $G$ is contracting if and only if $R_G(x, x, x)$ for all $x \in X_G$.*

*Proof.* "$\Rightarrow$": Suppose that $a \leq a \circ a$ for all $a \in G$. Let $x \in X_G$, and $c \in x \circ^{\mathrm{c}} x$; then, there are $a, b \in x$ such that $a \circ b = c$. Since $x$ is down directed, there is some $d \in x$ such that $d \leq a, b$, and $d = d \circ d \leq a \circ b \leq c$, and thus, $c \in x$ because $x$ is an order filter.

"$\Leftarrow$": Suppose that $R_G(x, x, x)$, i.e. $x \circ^{\mathrm{c}} x \subseteq x$, and $a \in G$. Then, $\uparrow a \circ^{\mathrm{c}} \uparrow a \subseteq \uparrow a$ which implies $a \leq a \circ a$. Note that this direction does not require $x$ to be down directed. $\square$

Thus, if $X_G$ is the set of all down directed filters, the frame condition $R(x, x, x)$ assures that $a \leq a \circ a$ holds in $G$ if and only if $A \subseteq A \circ_{X_{G_X}} A$ holds in $G_{X_G}$. Together with Lemma 5.4, we have shown

Theorem 5.5. *There is a discrete duality between O-groupoids $\langle G, \circ, \leq \rangle$ with $\circ$ expanding and $X_G = \mathcal{F}_G^{\mathsf{D}}$, and O-frames with $G_X = \mathcal{F}_X^{\mathsf{D}}$ that satisfy $F_6$.*

Next, we consider the case that $\circ$ is contracting, starting with an example which shows that the condition $F_6$ has no connection to $a \circ a \leq a$:

Example 5.6. Let $G$ be the set of negative integers with the natural order, and set $a \circ b \overset{\text{df}}{=} \min\{a, b\} - 1$. Then, $\circ$ is compatible to $\leq$, and satisfies $a \circ a \lneq a$. If $x$ is a proper order filter, then $x$ is principal, say, $x = \uparrow a$. Since $a - 1 \in x \circ x$, we have $x \circ^{\mathrm{c}} x \not\subseteq x$, therefore, $\neg R_G(x, x, x)$.

Theorem 5.5 shows that choosing $X_G$ as the set of down directed order filters establishes the discrete duality for expanding $\circ$ and frames satisfying $F_6$. A similar situation occurs when we consider a contracting $\circ$. First, we shall show that $A \circ_{X_G} A \subseteq A$ need not hold if $X_G$ is closed under unions of principal filters such as $\mathcal{F}^{\mathsf{O}}_G$.

EXAMPLE 5.7. Let $\langle G, \circ \rangle$ be the groupoid of Example 5.3, and order $G$ by the converse order of Example 5.3, that is, $a \leq c, b \leq c$. Then, $\circ$ is compatible with $\leq$, and $\langle G, \circ, \leq \rangle$ is a join semilattice. The order filters of $G$ are

$$u \stackrel{\text{df}}{=} \{a, b, c\}, x \stackrel{\text{df}}{=} \{a, c\}, y \stackrel{\text{df}}{=} \{b, c\}, z \stackrel{\text{df}}{=} \{c\}.$$

Note that $\{x, y, u\}$ is not down directed, and that $\{x, y, u\} \circ_{X_G} \{x, y, u\} \not\subseteq \{x, y, u\}$. Hence, $\circ_{X_G}$ is not contracting. If we allow only down directed order filters in $G_{X_G}$, then $\circ_{X_G}$ is contracting.

Consider the frame condition

$F_7$ $R(x, x, z) \Rightarrow x \leq z$.

LEMMA 5.8. *1. Suppose that $G_X$ is the set of all down directed order filters of $X$, and that $R$ satisfies $F_7$. Then, $\circ_X$ is contracting.*
*2. If $\circ$ is contracting on $G$, then $R_G$ satisfies $F_7$.*

*Proof.* 1. Let $A \in G_X$ and $z \in A \circ_X A$; then, there are $x, y \in A$ such that $R(x, y, z)$. Since $A$ is down directed, there is some $u \in A$ such that $u \leq x, y$. Now, $F_4$ implies $R(u, u, z)$.

2. Suppose that $x, z \in X_G$, and $R_G(x, x, z)$. Let $a \in x$. Since $R_G(x, x, z)$ it follows that $x \circ^{\mathrm{c}} x \subseteq z$, in particular, that $a \circ a \in x$. Now, $a \circ a \leq a$ and the fact that $z$ is an order filter imply that $a \in z$. $\square$

The discrete duality is now immediate:

THEOREM 5.9. *There is a discrete duality between O-groupoids $\langle G, \circ, \leq \rangle$ with $\circ$ contracting and $X_G = \mathcal{F}^{\mathsf{D}}_G$, and O-frames for which $G_X = \mathcal{F}^{\mathsf{D}}_X$ and that satisfy $F_7$.*

Theorem 5.5 and Theorem 5.9 lead to the discrete duality of idempotent O-groupoids.

Table 1 shows the resources required for the discrete dualities. Sadly, we do not know any logics whose algebraic semantics consists of groupoids discussed in the previous sections.

## 6. O-groupoids with identity

In this section we suppose that $X_G = \mathcal{F}^{\mathsf{O}}_G$ and $G_X = \mathcal{F}^{\mathsf{O}}_X$. As the next step, let us consider an $O$-groupoid $\langle G, \circ, \leq, e \rangle$ with a left identity element $e$. Its

| | $X_G$ | $G_X$ | Frame condition |
|---|---|---|---|
| Commutative | $\mathcal{F}_G^{\mathsf{O}}$ | $\mathcal{F}_X^{\mathsf{O}}$ | $R(x,y,z) \Rightarrow R(y,x,z)$ |
| Associative | | | $R(x,y,z)\&R(z,y',z')$ $\Rightarrow (\exists u)[R(y,y',u)$ and $R(x,u,z')]$ |
| | | | $R(x,y,z)\&R(x',z,z')$ $\Rightarrow (\exists u)[R(x',x,u)$ and $R(u,y,z')]$ |
| Expanding | $\mathcal{F}_G^{\mathsf{D}}$ | $\mathcal{F}_X^{\mathsf{O}}$ | $R(x,x,x)$ |
| Contracting | $\mathcal{F}_G^{\mathsf{O}}$ | $\mathcal{F}_X^{\mathsf{D}}$ | $R(x,x,z) \Rightarrow x \leq z.$ |

Table 1: Conditions for axiomatic extensions

canonical frame is the structure $\langle X_G, R_G, \subseteq, \uparrow e\rangle$. An *LO-frame* is an *O*-frame with an added designated element $i$ satisfying the frame condition

$F_8$ $(\forall y,z)[y \leq z \iff R(i,y,z))]$.

This is the condition $\mathsf{d1}$ in [19] and $\mathsf{No.\ 5}$ in [30], adjusted in the first component by the monotonicity condition $F_4$.

Lemma 6.1. *1. If $\langle X, R, \subseteq, i\rangle$ is an O-frame with designated element $i$ satisfying $F_8$, then $\uparrow i \circ_X A = A$ for all $A \in G_X$.*

*2. If $\langle G, \circ, \leq, e\rangle$ is an ordered groupoid with left identity, then its canonical frame satisfies $F_8$ with $i = \uparrow e$.*

*Proof.* 1. Suppose that $A \in G_X$; we need to show that $\uparrow i \circ_X A = A$:

"$\subseteq$": Let $z \in \uparrow i \circ_X A$; there are $x \geq i, y \in A$ such that $R(x,y,z)$ by definition of $\circ_X$. $(F_8)$ implies that $y \leq z$, and $y \in A$ and the fact that $A$ is increasing imply $z \in A$.

"$\supseteq$": Let $z \in A$. Setting $y = z$ in $(F_8)$, we have $R(i,z,z)$, and thus, $z \in \uparrow i \circ_X A$.

2. Let $y, z \in X_G$.

"$\Rightarrow$": Let $y \subseteq z$, and $c \in \uparrow e \circ^{\mathrm{c}} y$. Then, there are $a, b$ such that $e \leq a$, $b \in y$, and $a \circ b = c$. Since $e \leq a$, the compatibility of $\leq$ implies that $b = e \circ b \leq a \circ b = c$, and therefore, $c \in y$ because $y$ is an order filter. The hypothesis $y \subseteq z$ now implies that $c \in z$.

"$\Leftarrow$": Conversely, suppose that $R_G(\uparrow e, y, z)$ for some $y, z \in X_G$; then, $\uparrow e \circ^{\mathrm{c}} y \subseteq z$ by definition of $R_G$. If $a \in y$, then $a = e \circ a \in \uparrow e \circ^{\mathrm{c}} y$, hence, $a \in z$. □

The discrete duality theorem follows immediately:

Theorem 6.2. *There is a discrete duality between O-groupoids $\langle G, \circ, \leq, e\rangle$ with a left identity element $e$, and LO-frames.*

For the right identity, we consider the frame condition

$F_9$ $(\forall y, z)[y \leq z \Longleftrightarrow R(y, i, z))]$.

An *RO-frame* is an *O*-frame with an added designated element $i$ satisfying the frame condition $F_9$. The proof of the discrete duality theorem is analogous to the previous one:

THEOREM 6.3. *There is a discrete duality between O-groupoids* $\langle G, \circ, \leq, e\rangle$ *with a right identity element* $e$*, and RO-frames.*

## 7. Residuated O-groupoids

Finally, we enhance an O-groupoid by residuals. For this, we suppose that $X_G$ and $G_X$ are the sets of all order filters in the respective orderings. The *left residual* with respect to $\circ$, denoted by $\lhd$, is a binary operator on $G$ such that

$$(\forall a, b, c)[a \circ b \leq c \Longleftrightarrow a \leq b \lhd c]. \tag{11}$$

In the complex algebra of an O-frame we define the corresponding operation $\lhd_X$ by

$$A \lhd_X B \stackrel{\text{df}}{=} \{x : (\forall y, z)[(R(x, y, z) \text{ and } y \in A) \Rightarrow z \in B]\}. \tag{12}$$

This is the definition for dual algebras of relevant spaces in [30], for complex algebras of R-frames given in [20, Chapter 12.2], and also in [19] in a slightly different form. It also works in our reduced setup:

THEOREM 7.1. *There is a discrete duality between O-groupoids with a left residual* $\lhd$ *and O-frames with* $\lhd_X$ *defined by* (12)*.*

*Proof.* Taking into account our previous results, it is enough to show

1. $\lhd_X$ is the left residual of $\circ_X$.
2. $h$ preserves $\lhd$.

1. Suppose that $A, B, C \in G_X$; we are going to show that (11) holds for $\circ_X$ and $\lhd_X$.

"$\Rightarrow$": Let $A \circ_X B \subseteq C$, and choose some $x \in A$. Suppose that $R(x, y, z)$ and $y \in B$. Since $x \in A$, $y \in B$, and $R(x, y, z)$, we have $z \in A \circ_X B$. The hypothesis now implies $z \in C$.

"$\Leftarrow$": Let $z \in A \circ_X B$; then, there are $x \in A, y \in B$ such that $R(x, y, z)$. The hypothesis implies that $x \in B \lhd_X C$; $y \in B$, $R(x, y, z)$ and the definition of $\lhd_X$ imply that $z \in C$.

2. Let $a, b, c \in G$. We show that $h(b) \lhd_{X_G} h(c) = h(b \lhd c)$. First, note that by (12)

$$x \in h(b) \lhd_{X_G} h(c) \Longleftrightarrow (\forall y, z \in X_G)[x \circ^{\mathrm{c}} y \subseteq z \text{ and } b \in y \Rightarrow c \in z]. \tag{13}$$

"$\subseteq$": Suppose that $b \lhd c \notin x$; then, $c \notin x \circ^{c} \uparrow b$. Setting $y \stackrel{\text{df}}{=} \uparrow b$ and $z \stackrel{\text{df}}{=} x \circ^{c} \uparrow b$, (13) shows that $x \notin h(b) \lhd_{X_G} h(c)$.

"$\supseteq$": Let $b \lhd c \in x$. Suppose that $x \circ^{c} y \subseteq z$ and $b \in y$; then, $a \circ b \in z$ for all $a \in x$. Assume that $c \notin z$. Then, $a \circ b \not\leq c$ for all $a \in x$ since $z$ is increasing and $a \circ b \in z$. By (11), $a \not\leq b \lhd c$, contradicting $b \lhd c \in x$. □

The *right residual of* $\circ$ is a binary operator, denoted by $\rhd$, defined by

$$(\forall a, b, c)[a \circ b \leq c \iff b \leq a \rhd c]. \tag{14}$$

The corresponding complex algebra operation is

$$A \rhd_X B \stackrel{\text{df}}{=} \{x : (\forall y, z)[(R(y, x, z) \text{ and } y \in A) \Rightarrow z \in B]\}. \tag{15}$$

THEOREM 7.2. *There is a discrete duality between O-groupoids with a right residual* $\rhd$ *and O-frames with* $\rhd_X$ *defined by* (15).

*Proof.* That $\rhd_X$ is the right residual of $\circ_X$ was shown in [20, Proposition 12.2.2], the proof of which does not use any resources apart from the ordering and the definition of $\circ_X$ by (3) and of $\rhd_X$ by (15). The proof that $h$ preserves $\rhd$ is analogous to the one for $\lhd$ in Theorem 7.1. □

Such signature extensions of ordered groupoids have a place in logical systems. An *Ackermann groupoid* is a structure $\langle G, \circ, \leq, \lhd, e\rangle$ where $\langle G, \circ, \leq\rangle$ is an O-groupoid, $e$ is a left identity, and $\lhd$ is the left residual with respect to $\circ$. These were considered in [19] as the "most basic relevant algebra, ... introduced for the specific purpose of explicating pure implicational calculi." A discrete duality for Ackermann groupoids can be obtained from the discrete dualities presented in Section 6 and this section.

## 8. Some remarks on further axiomatic or signature extensions of groupoids and their discrete dualities

Groupoids are the basic component in semantic structures of a large variety of non-classical logics. In Sections 3–7 we studied the very basic structures whose signatures did not include lattice operations. In the present section we briefly describe groupoids endowed with the lattice operations and their discrete dualities which play an important role in algebra and logic.

A *positive Ackermann groupoid* is a signature extension of an Ackermann groupoid by the operations $\wedge, \vee$ such that $\langle G, \wedge, \vee\rangle$ is a distributive lattice with $\leq$ as its natural order. They are reducts of the algebras of relevant logics which were investigated in [30]. It was shown in [19] that positive Ackermann groupoids provide algebraic models for the logic **B+** which they introduced in [25]. It is interesting to note that in their setup the universe of the complex algebras of the ternary frames for the logic are the order filters of the frame universe. A discrete duality for positive Ackermann groupoids, that is, distributive-lattice-ordered Ackermann groupoids, can be obtained based the discrete duality for distributive residuated lattices [20, Chapter 12] and the discrete dualities from Sections 6 and 7.

A *Church monoid*, also introduced in [19], is an axiomatic extension of Ackermann groupoids which, in addition, are commutative and associative, and $\circ$ is expanding. A signature extension of a Church monoid with $\wedge$ and $\vee$ as for positive Ackermann groupoids is called a *Dunn monoid*.

A discrete duality for Church monoids can be obtained based on discrete duality for Ackermann groupoids and Theorem 5.1. A discrete duality for Dunn monoids can be obtained based on discrete dualities for positive Ackermann groupoids.

A *residuated-lattice-ordered monoid* is a lattice-ordered monoid endowed with right and left residuals of its groupoid operator. Algebraic signature extensions or axiomatic extensions of such lattices provide algebraic semantics for a number of non-classical logics, in particular substructural logics, multiple-valued logics, and fuzzy logics, see, for example, [3, 10, 11]. Discrete dualities for residuated lattices were studied both for distributive lattices and not necessarily distributive ones. The representation theorem for distributive lattices has been well known since the fundamental work by Stone [29]; a modern form based on ordered topological spaces was presented by Priestley [23]. Its topology-free version, extended to residuals as in Section 7, easily leads to a discrete duality for residuated distributive-lattice-ordered monoids.

For a number of their axiomatic extensions discrete dualities exist, for example: Integral distributive-residuated-lattice-ordered monoids where the unit element of the lattice coincides with the unit element of the monoid, commutative distributive-residuated-lattice-ordered monoids, where the operator of the monoid is commutative, monoidal t-norm fuzzy logic algebras (MTL) which are both integral and commutative distributive-residuated-lattice-ordered monoids with the axiom of prelinearity, MTL algebras with a negation. A Tarski relation algebra is a Boolean-algebra-ordered monoid with some additional axioms which reflect relationships of Boolean operators with monoid operators. A discrete duality for relation algebras and their frames follows from representation theorems in [28] and [16] and from developments in [18].

In case of non-distributive lattices, a topology free version of Urquhart's representation [8] or the representation of general lattices in [13] may be a starting point for developing discrete dualities. Based on Urquhart's lattice representation [30] discrete dualities for general-lattice-ordered groupoids exist, among others, for general-lattice-ordered monoids with left and right residuals of their operators, residuated general-lattice-ordered monoids with involution, general-lattice-ordered relation algebras, the class of $FL$ algebras corresponding to full Lambek Calculus and its axiomatic extensions $FL_e$, $FL_c$, $FL_w$, $FL_{ew}$, among others. The class $FL_{ew}$ provides a basis for the Esteva-Godo-Ono hierarchy of substructural and fuzzy logics presented in [10]. Details of the discrete dualities mentioned above can be found in [20].

## 9. Conclusion

In this paper discrete dualities for groupoids were studied. For every chosen class of groupoids the task of proving discrete duality included establishing the appropriate class of frames and then proceeding as it is described in steps $\mathtt{S}_1$, $\mathtt{S}_2$, and $\mathtt{S}_3$ in the introduction. It was observed that for the class of plain groupoids the definition of canonical frame analogous to the definition given in [15] did not enable us proving preservation of the groupoid operator under the embedding due to lack of any ordering in the groupoids. Thus in the subsequent sections ordered groupoids are considered and some of their axiomatic or signature extensions. For the class of groupoids with a partial ordering compatible with their operators the class of frames was proposed based on the frame semantics of relevant logic presented in [27] and discrete dualities were proved for these classes of structures. In subsequent sections discrete dualities were presented for several axiomatic extension of the classes of ordered groupoids including classes of commutative, associative, and idempotent groupoids; signature extensions of the classes of ordered groupoids included ordered groupoids with identity and ordered groupoids with residuals of their operators. Finally, in Section 8 further axiomatic or signature extensions of groupoids were mentioned together with bibliographical information on discrete duality results for these.

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