

Doubling inequality at the boundary for the Kirchhoff-Love plate's equation with supported conditions

ANTONINO MORASSI, EDI ROSSET, EVA SINCICH,
 AND SERGIO VESSELLA

ABSTRACT. *In this article we derive a doubling inequality at the boundary for solutions to the Kirchhoff-Love isotropic plate's equation satisfying supported boundary conditions. To this end, we combine the use of a suitable conformal mapping which flattens the boundary and a reflection argument which guarantees the needed regularity of the extended solution. We finally apply inequalities of Carleman type in order to derive the result. The latter implies Strong Unique Continuation Property at the boundary (SUCPB).*

Keywords: Kirchhoff–Love elastic plates, doubling inequality at the boundary, unique continuation, supported conditions.

MS Classification 2020: 35B60, 35J30, 74K20, 35R25, 35R30.

1. Introduction

In this paper we are mainly concerned with a Strong Unique Continuation Property at the Boundary for the Kirchhoff-Love isotropic plate's equation satisfying the so-called supported conditions. We denote by $\Omega \subset \mathbb{R}^2$ the middle surface of a thin plate having uniform thickness h . Working in the framework of linear elasticity for infinitesimal deformations and under the assumptions of the Kirchhoff-Love theory, the transverse displacement u of the plate satisfies the following fourth order partial differential equation

$$\operatorname{div} (\operatorname{div} (B(1 - \nu)\nabla^2 u + B\nu\Delta u I_2)) = 0, \quad \text{in } \Omega, \quad (1)$$

where B is the bending stiffness and ν is the Poisson's coefficient. We are interested in analyzing the case of supported conditions, namely

$$\begin{cases} u = 0, & \text{on } \Gamma, \\ B(1 - \nu)\partial_{ij}^2 u n_i n_j + B\nu\Delta u = 0 & \text{on } \Gamma, \end{cases} \quad (2)$$

$$(3)$$

where $n = (n_1, n_2)$ denotes the outer unit normal to $\partial\Omega$ and Γ is an open portion of the boundary of Ω . These boundary conditions occur when the constraint on Γ prevents the transverse displacement of the plate and, simultaneously, leaves free the rotation of the transverse material fibers $x \times [-\frac{h}{2}, \frac{h}{2}]$, $x \in \Gamma$, around the direction of the tangent to the boundary of the plate. Situations of this type are quite common in engineering structures, for example for slab plates or roof plates [14].

Assuming $B, \nu \in C^2(\bar{\Omega})$ and Γ of $C^{4,\alpha}$ class, we obtain the following doubling inequality at the boundary for problem (1)–(3)

$$\int_{B_s(P) \cap \Omega} |u|^2 \leq CN^k \left(\frac{s}{r}\right)^{\log_2(CN^k)} \int_{B_r(P) \cap \Omega} |u|^2, \quad (4)$$

for every $P \in \Gamma$ such that $B_{r_0}(P) \cap \partial\Omega \subset \Gamma$, where $C, k > 1, 0 < r < s < \frac{r_0}{C}$ and N is the frequency of the solution u defined as follows

$$N = \frac{\int_{B_{r_0}(P) \cap \Omega} |u|^2}{\int_{B_{\frac{r_0}{C}}(P) \cap \Omega} |u|^2}.$$

Inequalities as (4) are classical tools of Strong Unique Continuation at the Boundary arising in the quantification of the local vanishing rate on the boundary of solutions to PDEs (see, for instance, [1, 4, 6–8, 12]). The quantification of this property poses significant challenges, in fact one needs to infer quantitative control on the zero set of u . Moreover, from a more applied side, it has been proved useful in the context of stability estimates for inverse problems consisting in the determination of unknown coefficients and boundaries [3, 9, 11, 13].

While the overall strategy follows the ideas introduced in [12], we emphasize that the presence of supported boundary conditions displays several new features and difficulties that we outline below. As first step, following [12], we flatten the boundary by means of a conformal map which has been introduced in [4]. Such a change of variables on one hand simplifies the geometry of the problem and preserves the structure of the equation, but on the other hand leads to a quite demanding computation in order to achieve the underlying boundary conditions. A further obstruction in dealing with supported boundary conditions is due to the fact that, in order to apply a reflection argument, we need to operate another transformation (see (45)–(46) for a precise definition) that fits our problem into a new one for a solution v satisfying more elementary boundary conditions, namely

$$v = \Delta v = 0, \quad \text{in } (-1, 1) \times \{0\}.$$

The latter allows us to apply an odd reflection to v , with extended function \bar{v} belonging to $H^4(B_1)$ and satisfying a suitable partial differential equation in B_1 .

Taking advantage of the regularity of the extended solution \bar{v} , we are now in position to deal with the following Carleman estimate [12]. For every $\tau \geq \bar{\tau}$ and for every $r \in (0, 1)$ the following inequality holds

$$\tau^4 r^2 \int \rho^{-2-2\tau} |U|^2 + \sum_{k=0}^3 \tau^{6-2k} \int \rho^{2k+1-2\tau} |D^k U|^2 \leq C \int \rho^{8-2\tau} (\Delta^2 U)^2, \quad (5)$$

for every $U \in C_0^\infty(B_1 \setminus \bar{B}_{r/4})$, where $\rho(x, y) \sim \sqrt{x^2 + y^2}$ as $(x, y) \rightarrow (0, 0)$ (see Proposition 4.2 for a precise statement). Indeed we finally apply the estimate (5) to $U = \eta \bar{v}$ where η is a cut-off function and we conclude by using a standard procedure. We also remark that the Carleman estimate (5) for the bi-laplacian operator is analogous to the estimate (6.32) derived in [10], with the exception of the presence of the first extra integral on the left hand side of (5) (see [5] for the first derivation of a doubling inequality in the interior from a Carleman estimate of the form (5)). It can be shown that this term plays a crucial role in obtaining the doubling inequality (4).

The outline of the article is as follows. After briefly recalling our notation in Section 2, in Section 3 we introduce and state our main result, Theorem 3.1. In Section 4, we recall two main ingredients of our strategy, namely the conformal mapping in Theorem 4.1 and the Carleman estimate in Proposition 4.2. In Section 5, which contains the central estimates of the paper, we deduce the doubling inequality at the boundary. The proof is divided into three steps, namely the reduction to a flat boundary, the extension by odd reflection and the application of the Carleman estimate.

2. Notation

We shall generally denote points in \mathbb{R}^2 by $x = (x_1, x_2)$ or $y = (y_1, y_2)$, except for Sections 4 and 5 where we rename x, y the coordinates in \mathbb{R}^2 .

We shall denote by $B_r(P)$ the disc in \mathbb{R}^2 of radius r and center P , by B_r the disk of radius r and center O , by B_r^+ , B_r^- the hemidisks in \mathbb{R}^2 of radius r and center O contained in the halfplanes $\mathbb{R}_+^2 = \{x_2 > 0\}$, $\mathbb{R}_-^2 = \{x_2 < 0\}$ respectively, and by $R_{a,b}$ the rectangle $(-a, a) \times (-b, b)$.

Given a matrix $A = (a_{ij})$, we shall denote by $|A|$ its Frobenius norm $|A| = \sqrt{\sum_{i,j} a_{ij}^2}$.

DEFINITION 2.1 ($C^{k,\alpha}$ regularity). *Let Ω be a bounded domain in \mathbb{R}^2 . Given k, α , with $k \in \mathbb{N}$, $0 < \alpha \leq 1$, we say that a portion S of $\partial\Omega$ is of class $C^{k,\alpha}$ with constants $r_0, M_0 > 0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and*

$$\Omega \cap R_{r_0, 2M_0 r_0} = \{x \in R_{r_0, 2M_0 r_0} \mid x_2 > g(x_1)\},$$

where g is a $C^{k,\alpha}$ function on $[-r_0, r_0]$ satisfying

$$g(0) = g'(0) = 0,$$

$$\|g\|_{C^{k,\alpha}([-r_0, r_0])} \leq M_0 r_0,$$

where

$$\|g\|_{C^{k,\alpha}([-r_0, r_0])} = \sum_{i=0}^k r_0^i \sup_{[-r_0, r_0]} |g^{(i)}| + r_0^{k+\alpha} |g|_{k,\alpha},$$

$$|g|_{k,\alpha} = \sup_{\substack{t,s \in [-r_0, r_0] \\ t \neq s}} \left\{ \frac{|g^{(k)}(t) - g^{(k)}(s)|}{|t - s|^\alpha} \right\}.$$

Throughout the paper, summation over repeated indexes is assumed.

The cartesian components of the divergence of a second order tensor field $T(x)$ are defined, as usual, by

$$(\operatorname{div} T(x))_i = T_{ij,j}(x), \quad i = 1, 2.$$

In places we will use equivalently the symbols D and ∇ to denote the gradient of a function. Moreover, setting $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$, the multi-index notation D^β represents the partial derivative

$$\frac{\partial^{\beta_1 + \beta_2}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}.$$

Throughout the paper, C shall denote a positive constant which may change from line to line, and p_2, q_2, P_2, Q_2 shall denote second order differential operators which may also change from line to line.

3. Main results

In this paper we consider an isotropic thin elastic plate $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$, having middle plane Ω and thickness h . According to the Kirchhoff-Love theory, the transverse displacement u of the plate satisfies the following fourth-order partial differential equation

$$L(u) := \operatorname{div} (\operatorname{div} (B(1 - \nu) \nabla^2 u + B\nu \Delta u I_2)) = 0, \quad \text{in } \Omega, \quad (6)$$

where the *bending stiffness* B is given by

$$B(x) = \frac{h^3}{12} \left(\frac{E(x)}{1 - \nu^2(x)} \right), \quad (7)$$

with the *Young's modulus* E and the *Poisson's coefficient* ν given by

$$E(x) = \frac{\mu(x)(2\mu(x) + 3\lambda(x))}{\mu(x) + \lambda(x)}, \quad \nu(x) = \frac{\lambda(x)}{2(\mu(x) + \lambda(x))},$$

where λ and μ are the Lamé moduli. Therefore, we may also rewrite (7) as

$$B(x) = \frac{h^3}{3} \cdot \frac{\mu(x)(\mu(x) + \lambda(x))}{2\mu(x) + \lambda(x)}.$$

We assume that the Lamé moduli satisfy the strong convexity assumptions

$$\mu(x) \geq \alpha_0 > 0, \quad 2\mu(x) + 3\lambda(x) \geq \gamma_0 > 0, \quad \text{in } \Omega, \quad (8)$$

where α_0, γ_0 are positive constants. Let us notice that the above assumptions imply that

$$\mu(x) + \lambda(x) \geq \min \left\{ \alpha_0, \frac{\gamma_0}{2} \right\}, \quad \text{in } \Omega. \quad (9)$$

Moreover, we shall assume the following regularity on λ and μ :

$$\|\lambda\|_{C^2(\bar{\Omega}_{r_0})}, \quad \|\mu\|_{C^2(\bar{\Omega}_{r_0})} \leq \Lambda_0, \quad (10)$$

with Λ_0 a positive constant.

We found it convenient to represent the plate equation $L(u)$ in the compact form

$$L(u) = \partial_{ij}^2 (c_{ijkl} \partial_{lk}^2 u) = 0 \quad \text{in } \Omega, \quad (11)$$

with

$$c_{ijkl} = B(1 - \nu)\delta_{il}\delta_{jk} + B\nu\delta_{ij}\delta_{lk}, \quad \text{in } \Omega, \quad (12)$$

so that, by a straightforward computation, equation (6) can be rewritten as

$$L(u) = B(\Delta^2 u + \tilde{a} \cdot \nabla(\Delta u) + \tilde{q}_2(u)) = 0, \quad \text{in } \Omega, \quad (13)$$

where

$$\tilde{a} = 2 \frac{\nabla B}{B}, \quad (14)$$

$$\tilde{q}_2(u) = \sum_{i,j=1}^2 \frac{1}{B} \partial_{ij}^2 (B(1 - \nu) + \nu B \delta_{ij}) \partial_{ij}^2 u. \quad (15)$$

Let us define

$$\Omega_{r_0} = \Omega \cap R_{r_0, 2M_0 r_0} = \{x \in R_{r_0, 2M_0 r_0} \mid x_2 > g(x_1)\},$$

$$\Gamma_{r_0} = \partial\Omega \cap R_{r_0, 2M_0 r_0} = \{(x_1, g(x_1)) \mid x_1 \in (-r_0, r_0)\},$$

with g as in Section 2 and such that

$$\|g\|_{C^{4,\alpha}([-r_0, r_0])} \leq M_0 r_0, \quad (16)$$

for some $\alpha \in (0, 1)$.

We notice that, by (8)–(10), the coefficient \tilde{a} and the second order operator $\tilde{q}_2 = \sum_{|\alpha| \leq 2} c_\alpha D^\alpha$ satisfy

$$\|\tilde{a}\|_{C^1(\bar{\Omega}_{r_0}, \mathbb{R}^2)} \leq L, \quad \|c_\alpha\|_{C^0(\bar{\Omega}_{r_0})} \leq L,$$

with $L > 0$ only depending on $\alpha_0, \gamma_0, \Lambda_0$.

We restrict our attention to the set Ω_{r_0} , and we consider the following boundary conditions on Γ_{r_0} :

$$\begin{cases} u = 0, & \text{on } \Gamma_{r_0}, \\ c_{ijkl} \partial_{lk}^2 u n_i n_j = 0, & \text{on } \Gamma_{r_0}, \end{cases}$$

where $n = (n_1, n_2)$ denotes the outer unit normal to $\partial\Omega$. By setting

$$\partial_{nn}^2 u = \partial_{ij}^2 u n_i n_j,$$

the second boundary condition can be rewritten as

$$c_{ijkl} \partial_{lk}^2 u n_i n_j = B(1 - \nu) \partial_{nn}^2 u + B\nu \Delta u = 0, \quad \text{on } \Gamma_{r_0}.$$

Therefore, we study the following problem

$$\begin{cases} L(u) = 0, & \text{in } \Omega_{r_0}, \end{cases} \quad (17)$$

$$\begin{cases} u = 0, & \text{on } \Gamma_{r_0}, \end{cases} \quad (18)$$

$$\begin{cases} B(1 - \nu) \partial_{nn}^2 u + B\nu \Delta u = 0, & \text{on } \Gamma_{r_0}. \end{cases} \quad (19)$$

Let us derive the *variational formulation* to the problem (17)–(19). Let

$$\mathcal{H} = \{v \in C^\infty(\bar{\Omega}_{r_0}) \mid D^\alpha v = 0 \text{ on } \partial\Omega_{r_0} \setminus \Gamma_{r_0}, \forall \alpha \in \mathbb{N}^2, v = 0 \text{ on } \Gamma_{r_0}\}.$$

A *weak solution* to problem (17)–(19) is a function $u \in H^2(\Omega_{r_0})$ satisfying

$$\begin{cases} \int_{\Omega_{r_0}} c_{ijkl} \partial_{lk}^2 u \partial_{ij}^2 v = 0, & \text{for every } v \in \mathcal{H}, \end{cases} \quad (20)$$

$$\begin{cases} u = 0, & \text{on } \Gamma_{r_0}. \end{cases} \quad (21)$$

We notice that the elliptic and regularity assumptions on the Lamé moduli (8) and (10), and the regularity condition (16) on g guarantee that $u \in H^4(\Omega_{r_0})$, see, for instance, [2].

Hence, by applying the integration by parts we have that, for every $v \in \mathcal{H}$,

$$\begin{aligned} \int_{\Omega_{r_0}} c_{ijkl} \partial_{lk}^2 u \partial_{ij}^2 v &= \int_{\Omega_{r_0}} \partial_i (c_{ijkl} \partial_{lk}^2 u \partial_j v) - \partial_i (c_{ijkl} \partial_{lk}^2 u) \partial_j v = \\ &= \int_{\partial\Omega_{r_0}} c_{ijkl} \partial_{lk}^2 u \partial_j v n_i - \int_{\Omega_{r_0}} \partial_i (c_{ijkl} \partial_{lk}^2 u) \partial_j v = \\ &= \int_{\Gamma_{r_0}} c_{ijkl} \partial_{lk}^2 u \partial_j v n_i + \int_{\Omega_{r_0}} \partial_{ij}^2 (c_{ijkl} \partial_{lk}^2 u) v. \end{aligned}$$

Moreover, since $v \equiv 0$ on Γ_{r_0} , we have

$$\nabla v = (\nabla v \cdot n) n, \quad \text{on } \Gamma_{r_0},$$

$$\partial_j v = (\partial_n v) n_j, \quad j = 1, 2, \quad \text{on } \Gamma_{r_0},$$

so that

$$\int_{\Omega_{r_0}} c_{ijkl} \partial_{lk}^2 u \partial_{ij}^2 v = \int_{\Gamma_{r_0}} (c_{ijkl} \partial_{lk}^2 u n_i n_j) \partial_n v + \int_{\Omega_{r_0}} \partial_{ij}^2 (c_{ijkl} \partial_{lk}^2 u) v. \quad (22)$$

Therefore, given a weak solution $u \in H^2(\Omega_{r_0})$ to problem (17)–(19), by choosing the test functions in $C_0^\infty(\Omega_{r_0}) \subset \mathcal{H}$ in (20), we get the differential equation (11) and, consequently, we obtain

$$\int_{\Gamma_{r_0}} (c_{ijkl} \partial_{lk}^2 u n_i n_j) \partial_n v = 0, \quad \forall v \in \mathcal{H},$$

which implies the boundary condition (19). Viceversa, a classical solution to problem (17)–(19), in view of (22), must satisfy the weak formulation (20)–(21).

We are now in position to state the main result of this paper.

THEOREM 3.1 (Doubling inequality at the supported boundary). *Under the above notation, let us assume that the Lamé moduli satisfy (8)–(10) and the boundary Γ_{r_0} is of $C^{4,\alpha}$ -class, with g satisfying (16) for some $\alpha \in (0, 1)$. Let $u \in H^2(\Omega_{r_0})$ solve (17)–(19), then there exist $k > 1$ and $C > 1$ only depending on $\alpha_0, \gamma_0, \Lambda_0, M_0, \alpha$, such that, for every $0 < r < s < \frac{r_0}{C}$ we have that*

$$\int_{B_s \cap \Omega_{r_0}} |u|^2 \leq C N^k \left(\frac{s}{r} \right)^{\log_2(C N^k)} \int_{B_r \cap \Omega_{r_0}} |u|^2, \quad (23)$$

where

$$N = \frac{\int_{B_{r_0} \cap \Omega_{r_0}} |u|^2}{\int_{B_{\frac{r_0}{C}} \cap \Omega_{r_0}} |u|^2}.$$

4. Preliminary results

The following Proposition, obtained in [4], introduces a conformal map which flattens the boundary Γ_{r_0} and preserves the structure of equation (13).

PROPOSITION 4.1 (Conformal mapping [4, Proposition 3.1]). *Under the hypotheses of Theorem 3.1, there exists an injective sense preserving differentiable map*

$$\Phi = (\varphi, \psi) : [-1, 1] \times [0, 1] \rightarrow \overline{\Omega}_{r_0}$$

which is conformal and satisfies

$$\Phi((-1, 1) \times (0, 1)) \supset B_{\frac{r_0}{K}}(0) \cap \Omega_{r_0},$$

$$\Phi([-1, 1] \times \{0\}) = \{(x_1, g(x_1)) \mid x_1 \in [-r_1, r_1]\} \subset \Gamma_{r_0},$$

$$\Phi(0, 0) = (0, 0),$$

$$\frac{c_0 r_0}{2C_0} \leq |D\Phi(y)| \leq \frac{r_0}{2}, \quad \forall y \in [-1, 1] \times [0, 1],$$

$$\frac{4}{r_0} \leq |D\Phi^{-1}(x)| \leq \frac{4C_0}{c_0 r_0}, \quad \forall x \in \Phi([-1, 1] \times [0, 1]),$$

$$\frac{r_0}{K}|y| \leq |\Phi(y)| \leq \frac{r_0}{2}|y|, \quad \forall y \in [-1, 1] \times [0, 1],$$

with $K > 8$, $0 < c_0 < C_0$, $\frac{r_1}{r_0}$ being constants only depending on M_0 and α .

Another basic ingredient for our proof of the doubling inequality at the supported boundary of the plate is the following Carleman estimate, whose proof can be found in [12, Proposition 3.5].

PROPOSITION 4.2 (Carleman estimate). *Let us define*

$$\rho(x, y) = \phi\left(\sqrt{x^2 + y^2}\right),$$

where

$$\phi(s) = \frac{s}{(1 + \sqrt{s})^2}.$$

Then there exist absolute constants $\bar{\tau} > 1$, $C > 1$ such that

$$\begin{aligned} \tau^4 r^2 \int \rho^{-2-2\tau} |U|^2 + \sum_{k=0}^3 \tau^{6-2k} \int \rho^{2k+1-2\tau} |D^k U|^2 \\ \leq C \int \rho^{8-2\tau} (\Delta^2 U)^2, \end{aligned} \quad (24)$$

for every $\tau \geq \bar{\tau}$, for every $r \in (0, 1)$ and for every $U \in C_0^\infty(B_1 \setminus \overline{B}_{r/4})$.

5. Proof of the doubling inequality at the boundary

The strategy of the proof consists of several steps. Firstly, we flatten the boundary Γ_{r_0} by means of the conformal mapping defined in Proposition 4.1. The conformal mapping preserves the structure of the operator and, by the peculiar supported boundary conditions, the solution can be extended by an odd reflection with respect to the flattened boundary. Then, the desired doubling inequality (23) follows from the Carleman estimate (24) and a standard argument.

From now on, we shall simply denote by R the rectangle $(-1, 1) \times (0, 1)$.

First step. Reduction to a flat boundary.

Given a weak solution $u \in H^2(\Omega_{r_0})$ to problem (17)–(19), let us denote

$$w(y) = u(\Phi(y)),$$

where $\Phi = (\Phi_1, \Phi_2) = (\varphi, \psi)$ is the conformal mapping introduced in Proposition 4.1.

It is useful to notice that, since Φ is a conformal map,

$$\begin{aligned} D\Phi &= \begin{pmatrix} \partial_{y_1}\Phi_1 & \partial_{y_2}\Phi_1 \\ \partial_{y_1}\Phi_2 & \partial_{y_2}\Phi_2 \end{pmatrix} = \begin{pmatrix} \partial_{y_1}\varphi & \partial_{y_2}\varphi \\ -\partial_{y_2}\varphi & \partial_{y_1}\varphi \end{pmatrix}, \\ \det(D\Phi(y)) &= |\nabla\varphi(y)|^2, \\ (D\Phi)^{-1} &= \frac{1}{|\nabla\varphi|^2} \begin{pmatrix} \varphi_{y_1} & -\varphi_{y_2} \\ \varphi_{y_2} & \varphi_{y_1} \end{pmatrix}. \end{aligned}$$

For any function $z = z(x)$ defined in $\Phi(R)$, we can compute

$$(\nabla_x z)(\Phi(y)) = [(D\Phi(y))^{-1}]^T \nabla_y (z(\Phi(y))). \quad (25)$$

It follows that

$$(\Delta u)(\Phi(y)) = \frac{1}{|\nabla\varphi(y)|^2} (\Delta w)(y) \quad (26)$$

and

$$\begin{aligned} (\Delta^2 u)(\Phi(y)) &= \frac{1}{|\nabla\varphi(y)|^2} \Delta \left(\frac{1}{|\nabla\varphi(y)|^2} \Delta w(y) \right) = \\ &= \frac{1}{|\nabla\varphi(y)|^2} \left[\frac{1}{|\nabla\varphi(y)|^2} \Delta^2 w + 2\nabla \left(\frac{1}{|\nabla\varphi(y)|^2} \right) \cdot \nabla(\Delta w) + q_2(w) \right], \end{aligned}$$

where q_2 is a second order differential operator with C^1 norm of its coefficients bounded in terms of $\alpha_0, \gamma_0, \Lambda_0$.

In order to derive the differential equation satisfied by w in R , let us consider for the moment test functions having compact support in R . Precisely, given $\tilde{V} \in C_0^\infty(R)$, let

$$\tilde{v}(x) = \tilde{V}(\Phi^{-1}(x)), \quad \text{in } \Phi(R) \subset \Omega_{r_0}.$$

By integrating by parts (20) and recalling (13), we have

$$\begin{aligned} 0 &= \int_{\Omega_{r_0}} c_{ijkl} \partial_{lk}^2 u \partial_{ij}^2 \tilde{v} = \int_{\Phi(R)} c_{ijkl} \partial_{lk}^2 u \partial_{ij}^2 \tilde{v} = \int_{\Phi(R)} \partial_{ij}^2 (c_{ijkl} \partial_{lk}^2 u) \tilde{v} = \\ &= \int_{\Phi(R)} B(\Delta^2 u + \tilde{a} \cdot \nabla(\Delta u) + \tilde{q}_2(u)) \tilde{v}, \end{aligned}$$

where \tilde{a} and $\tilde{q}_2(u)$ are defined in (14) and (15), respectively.

By operating the change of variables $x = \Phi(y)$ in the last integral, we have that, for every $\tilde{V} \in C_0^\infty(R)$,

$$\begin{aligned} \int_R B(\Phi(y)) [(\Delta^2 u)(\Phi(y)) + \tilde{a}(\Phi(y)) \cdot (\nabla(\Delta u))(\Phi(y)) + (\tilde{q}_2(u))(\Phi(y))] \cdot \\ \cdot |det(D\Phi(y))| \tilde{V}(y) = 0. \end{aligned}$$

Since $det(D\Phi(y)) = |\nabla\varphi(y)|^2$, by the arbitrariness of the test functions $\tilde{V} \in C_0^\infty(R)$, we obtain that

$$(\Delta^2 u)(\Phi(y)) + \tilde{a}(\Phi(y)) \cdot (\nabla(\Delta u))(\Phi(y)) + (\tilde{q}_2(u))(\Phi(y)) = 0, \quad \text{in } R. \quad (27)$$

By applying (25) with $z = \Delta u$, and by (26), we compute

$$\begin{aligned} (\nabla_x(\Delta u))(\Phi(y)) &= [(D\Phi(y))^{-1}]^T \nabla_y((\Delta u)(\Phi(y))) = \\ &= [(D\Phi(y))^{-1}]^T \nabla_y \left(\frac{1}{|\nabla\varphi|^2} \Delta w(y) \right) = \\ &= [(D\Phi(y))^{-1}]^T \frac{1}{|\nabla\varphi|^2} \nabla_y \Delta w(y) + q_2(w), \end{aligned} \quad (28)$$

where q_2 is a second order differential operator with C^2 -norm of its coefficients only depending on $M_0, \alpha, \alpha_0, \gamma_0, \Lambda_0$.

By (27), (26) and (28), we have

$$\begin{aligned} \frac{1}{|\nabla\varphi|^2} \Delta^2 w + 2 \nabla \left(\frac{1}{|\nabla\varphi|^2} \right) \cdot \nabla_y(\Delta w) \\ + \tilde{a}(\Phi(y)) \cdot [(D\Phi(y))^{-1}]^T \nabla_y(\Delta w) + q_2(w) \\ = \left(2 \nabla \left(\frac{1}{|\nabla\varphi|^2} \right) + [(D\Phi(y))^{-1}]^T \tilde{a}(\Phi(y)) \right) \cdot \nabla_y(\Delta w) \\ + \frac{1}{|\nabla\varphi|^2} \Delta^2 w + q_2(w) = 0, \quad \text{in } R, \end{aligned} \quad (29)$$

where q_2 is a second order differential operator with C^0 -norm on \bar{R} of its coefficients only depending on $M_0, \alpha, \alpha_0, \gamma_0, \Lambda_0$. Therefore the function w satisfies

$$\mathcal{L}(w) = 0, \quad \text{in } R,$$

with

$$\mathcal{L}(w) = \Delta^2 w + b \cdot \nabla(\Delta w) + Q_2(w), \quad (30)$$

where $b \in C^1(\bar{R}, \mathbb{R}^2)$ is a vector field and $Q_2 = \sum_{|\alpha| \leq 2} C_\alpha D^\alpha$ is a second order differential operator such that $\|b\|_{C^1(\bar{R}, \mathbb{R}^2)}$ and $\|C_\alpha\|_{C^0(\bar{R})}$ are bounded in terms of $M_0, \alpha, \alpha_0, \gamma_0, \Lambda_0$.

In order to derive the boundary conditions satisfied by $w = u \circ \Phi$, with $u \in H^2(\Omega_{r_0})$ a weak solution to problem (17)–(19), let us define

$$\begin{aligned} \tilde{\mathcal{H}} = \{v \in C^\infty(\Phi(R)) \mid D^\alpha v = 0 \text{ on } \partial(\Phi(R)) \setminus (\Gamma_{r_0} \cap \Phi(R)), \forall \alpha \in \mathbb{N}^2, \\ v = 0 \text{ on } \Gamma_{r_0} \cap \Phi(R)\} \end{aligned}$$

It follows that

$$\int_{\Phi(R)} c_{ijkl} \partial_{lk}^2 u \partial_{ij}^2 v = 0, \forall v \in \tilde{\mathcal{H}}.$$

Given $v \in \tilde{\mathcal{H}}$, let us denote

$$V(y) = v(\Phi(y)).$$

By operating the change of variables $x = \Phi(y)$, we have that, for every $v \in \tilde{\mathcal{H}}$,

$$\begin{aligned} 0 &= \int_{\Phi(R)} c_{ijkl} \partial_{lk}^2 u \partial_{ij}^2 v \\ &= \int_R c_{ijkl}(\Phi(y)) |det(D\Phi(y))| (\partial_{lk}^2 u)(\Phi(y)) (\partial_{ij}^2 v)(\Phi(y)). \quad (31) \end{aligned}$$

Let us set

$$M(y) = (D\Phi(y))^{-1}$$

By applying (25) with $z = \partial_{x_k} u$, we obtain

$$\begin{aligned} (\partial_{x_l x_k}^2 u)(\Phi(y)) &= \{M^T(y) \nabla_y [(\partial_{x_k} u)(\Phi(y))]\}_l = \\ &= \{M^T(y) \nabla_y [M^T(y) \nabla_y w]_k\}_l = m_{hl}(y) \partial_{y_h} (m_{sk}(y) \partial_{y_s} w), \quad l, k = 1, 2. \end{aligned}$$

Similarly, given $v \in \tilde{\mathcal{H}}$, we have

$$(\partial_{x_i x_j}^2 v)(\Phi(y)) = m_{h'i}(y) \partial_{y_{h'}} (m_{s'j}(y) \partial_{y_{s'}} V), \quad l, k = 1, 2.$$

From (31), for every $v \in \tilde{\mathcal{H}}$, we have

$$\begin{aligned}
0 &= \int_{\Phi(R)} c_{ijkl} \partial_{lk}^2 u \partial_{ij}^2 v = \\
&= \int_R c_{ijkl}(\Phi(y)) |det(D\Phi(y))| m_{hl} \partial_{y_h} (m_{sk} \partial_{y_s} w) m_{h'i} \partial_{y_{h'}} (m_{s'j} \partial_{y_{s'}} V) = \\
&= \int_R \tilde{c}_{h'jhk} \partial_{y_h} (m_{sk} \partial_{y_s} w) \partial_{y_{h'}} (m_{s'j} \partial_{y_{s'}} V), \tag{32}
\end{aligned}$$

where

$$\tilde{c}_{h'jhk}(y) = m_{h'i}(y) c_{ijkl}(\Phi(y)) |det(D\Phi(y))| m_{hl}(y).$$

By integrating by parts twice, we obtain

$$\int_{\partial R} \mathcal{B}_1 - \mathcal{B}_2 + \int_R \partial_{y_{s'}} [m_{s'j} \partial_{y_{h'}} (\tilde{c}_{h'jhk} \partial_{y_h} (m_{sk} \partial_{y_s} w))] V = 0, \tag{33}$$

where

$$\mathcal{B}_1 = n_{h'} \tilde{c}_{h'jhk} \partial_{y_h} (m_{sk} \partial_{y_s} w) m_{s'j} \partial_{y_{s'}} V,$$

$$\mathcal{B}_2 = n_{s'} m_{s'j} \partial_{y_{h'}} (\tilde{c}_{h'jhk} \partial_{y_h} (m_{sk} \partial_{y_s} w)) V.$$

Next, by the definition of $\tilde{\mathcal{H}}$, $\mathcal{B}_2 \equiv 0$ on ∂R and

$$\int_{\partial R} \mathcal{B}_1 = \int_{[-1,1] \times \{0\}} \mathcal{B}_1 = \int_{[-1,1] \times \{0\}} -(\tilde{c}_{2jhk} \partial_{y_h} (m_{sk} \partial_{y_s} w) m_{2j}) \partial_{y_2} V. \tag{34}$$

Let us compute, by using (12),

$$\begin{aligned}
\tilde{c}_{2jhk} \partial_{y_h} (m_{sk} \partial_{y_s} w) m_{2j} &= |det(D\Phi(y))| m_{2i} m_{2j} m_{hl} c_{ijkl} \partial_{y_h} (m_{sk} \partial_{y_s} w) = \\
&= |det(D\Phi(y))| m_{2i} m_{2j} m_{hl} [B(1-\nu) \delta_{il} \delta_{jk} + B\nu \delta_{ij} \delta_{lk}] \partial_{y_h} (m_{sk} \partial_{y_s} w) = \\
&= B |det(D\Phi(y))| [m_{2l} m_{2k} m_{hl} (1-\nu) \\
&\quad + m_{2i} m_{2i} m_{hk} \nu] (m_{sk} \partial_{y_s y_h}^2 w + \partial_{y_h} (m_{sk}) \partial_{y_s} w) = \\
&= B |det(D\Phi(y))| (\mathcal{A}_1 + \mathcal{A}_2),
\end{aligned}$$

where

$$\mathcal{A}_1 = [(1-\nu) m_{2l} m_{2k} m_{hl} m_{sk} + \nu m_{2i} m_{2i} m_{hk} m_{sk}] \partial_{y_s y_h}^2 w, \tag{35}$$

$$\mathcal{A}_2 = [(1-\nu) m_{2l} m_{2k} m_{hl} \partial_{y_h} (m_{sk}) + \nu m_{2i} m_{2i} m_{hk} \partial_{y_h} (m_{sk})] \partial_{y_s} w. \tag{36}$$

In order to evaluate \mathcal{A}_1 , let us notice that

$$MM^T = \frac{1}{|\nabla \varphi|^2} I_2.$$

Therefore,

$$m_{2l}m_{2k}m_{hl}m_{sk} = \frac{1}{|\nabla\varphi|^4}\delta_{2h}\delta_{2s}, \quad (37)$$

$$m_{2i}m_{2i}m_{hk}m_{sk} = \frac{1}{|\nabla\varphi|^4}\delta_{hs}. \quad (38)$$

By inserting (37) and (38) in (35), we have

$$\mathcal{A}_1 = \frac{1}{|\nabla\varphi|^4}[(1-\nu)\delta_{2h}\delta_{2s} + \nu\delta_{hs}]\partial_{y_s y_h}^2 w = \frac{1}{|\nabla\varphi|^4}[(1-\nu)\partial_{y_2 y_2}^2 w + \nu\Delta w]. \quad (39)$$

On the other hand, since $w(y_1, 0) \equiv 0$, we may rewrite (39) as

$$\mathcal{A}_1 = \frac{1}{|\nabla\varphi|^4}\Delta w. \quad (40)$$

Again by $w(y_1, 0) \equiv 0$, we compute

$$\begin{aligned} m_{2l}m_{2k}m_{hl}\partial_{y_h}(m_{sk})\partial_{y_s}w &= (m_{2l}m_{hl})m_{2k}\partial_{y_h}(m_{2k})\partial_{y_2}w = \\ &= \frac{1}{2}\frac{1}{|\nabla\varphi|^2}\delta_{2h}\partial_{y_h}(m_{2k}m_{2k})\partial_{y_2}w = \frac{1}{2}\frac{1}{|\nabla\varphi|^2}\partial_{y_2}\left(\frac{1}{|\nabla\varphi|^2}\right)\partial_{y_2}w. \end{aligned} \quad (41)$$

Next, let us notice that

$$(D\Phi)^T(D\Phi) = |\nabla\varphi|^2 I_2. \quad (42)$$

By using again $w(y_1, 0) \equiv 0$, by recalling that Φ_k are harmonic functions, $k = 1, 2$, and by (42), we compute

$$\begin{aligned} m_{2i}m_{2i}m_{hk}\partial_{y_h}(m_{sk})\partial_{y_s}w &= \frac{1}{|\nabla\varphi|^2}m_{hk}\partial_{y_h}(m_{2k})\partial_{y_2}w = \\ &= \frac{1}{|\nabla\varphi|^4}\partial_{y_h}(\Phi_k)\partial_{y_h}\left(\frac{\partial_{y_2}(\Phi_k)}{|\nabla\varphi|^2}\right)\partial_{y_2}w = \\ &= \frac{1}{|\nabla\varphi|^4}\left[\partial_{y_h}\left(\frac{\partial_{y_h}(\Phi_k)\partial_{y_2}(\Phi_k)}{|\nabla\varphi|^2}\right) - \frac{\Delta\Phi_k\partial_{y_2}(\Phi_k)}{|\nabla\varphi|^2}\right]\partial_{y_2}w = \\ &= \frac{1}{|\nabla\varphi|^4}\left[\partial_{y_h}\left(\frac{\partial_{y_h}(\Phi_k)\partial_{y_2}(\Phi_k)}{|\nabla\varphi|^2}\right)\right]\partial_{y_2}w = \\ &= \frac{1}{|\nabla\varphi|^4}\partial_{y_h}(\delta_{h2})\partial_{y_2}w = 0. \end{aligned} \quad (43)$$

From (41) and (43), we have

$$\mathcal{A}_2 = \frac{1}{2}(1-\nu)\frac{1}{|\nabla\varphi|^2}\partial_{y_2}\left(\frac{1}{|\nabla\varphi|^2}\right)\partial_{y_2}w. \quad (44)$$

By (34), (40) and (44), we can compute \mathcal{B}_1 on $[-1, 1] \times \{0\}$:

$$\begin{aligned}\mathcal{B}_1 &= -(\tilde{c}_{2jhk} \partial_{y_h} (m_{sk} \partial_{y_s} w) m_{2j}) \partial_{y_2} V = \\ &= -\frac{B}{|\nabla \varphi|^2} \left[\Delta w + \frac{1-\nu}{2} |\nabla \varphi|^2 \partial_{y_2} \left(\frac{1}{|\nabla \varphi|^2} \right) \partial_{y_2} w \right] \partial_{y_2} V.\end{aligned}$$

By (33), we have that

$$\begin{aligned}\int_R \partial_{y_{s'}} [m_{s'j} \partial_{y_{h'}} (\tilde{c}_{h'jkh} \partial_{y_h} (m_{sk} \partial_{y_s} w))] V &= \\ = \int_{[-1,1] \times \{0\}} \frac{B}{|\nabla \varphi|^2} \left[\Delta w + \frac{1-\nu}{2} |\nabla \varphi|^2 \partial_{y_2} \left(\frac{1}{|\nabla \varphi|^2} \right) \partial_{y_2} w \right] \partial_{y_2} V.\end{aligned}$$

for every $V = v(\Phi(y))$ with $v \in \tilde{\mathcal{H}}$. By choosing the test functions $v \in C_0^\infty(\Phi(R))$, that is $V \in C_0^\infty(R)$, we obtain that w satisfies the differential equation

$$\partial_{y_{s'}} [m_{s'j} \partial_{y_{h'}} (\tilde{c}_{h'jkh} \partial_{y_h} (m_{sk} \partial_{y_s} w))] = 0,$$

which coincides with the equilibrium equation (29), and therefore

$$\int_{[-1,1] \times \{0\}} \frac{B}{|\nabla \varphi|^2} \left[\Delta w + \frac{1-\nu}{2} |\nabla \varphi|^2 \partial_{y_2} \left(\frac{1}{|\nabla \varphi|^2} \right) \partial_{y_2} w \right] \partial_{y_2} V = 0,$$

for every $V \in C_0^\infty(R)$. By recalling (30) and the boundary condition implied by the last identity, we have that $w = u(\Phi(y)) \in H^4(R)$ satisfies the following problem

$$\begin{cases} \Delta^2 w + b \cdot \nabla \Delta w + Q_2(w) = 0, & \text{in } R, \\ w(y_1, 0) = 0, & \text{in } [-1, 1] \times \{0\}, \\ \Delta w + \gamma(y_1) \partial_{y_2} w = 0, & \text{in } [-1, 1] \times \{0\}, \end{cases}$$

where

$$\gamma(y_1) = \frac{1-\nu}{2} |\nabla \varphi|^2 \partial_{y_2} \left(\frac{1}{|\nabla \varphi|^2} \right) \Big|_{(y_1, 0)}. \quad (45)$$

In order to get a simpler boundary condition, let us introduce the function

$$v = e^{\frac{1}{2} y_2 \gamma(y_1)} w. \quad (46)$$

Denoting

$$a = e^{-\frac{1}{2} y_2 \gamma(y_1)},$$

we have

$$w = av, \quad (47)$$

$$\Delta^2 w = \Delta^2 (av) = a \Delta^2 v + 4 \nabla a \cdot \nabla (\Delta v) + p_2(v), \quad (48)$$

$$\nabla (\Delta w) = a \nabla (\Delta v) + p_2(v). \quad (49)$$

Substituting (47), (48), (49) in equation (30), we obtain

$$\Delta^2 v + \left(\frac{4}{a} \nabla a + b \right) \cdot \nabla (\Delta v) + p_2(v) = 0.$$

Noticing that $v(y_1, 0) = 0$, $\partial_{y_1} v(y_1, 0) = 0$ and $a(y_1, 0) = 1$ for every $y_1 \in [-1, 1]$, we can compute for every $(y_1, y_2) \in [-1, 1] \times \{0\}$,

$$\begin{aligned} \Delta w + \gamma(y_1) \partial_{y_2} w &= \Delta(av) + \gamma(y_1) \partial_{y_2} v = 2\nabla a \cdot \nabla v + \Delta v + \gamma(y_1) \partial_{y_2} v = \\ &= 2\partial_{y_2} a \partial_{y_2} v + \Delta v + \gamma(y_1) \partial_{y_2} v = \Delta v. \end{aligned}$$

Therefore v satisfies the following problem

$$\begin{cases} \Delta^2 v + \left(\frac{4}{a} \nabla a + b \right) \cdot \nabla (\Delta v) + p_2(v) = 0, & \text{in } R, \\ v(y_1, 0) = 0, & \text{in } [-1, 1] \times \{0\}, \\ \Delta v(y_1, 0) = 0, & \text{in } [-1, 1] \times \{0\}, \end{cases}$$

with C^1 and C^0 norm on R of a and of the coefficients of p_2 respectively depending on $M_0, \alpha, \alpha_0, \gamma_0, \Lambda_0$ only.

Second step. Extension by odd reflection.

Let us introduce the following extension of v to B_1 :

$$\bar{v}(x, y) = \begin{cases} v(x, y), & \text{in } B_1^+, \\ -v(x, -y), & \text{in } B_1^-. \end{cases}$$

Moreover, let

$$f = - \left(\frac{1}{a} (4\nabla a + ab) \cdot \nabla (\Delta v) + p_2(v) \right), \quad f \in L^2(B_1^+),$$

and let us define

$$\bar{f}(x, y) = \begin{cases} f(x, y), & \text{in } B_1^+, \\ -f(x, -y), & \text{in } B_1^-. \end{cases}$$

Then $\bar{f} \in L^2(B_1)$, $\bar{v} \in H^4(B_1)$ and

$$\Delta^2 \bar{v} = \bar{f}, \quad \text{in } B_1. \tag{50}$$

The proof of (50) can be obtained by adapting the arguments used in the proof of Proposition 4.1 in [4].

Third step. Application of Carleman's estimate and conclusion.

Next, by a density argument, we apply the Carleman estimate in Proposition 4.2 to $U = \eta(\sqrt{x^2 + y^2})\bar{v}$, where $\eta \in C_0^\infty((0, 1))$ is a suitable cut-off

function; see, for instance, the proof of Lemma 4.1 in [12]. We obtain the following result: there exists a positive number $\bar{r}_0 \in (0, 1)$, only depending on $M_0, \alpha, \alpha_0, \gamma_0, \Lambda_0$, such that, for every \bar{r} and for every r such that $0 < 2r < \bar{r} < \frac{\bar{r}_0}{2}$, we have

$$\begin{aligned} \bar{r}(2r)^{-2\tau} \int_{B_{2r}^+} |v|^2 + \bar{r}^{1-2\tau} \int_{B_{\bar{r}}^+} |v|^2 &\leq \\ &\leq C \left(\left(\frac{r}{4} \right)^{-2\tau} \int_{B_r^+} |v|^2 + \left(\frac{\bar{r}_0}{2} \right)^{-2\tau} \int_{B_{\bar{r}_0}^+} |v|^2 \right), \end{aligned}$$

for every $\tau \geq \bar{\tau}$, with $\bar{\tau}$ a positive absolute constant and C a positive constant depending on $M_0, \alpha, \alpha_0, \gamma_0, \Lambda_0$ only.

Finally, by using a standard procedure (see, for instance, the proof of Theorem 2.2 in [12]) and coming back to the original coordinates via Proposition 4.1, we obtain the desired doubling inequality (23) at the supported boundary of the plate.

Acknowledgements

The work of Edi Rosset, Eva Sincich, and Sergio Vessella was partly performed under the PRIN grant No. 201758MTR2-007.

REFERENCES

- [1] V. ADOLFSSON AND L. ESCAURIAZA, *$C^{1,\alpha}$ domains and unique continuation at the boundary*, Comm. Pure Appl. Math. **50** (1997), 935–969.
- [2] S. AGMON, *Lectures on elliptic boundary value problems*, Van Nostrand, New York, 1965.
- [3] G. ALESSANDRINI, E. BERETTA, E. ROSSET, AND S. VESSELLA, *Optimal stability for inverse elliptic boundary value problems with unknown boundaries*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), 755–806.
- [4] G. ALESSANDRINI, E. ROSSET, AND S. VESSELLA, *Optimal three spheres inequality at the boundary for the Kirchhoff-Love plate's equations with Dirichlet conditions*, Arch. Ration. Mech. Anal. **231** (2019), 1455–1486.
- [5] L. BAKRI, *Quantitative uniqueness for Schrödinger operator*, Indiana Univ. Math. J. **61** (2012), no. 4, 1565–1580.
- [6] A. BANERJEE AND N. GAROFALO, *Quantitative uniqueness for elliptic equations at the boundary of C^1 Dini domains*, J. Differential Equations **261** (2016), no. 12, 6718–6757.
- [7] M. DI CRISTO, E. SINCICH, AND S. VESSELLA, *Size estimates of unknown boundaries with Robin type condition*, Proc. Roy. Soc. Edinburgh Sect. A **47** (2017), 727–741.

- [8] I. KUKAVICA AND K. NYSTRÖM, *Unique continuation on the boundary for Dini domains*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 441–446.
- [9] A. MORASSI, E. ROSSET, AND S. VESSELLA, *Optimal identification of a cavity in the Generalized Plane Stress problem in linear elasticity*, J. Eur. Math. Soc., to appear.
- [10] A. MORASSI, E. ROSSET, AND S. VESSELLA, *Size estimates for inclusions in an elastic plate by boundary measurements*, Indiana Univ. Math. J. **56** (2007), 2325–2384.
- [11] A. MORASSI, E. ROSSET, AND S. VESSELLA, *Optimal stability in the identification of a rigid inclusion in an isotropic Kirchhoff-love plate*, SIAM J. Math. Anal. **51** (2019), no. 2, 731–747.
- [12] A. MORASSI, E. ROSSET, AND S. VESSELLA, *Doubling inequality at the boundary for the Kirchhoff-Love plate’s equation with Dirichlet conditions*, Le Matematiche **75** (2020), 27–55.
- [13] E. SINCICH, *Stability for the determination of unknown boundary and impedance with a Robin boundary condition*, SIAM J. Math. Anal. **42** (2010), no. 6, 2922–2943.
- [14] S. TIMOSHENKO AND S. WOINOWSKY-KRIEGER, *Theory of plates and shells*, McGraw-Hill, New York, 1959.

Authors’ addresses:

Antonino Morassi
 Dipartimento Politecnico di Ingegneria e Architettura
 Università degli Studi di Udine
 via Cottonificio 114, 33100 Udine, Italy
 E-mail: antonino.morassi@uniud.it

Edi Rosset
 Dipartimento di Matematica e Geoscienze
 Università degli Studi di Trieste
 via Valerio 12/1, 34127, Trieste, Italy
 E-mail: rossedi@units.it

Eva Sincich
 Dipartimento di Matematica e Geoscienze
 Università degli Studi di Trieste
 via Valerio 12/1, 34127, Trieste, Italy
 E-mail: esincich@units.it

Sergio Vessella
 Dipartimento di Matematica e Informatica “Ulisse Dini”
 Università degli Studi di Firenze,
 Viale Morgagni 67/a, 50134 Firenze, Italy
 E-mail: sergio.vessella@unifi.it

Received May 13, 2021
 Accepted May 25, 2021