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A remark on the derivation of an effective model describing the flow of fluid in a reservoir with small holes

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Abstract. *We study the flow of the fluid in a reservoir with several small holes. The velocities on holes (injection and ejection) are prescribed and high. As the size of each holes shrinks to one point, the boundary value degenerates to a Dirac masses concentrated on the boundary. We prove that the solution of the Stokes system, describing the original flow, converges to the very-weak solution of the Stokes system with Dirac measures on the boundary.*

Keywords: Stokes problem, very weak solution, concentrated boundary condition, Dirac measure.
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1. Introduction

The situation when a viscous fluid is injected to a reservoir through a small hole is common in real life. When the size of the hole is small compared with the size of the reservoir it is reasonable to model the physical phenomenon by replacing the hole with one point. However, imposing the boundary value of the velocity in one point is not feasible in variational setting as the trace of a function from $H^1(\Omega)$ is not defined in a single point. Thus we need another approach. Like in [7] and [8], we choose to work with very-weak setting that turns out to be convenient due to the fact that it allows irregular boundary values and the boundary value appears explicitly in the formulation (unlike in the standard variational formulation). We start with delta-like boundary values (concentrated near given set of points, with shrinking support and high magnitude). Using the rigorous asymptotic analysis we prove that, as the supports of the boundary values shrink to one points (and their magnitudes tend to infinity), the weak solution of the Stokes system tends to the very-weak solution of the same system, but with Dirac delta measures concentrated in those points. The result is based on sharp a priori estimates, very-weak formulation, elliptic regularity and weak convergence.

1.1. Mathematical preliminaries

Let Ω be a bounded domain in $\mathbf{R}^n$, $n > 1$, with smooth boundary $\Gamma = \partial\Omega$ of class $C^{2,1}$.

As described before, we study the stationary Stokes problem (1) with Dirichlet's boundary condition (2), concentrated near of some boundary points $\mathbf{T}_k$

$$-\Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = 0\,, \quad \operatorname{div} \mathbf{u}^\varepsilon = 0\,, \quad \text{in } \Omega\,, \tag{1}$$

$$\mathbf{u}^\varepsilon = \mathbf{g}_\varepsilon\,, \qquad\qquad \text{on } \Gamma\,. \tag{2}$$

The idea is to choose (in the next section) the boundary value that converges weak* in the sense of measures to

$$\mathbf{g}_\varepsilon \rightharpoonup \sum_{k=1}^{m} \lambda_k\ \delta_{\mathbf{T}_k}\,,$$

as $\varepsilon \to 0$. Here, $\lambda_k \in \mathbf{R}$ are some numbers and $\delta_{\mathbf{T}_k}$ are the Dirac masses concentrated in points $\mathbf{T}_k \in \Gamma$, defined as the distributions that act as

$$\langle \delta_{\mathbf{T}_k} \,|\, \phi \rangle = \phi(\mathbf{T}_k)\,,$$

on any continuous test-function ϕ.

The object of our study is the asymptotic behavior of the solution $(\mathbf{u}^\varepsilon, p^\varepsilon)$ with respect to the small parameter ε. Obviously, since the boundary value degenerates to a measure, we cannot use the notion of a weak solution. Thus, our strategy is to use the very-weak solutions. To do so, we first define the notion of the very-weak solution and its basic properties.

Let $\mathbf{g}_\varepsilon \in W^{-1/r,r}(\Gamma)$. We recall that the space $W^{-1/r,r}(\Gamma)$ is a dual space of the $W^{1/r,r'}(\Gamma)$, where $\frac{1}{r'} + \frac{1}{r} = 1$. To solve the problem (1)-(2), we usually assume that compatibility condition holds:

$$\int_\Gamma \mathbf{g}_\varepsilon \cdot \mathbf{n}\, dS = 0\,. \tag{3}$$

However, that does not quite make sense for our $\mathbf{g}_\varepsilon$, since it is not even a function.

The notion of the very weak solution for the stationary Stokes system was introduced by Conca in [2] (using the idea from [5]) and extended by Marušić-Paloka in [6] for the Navier-Stokes system. The case of boundary conditions involving the pressure was studied by Conca, Murat and Pirroneau in [3] and the case of measure boundary condition, that we need here, was studied by Galdi, Simader and Sohr in [4]. We use their existence and uniqueness result.

We briefly recall the definition.

We say that $(\mathbf{u}^\varepsilon, p^\varepsilon) \in L^r(\Omega)^n \times W^{-1,r}(\Omega)$ is the very weak solution to the Stokes problem (1) with Dirichlet's boundary condition (2) if

$$\operatorname{div} \mathbf{u}^\varepsilon = 0$$

in the sense of distributions, and if for any $(\mathbf{w}, \pi) \in (W^{2,r'}(\Omega)^n \cap W_0^{1,r'}(\Omega)^n) \times W^{1,r'}(\Omega)$, where $\frac{1}{r'} + \frac{1}{r} = 1$, the following equation holds

$$\int_\Omega \mathbf{u}^\varepsilon \cdot (-\Delta \mathbf{w} + \nabla \pi) \, d\Omega = \Big\langle \mathbf{g}_\varepsilon \,\Big|\, -\frac{\partial \mathbf{w}}{\partial \mathbf{n}} + \pi \, \mathbf{n} \Big\rangle_\Gamma .$$

Here $\langle \cdot \mid \cdot \rangle_\Gamma$ stands for the duality pairing between $W^{-1/r,r}(\Gamma)$ and $W^{1/r',r'}(\Gamma)$.

In this setting the compatibility condition (3) translates as

$$\langle \mathbf{g}_\varepsilon \mid \mathbf{n} \rangle_\Gamma = 0 . \tag{4}$$

Stationary Stokes (and even Navier-Stokes) system with the boundary value $\mathbf{g}_\varepsilon$ from the dual space $W^{-1/r,r}(\Gamma)$, was studied in [4] and it was proved that very weak solution exists and is unique (in case of the Navier-Stokes system, for small data). Their result states as follows:

THEOREM 1.1. *Let Ω be a bounded domain in $\mathbf{R}^n$ with $C^{2,1}$ boundary $\Gamma = \partial\Omega$. Let $\mathbf{g}_\varepsilon \in W^{-1/r,r}(\Gamma)$, with $1 < r < \infty$ be such that it satisfies the compatibility condition (4). Then the problem (1)-(2) has a unique very weak solution $(\mathbf{u}^\varepsilon, p^\varepsilon) \in L^r(\Omega)^n \times W^{-1,r}(\Omega)$. Furthermore, the following estimate holds*

$$|\mathbf{u}^\varepsilon|_{L^r(\Omega)} + |p^\varepsilon|_{W^{-1,r}(\Omega)} \le C |\mathbf{g}_\varepsilon|_{W^{-\frac{1}{r},r}(\Gamma)} , \tag{5}$$

where $C = C(\Omega, r) > 0$.

2. Concentrated boundary condition

In this section we precise the form of the delta-like boundary condition and prove the convergence.

2.1. The geometry

To define the injection and ejection holes we first pick their central points $\mathbf{T}_k = (t_1^k, \cdots, t_n^k) \in \Gamma$, $k = 1, \ldots, m$.

For each point $\mathbf{T}_k$ we choose the local orthonormal coordinate system $\{\mathbf{0}^k, (\mathbf{e}_j^k)_{j=1}^n\}$ such that the origin is at point $\mathbf{T}_k$. Next, the n-th coordinate vector $\mathbf{e}_n^k = \mathbf{n}$ equals the unit exterior normal at point $\mathbf{T}_k = \mathbf{0}^k$ and the rest of the vectors $\mathbf{e}_\alpha^k$, $\alpha = 1, \ldots, n-1$, are tangential to Γ.

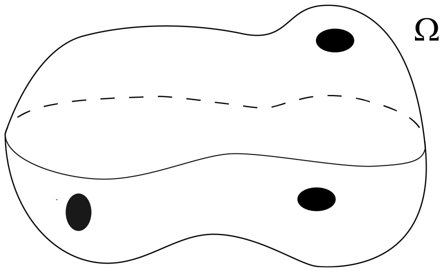


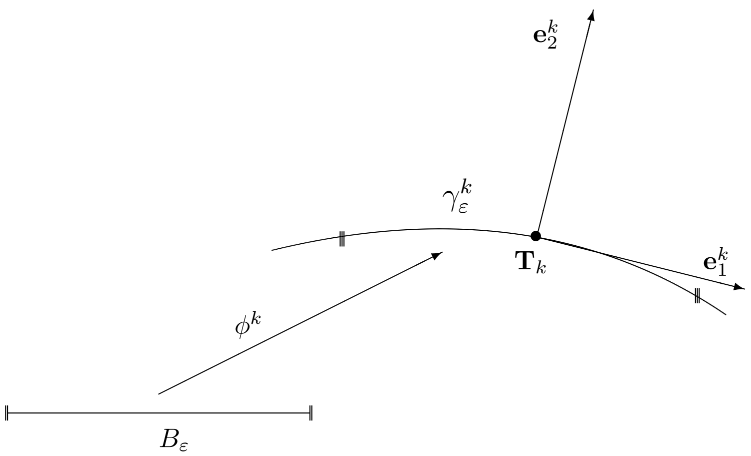
Figure 1: An example of a domain Ω

We proceed by picking the local chart. More precisely, we assume that Γ can be described by the equation

$$x_n^k = \phi^k(\overline{\mathbf{x}}^k)\ ,\ \ \overline{\mathbf{x}}^k = (x_1^k, \cdots, x_{n-1}^k)$$

in vicinity of $\mathbf{T}_k$ for some smooth function $\phi^k : B_\varepsilon = \{\overline{\mathbf{x}}^k \in \mathbf{R}^{n-1}\ ;\ \left|\overline{\mathbf{x}}^k\right| < \varepsilon\ \} \to \mathbf{R}$. Now we define the hole γ_ε^k as the set (slightly distorted ball of radius ε)

$$\gamma_\varepsilon^k = \{\mathbf{x} = (\overline{\mathbf{x}}^k, \phi^k(\overline{\mathbf{x}}^k)) \in \Gamma\ ;\ \overline{\mathbf{x}}^k \in B_\varepsilon\ \}\,.$$



For small ε, since ϕ^k is smooth, the surface γ_ε^k is almost flat and

$$\phi^k(\overline{\mathbf{x}}^k) = \phi^k(\overline{\mathbf{0}}^k) + O(\varepsilon)\,, \tag{6}$$

while the unit exterior normal on γ_ε^k

$$\mathbf{n} = \mathbf{e}_n^k + O(\varepsilon)\,. \tag{7}$$

For the sake of simplicity, we choose the boundary value $\mathbf{g}_\varepsilon$ to be zero outside each hole γ_ε^k (impermeable boundary).

On k-th hole we define the boundary value, using the local chart, as

$$\mathbf{g}_\varepsilon(\overline{\mathbf{x}}^k) = \mathbf{g}_\varepsilon^k(\overline{\mathbf{x}}^k) \equiv \begin{cases} \frac{1}{\varepsilon^{n-1}}\mathbf{g}^k\left(\frac{\overline{\mathbf{x}}^k}{\varepsilon}\right), & \text{for } |\overline{\mathbf{x}}^k| < \varepsilon\,, \\ 0\,, & \text{for } |\overline{\mathbf{x}}^k| > \varepsilon\,, \end{cases} \tag{8}$$

where $\mathbf{g}^k(\overline{\mathbf{y}}^k)$ is continuous function, compactly supported in a unit ball $|\overline{\mathbf{y}}^k| < 1$ and $\overline{\mathbf{y}}^k = (y_1^k, , \ldots, y_{n-1}^k)$.

Our problem now reads

$$\begin{aligned} -\Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = 0\,, \quad \operatorname{div} \mathbf{u}^\varepsilon = 0\,, \quad & \text{in } \Omega \\ \mathbf{u}^\varepsilon = \mathbf{g}_\varepsilon^k\,, \quad & \text{on } \gamma_\varepsilon^k\ ,\ k = 1, \ldots, m\,, \\ \mathbf{u}^\varepsilon = 0\,, \quad & \text{on } \Gamma \setminus \bigcup_{k=1}^m \gamma_\varepsilon^k\,. \end{aligned}$$

We turn our attention to the compatibility condition (3). It reads

$$\sum_{k=1}^m \int_{\gamma_\varepsilon^k} \mathbf{g}_\varepsilon^k \cdot \mathbf{n}\, dS_\varepsilon = 0\,.$$

Due to (6) and (7) we have

$$\int_{\gamma_\varepsilon^k} \mathbf{g}_\varepsilon^k \cdot \mathbf{n}\, dS_\varepsilon = \frac{1}{\varepsilon^{n-1}} \int_{B_\varepsilon} \mathbf{g}_\varepsilon^k \cdot \mathbf{n}\, dS = \int_{B_1} \mathbf{g}^k(\overline{\mathbf{y}}^k) \cdot \mathbf{e}_n^k\, d\overline{\mathbf{y}}^k + O(\varepsilon)\,.$$

Defining

$$\lambda_k = \int_{B_1} \mathbf{g}^k(\overline{\mathbf{y}}^k) \cdot \mathbf{e}_n^k\, d\overline{\mathbf{y}}^k$$

the compatibility condition (since $\varepsilon \ll 1$ is small) implies that we must impose the condition

$$\sum_{k=1}^m \lambda_k = 0\,. \tag{9}$$

2.2. The convergence

Our main result is the following convergence theorem:

THEOREM 2.1. *Let Ω be a smooth bounded domain in $\mathbf{R}^n$ with holes γ_ε^k as described above. Let $\mathbf{g}_\varepsilon$ be defined as in (8) and let the compatibility conditions*

(3) and (9) be satisfied. Let $(\mathbf{u}^\varepsilon, p^\varepsilon) \in H^1(\Omega)^n \times L^2(\Omega)$ *be the weak solution to the problem (1)-(2). Then, for any* r*, such that* $1 < r < \frac{n}{n-1}$

$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}^0 \ \textit{ weakly in } \ L^r(\Omega)^n \quad \textit{and} \quad \mathbf{p}^\varepsilon \rightharpoonup \mathbf{p}^0 \quad \textit{weakly in } \ W^{-1,r}(\Omega)\,,$$

where $(\mathbf{u}^0, p^0) \in L^r(\Omega)^n \times W^{-1,r}(\Omega)$ *is the unique very-weak solution to the problem*

$$-\Delta \mathbf{u}^0 + \nabla p^0 = 0\,, \quad \operatorname{div} \mathbf{u}^0 = 0\,, \quad \textit{in } \Omega\,, \tag{10}$$

$$\mathbf{u}^0 = \sum_{k=1}^m \lambda_k \ \delta_{\mathbf{T}_k}\,, \qquad\qquad \textit{on } \Gamma\,. \tag{11}$$

Proof. Let $r' > n$ and let $(\mathbf{w}, \pi) \in W^{2,r'}(\Omega)^n \times W^{1,r'}(\Omega)$. The Sobolev embedding theorem implies that $\mathbf{w} \in C^1(\overline{\Omega})^n$ and $\pi \in C(\overline{\Omega})$ (see e.g. [1], for Stokes system). Next, we need a priori estimates, in order to extract convergent subsequences. By definition

$$|\mathbf{g}_\varepsilon|_{W^{-\frac{1}{r},r}(\Gamma)} = \sup_{|\mathbf{z}|_{W^{\frac{1}{r},r'}(\Gamma)}=1} \left| \int_\Gamma \mathbf{g}_\varepsilon \ \cdot \ \mathbf{z}\, dS \right| .$$

Functions from $W^{\frac{1}{r},r'}(\Gamma)$ are the traces of functions from $W^{1,r'}(\Omega)$ which is embedded in $C(\overline{\Omega})$. Thus, if $|\mathbf{z}|_{W^{\frac{1}{r},r'}(\Gamma)} = 1$, then $|\mathbf{z}|_{L^\infty(\Gamma)} \le C$, and obviously $|\mathbf{z}|_{L^\infty(\gamma_\varepsilon^k)} \le C$ for any $k = 1, \ldots, m$ and some $C > 0$. Furthermore

$$\int_\Gamma \mathbf{g}_\varepsilon \ \cdot \ \mathbf{z}\, dS = \sum_{k=1}^m \int_{\gamma_\varepsilon^k} \mathbf{g}_\varepsilon^k \ \cdot \ \mathbf{z}\, dS_\varepsilon\,,$$

implying

$$\left| \int_{\gamma_\varepsilon^k} \mathbf{g}_\varepsilon^k \ \cdot \ \mathbf{z}\, dS_\varepsilon \right| \le C |\mathbf{g}_\varepsilon^k|_{L^1(\gamma_\varepsilon^k)} = \int_{B_1} |\, \mathbf{g}^k \left(\overline{\mathbf{y}}^k\right) |\, d\overline{\mathbf{y}}^k \le C\,.$$

Thus, adding up the above

$$|\mathbf{g}_\varepsilon|_{W^{-\frac{1}{r},r}(\Gamma)} \le C\,. \tag{12}$$

Now (12) and (5) imply that

$$|\mathbf{u}^\varepsilon|_{L^r(\Omega)} + |p^\varepsilon|_{W^{-1,r}(\Omega)} \le C\,,$$

with $C > 0$, independent from ε.

The compactness theorem now implies the existence of $(\mathbf{u}^0, p^0) \in L^r(\Omega)^n \times W^{-1,r}(\Omega)$ such that

$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}^0 \quad \text{in} \quad L^r(\Omega)^n \quad \text{and} \quad \mathbf{p}^\varepsilon \rightharpoonup \mathbf{p}^0 \quad \text{in} \quad W^{-1,r}(\Omega)$$

as $\varepsilon \to 0$ (after possible extraction of a subsequence). We need to identify that limit. Due to the continuity of the weak derivative, we have

$$\operatorname{div} \mathbf{u}^0 = 0$$

in the sense of distributions. Starting from the very weak formulation we get

$$\begin{aligned}
\int_\Omega \mathbf{u}^\varepsilon \cdot (-\Delta \mathbf{w} + \nabla \pi)\, d\Omega &= \sum_{k=1}^m \int_{\gamma_\varepsilon^k} \mathbf{g}_\varepsilon^k \left(-\frac{\partial \mathbf{w}}{\partial \mathbf{n}} + \pi\, \mathbf{n} \right) dS_\varepsilon = \\
&= \frac{1}{\varepsilon^{n-1}} \sum_{k=1}^m \int_{B_\varepsilon} \mathbf{g}^k \left(\frac{\overline{\mathbf{x}}^k}{\varepsilon} \right) \left(-\frac{\partial \mathbf{w}}{\partial \mathbf{n}} + \pi\, \mathbf{n} \right) (\overline{\mathbf{x}}^k, 0)\, d\overline{\mathbf{x}}^k + O(\varepsilon) = \\
&= \sum_{k=1}^m \int_{B_1} \mathbf{g}^k \left(\overline{\mathbf{y}}^k \right) \left(-\frac{\partial \mathbf{w}}{\partial \mathbf{n}} + \pi\, \mathbf{n} \right) (\mathbf{T}_k)\, d\overline{\mathbf{y}}^k + O(\varepsilon) \to \\
&\to \sum_{k=1}^m \left(-\frac{\partial \mathbf{w}}{\partial \mathbf{n}} (\mathbf{T}_k) + \pi(\mathbf{T}_k)\, \mathbf{n} \right) \int_{B_1} \mathbf{g}^k \left(\overline{\mathbf{y}}^k \right) d\overline{\mathbf{y}}^k = \\
&= \sum_{k=1}^m \lambda_k \left(-\frac{\partial \mathbf{w}}{\partial \mathbf{n}} (\mathbf{T}_k) + \pi(\mathbf{T}_k)\, \mathbf{n} \right) .
\end{aligned}$$

It proves that the limit $\mathbf{u}^0$ satisfies the Stokes system (10) and that it satisfies the boundary condition (11) (both in the very-weak sense), for some pressure p^0. We still have to prove the convergence of the pressure and show that p^0 is the limit of p^ε. We obviously have

$$\langle p^\varepsilon | \operatorname{div} \mathbf{h} \rangle = -\int_\Omega \mathbf{u}^\varepsilon \cdot \Delta \mathbf{h}\, d\Omega\,, \quad \forall\, \mathbf{h} \in C_0^\infty(\Omega)^n\,.$$

Passing to the limit implies

$$\langle p^0 | \operatorname{div} \mathbf{h} \rangle = -\int_\Omega \mathbf{u}^0 \cdot \Delta \mathbf{h}\, d\Omega\,, \quad \forall\, \mathbf{h} \in C_0^\infty(\Omega)^n$$

proving that $(\mathbf{u}^0, p^0)$ is the very weak solution to the problem (10)-(11). Since it has a unique solution, the whole sequence $(\mathbf{u}^\varepsilon, p^\varepsilon)$ converges to $(\mathbf{u}^0, p^0)$ in the above weak sense. □

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