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Vector-valued Sobolev multiplier spaces and their preduals

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Abstract. *We study the properties of the Sobolev Multiplier Spaces of X-valued functions and their preduals, where X is a Banach space. It is shown that a necessary and sufficient condition for the duality to be valid is that X^* possesses the Radon-Nikodym property. Weak vector-valued and the projective tensor type spaces regarding of the preduals will also be taken into account and it is shown that they differ to each other when X is infinite-dimensional.*

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1. Introduction

Let $\alpha > 0$, $s > 1$ be real numbers and X be a Banach space. We define the Sobolev Space $W^{\alpha,s} = W^{\alpha,s}(\mathbb{R}^n)$, $n \geq 1$ to be the set of functions u of the type

$$u = G_\alpha * f$$

for some $f \in L^s$. Here G_α is the Bessel kernel of order α defined by

$$G_\alpha(x) := \mathcal{F}^{-1}[(1 + |\cdot|^2)^{-\alpha/2}](x),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform in $\mathbb{R}^n$. The norm of $u = G_\alpha * f \in W^{\alpha,s}$ is defined to be $\|u\|_{W^{\alpha,s}} = \|f\|_{L^s}$. Recall also that the Bessel capacity $\mathrm{Cap}_{\alpha,s}(\cdot)$ associated to $W^{\alpha,s}$ is defined to be

$$\mathrm{Cap}_{\alpha,s}(E) = \inf\{\|f\|_{L^s}^s : f \geq 0, G_\alpha * f \geq 1 \text{ on } E\}$$

for any set $E \subseteq \mathbb{R}^n$.

In this paper we are concerned with the X-valued Sobolev multiplier space $M_p^{\alpha,s}(\mathbb{R}^n, X)$, $1 < p < \infty$, which consists of locally p-Bochner-integrable functions $f : \mathbb{R}^n \to X$ such that the trace inequality

$$\left(\int_{\mathbb{R}^n} |u(t)|^s \|f(t)\|_X^p dt\right)^{1/p} \leq C\|u\|_{W^{\alpha,s}}^{s/p}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$. For the sake of simplicity, let us write $M(X)$ for $M_p^{\alpha,s}(\mathbb{R}^n, X)$. It is proved in [6, Theorem 3.1.4] that the least possible constant C in the above inequality is equivalent to the quantity

$$\|f\|_{M(X)} := \sup_K \left(\frac{\int_K \|f(t)\|_X^p dt}{\mathrm{Cap}_{\alpha,s}(K)} \right)^{1/p},$$

where the supremum is taken for all compact sets $K \subseteq \mathbb{R}^n$ with non-zero capacity, and we will equip $M(X)$ with the norm $\|\cdot\|_{M(X)}$. The characterizations and preduals of the scalar case $M(\mathbb{C})$ have been greatly studied in [7] while this paper will deal with the vector-valued counterpart.

The predual that will be discussed later is of the block type space (the terminology is given in the last paragraph of page 32 in [2]), which we will label as $B(X) := B_q^{\alpha,s}(\mathbb{R}^n, X)$, where $1/p + 1/q = 1$. To begin with, by denoting L_X^q the space of q-Bochner-integrable functions, let us call a function $a : \mathbb{R}^n \to X$ an X-block if

1. There exists a bounded set A of X such that $a(x) = 0$ a.e. for $x \in \mathbb{R}^n \setminus A$.
2. $\|a\|_{L_X^q} \leq \mathrm{Cap}_{\alpha,s}(A)^{(1-q)/q}$.

Subsequently, the space $B(X)$ is defined to be the set of all functions $f : \mathbb{R}^n \to X$ of the form

$$f(x) = \sum_j c_j a_j(x),$$

where the convergence is in pointwise a.e. sense, taken under the norm of X, $\{c_j\}$ is a scalar sequence in ℓ^1, and each a_j is an X-block. The norm of a function $f \in B_q^{\alpha,s}(X)$ is defined as

$$\|f\|_{B_q^{\alpha,s}(X)} = \inf \left\{ \sum_j |c_j| : f = \sum_j c_j a_j \text{ a.e.} \right\}.$$

A basic fact regarding of the space $B(X)$ is the following, cf. [7, Remark 2.1]:

LEMMA 1.1. $B(X) \hookrightarrow L_X^1$ *in the sense of continuous embedding.*

To see that the space $M(X^*)$ is dual to $B(X)$, we first recall the following general definition of Radon-Nikodym property, cf. [4, Definition 3].

> A Banach space X has the Radon-Nikodym property with respect to a finite measure space (Ω, Σ, μ) if for each μ-continuous vector measure $G : \Sigma \to X$ of bounded variation there exists $g \in L^1(X, \mu)$ such that
>
> $$G(E) = \int_E g d\mu$$

for all $E \in \Sigma$. A Banach space X has the Radon-Nikodym property if X has the Radon-Nikodym property with respect to every finite measure space.

In addition, we recall a standard fact from functional analysis that

$(L^r_X)^* = L^{r'}_{X^*}$ if and only if X^* possesses the Radon-Nikodym property, where $1/r + 1/r' = 1$ and $1 \le r < \infty$, cf. [4, Theorem 1].

We also recall a result obtained in [1, Theorem 3] regarding the duality between the atomic Hardy space $H^{1,\infty}_X$ and $\mathbf{BMO}_{X^*}$ that

$(H^{1,\infty}_X)^* = \mathbf{BMO}_{X^*}$ if and only if X^* possesses the Radon-Nikodym property.

Hence, the above facts suggest that the duality in our case is also valid:

Theorem 1.2. *$M(X^*)$ is isomorphic to $B(X)^*$ if and only if X^* possesses the Radon-Nikodym property.*

The following proposition is needed for proving Theorem 1.2, where its proof can be found in the proof of [7, Theorem 1.6] by simple modification of the scalar case to the vector-valued case:

Proposition 1.3. *$M(X^*) \hookrightarrow B(X)^*$ in the sense of continuous embedding.*

Since we can always consider the weak topology for any dual space X^*, there should be a weak vector-valued counterpart of the spaces $M(X)$ and $B(X)$. We define the space $wM(X)$ to be the set of all functions $f : \mathbb{R}^n \to X$ such that $x^* \circ f \in M(\mathbb{C})$ for all $x^* \in X^*$. The norm equipped with $wM(X)$ is taken to be

$$\|f\|_{wM(X)} = \sup_{\|x^*\| \le 1} \|x^* \circ f\|_{M(\mathbb{C})}.$$

The space $wB(X)$ is then defined in the same fashion as above. Due to the standard facts of functional analysis, we shall expect that the spaces $M(X)$ and $wM(X)$, $B(X)$ and $wB(X)$ coincide exactly when X is finite-dimensional:

Proposition 1.4. *$wM(X)$ is isomorphic to $M(X)$ if and only if X is finite-dimensional.*

Proposition 1.5. *$wB(X)$ is isomorphic to $B(X)$ if and only if X is finite-dimensional.*

By Lemma 1.1 and the standard functional analysis fact that

$$L^1(\mathbb{R}^n)\widehat{\otimes}_\pi X = L^1_X,$$

where $\widehat{\otimes}_\pi$ denotes the projective tensor product (the definition can be found in the first paragraph of page 17 in [8]), one may suspect that $B(\mathbb{C})\widehat{\otimes}_\pi X$ is isomorphic to $B(X)$. First of all, the following basic claim is valid:

PROPOSITION 1.6. *$B(\mathbb{C}) \otimes X$ is dense in $B(X)$.*

Unfortunately, the general projective tensor product $B(\mathbb{C})\widehat{\otimes}_\pi X$ is not isomorphic to $B(X)$ for general infinite dimensional Hilbert space X. More generally, we have the following theorem:

THEOREM 1.7. *If X^* has the Radon-Nikodym property, then the projective tensor product $B(\mathbb{C})\widehat{\otimes}_\pi X$ is isomorphic to $B(X)$ if and only if X is finite-dimensional. In which case, $B(\mathbb{C})\widehat{\otimes}_\pi X$, $B(X)$, and $wB(X)$ are all isomorphic.*

As a result, we have

COROLLARY 1.8. *$B(\mathbb{C})\widehat{\otimes}_\pi \ell^2$ is not isomorphic to $B(\ell^2)$.*

We are ready to give the proofs of the aforementioned propositions and theorems in the next section. In what follows, for any two quantities A and B, we write $A \lesssim B$ to abbreviate $A \leq CB$ for some constant C, a similar convention is used for $A \gtrsim B$. Meanwhile, the notation $A \approx B$ will denote both $A \lesssim B$ and $A \gtrsim B$.

2. Proofs

Proof of Lemma 1.1. Let $f = \sum_j c_j a_j$ be such a representation as in the definition of $B(X)$, then

$$\|a_j\|_{L^1_X} \leq |A_j|^{1/p}\|a_j\|_{L^q_X} \leq |A_j|^{1/p}\mathrm{Cap}_{\alpha,s}(A_j)^{(1-q)/q} \lesssim 1$$

by the standard Sobolev's Embedding that $|A_j| \lesssim \mathrm{Cap}_{\alpha,s}(A_j)$. As a result, $\|f\|_{L^1_X} \leq \sum_j |c_j|\|a_j\|_{L^1_X} \lesssim \sum_j |c_j|$. $\square$

Proof of Theorem 1.2. We first prove the sufficiency. Let $L \in B(X)^*$ be given. For any fixed compact set $K \subseteq \mathbb{R}^n$ and $f \in L^q_X$ supported in K, then $\|f\|_{B(X)} \lesssim \mathrm{Cap}_{\alpha,s}(K)^{1/p}\|f\|_{L^q_X}$ and hence

$$|L(f)| \leq \|L\|\|f\|_{B(X)} \lesssim \|L\|\|f\|_{L^q_X}.$$

Since X^* has the Radon-Nikodym property, there exists a $g_K \in L^p_{X^*}$ supported in K such that

$$L(f) = \int_{\mathbb{R}^n} \langle f(t), g_K(t) \rangle \, dt.$$

Consider a compact exhaustion $\{K\}$ of $\mathbb{R}^n$ and let $g = g_K$ locally, one can argue as in the proof of [7, Theorem 1.3] that $g \in M(X^*)$ with the equivalent norm $\|g\|_{M(X^*)} \approx \|L\|$, this shows that $B(X)^* \hookrightarrow M(X^*)$.

We now prove the necessity. To show that X^* possesses the Radon-Nikodym property, it is equivalent to establish the following assertion, cf. [4, p. 107].

For every bounded measurable set K and bounded linear operator

$$T \in \mathcal{L}(L^1(K), X^*)$$

there exists a function $g \in L^\infty_{X^*}(K)$ such that

$$T(f) = \int_K f(t)g(t)dt, \qquad f \in L^1(K).$$

Let such a T be given. Define $\overline{T} : L^1_X(K) \to \mathbb{C}$ as

$$\overline{T}\left(\sum_{i=1}^N x_i \chi_{E_i}\right) = \sum_{i=1}^N \langle T(\chi_{E_i}), x_i \rangle ,$$

where $\{x_i\}_{i=1}^N \subseteq X$ and $\{E_i\}_{i=1}^N$ is a disjoint sequence of measurable subsets of K, and $N \in \mathbb{N}$. It is immediate that

$$\left|\overline{T}\left(\sum_{i=1}^N x_i \chi_{E_i}\right)\right| \leq \sum_{i=1}^N \|x_i\|_X \|T\| \cdot |E_i| = \|T\| \cdot \left\|\sum_{i=1}^N x_i \chi_{E_i}\right\|_{L^1_X(K)},$$

hence T is extended to $(L^1_X(K))^*$ by density. By Lemma 1.1, T extends to $(B(X))^*$ and is given by the integral representation of a function

$$g_K \in M(X^*)$$

by the necessity assumption. The final task is to show that $g_K \in L^\infty_{X^*}(K)$. Let $B_r(t_0)$ be an open ball in $\mathbb{R}^n$ with center t_0 and radius r. We have

$$\begin{aligned}
\left\|\int_{B_r(t_0)} g_K(t)dt\right\|_{X^*} &= \sup_{\|x\|_X \leq 1} \left|\int_{B_r(t_0)} \langle x, g_K(t) \rangle \, dt\right| \\
&= \sup_{\|x\|_X \leq 1} \left|\overline{T}(x\chi_{B_r(t_0)\cap K})\right| \\
&= \sup_{\|x\|_X \leq 1} \left|\left\langle x, T(\chi_{B_r(t_0)\cap K})\right\rangle\right| \leq \|T\| \cdot |B_r(t_0)|.
\end{aligned}$$

So, we deduce by the vector-valued Lebesgue's Differentiation Theorem that $\|g(t)\|_{X^*} \le \|T\|$ a.e. $t \in \mathbb{R}^n$, and hence $\|g_K\|_{L^\infty_{X^*}(K)} \le \|T\| < \infty$, the proof is now complete. $\square$

Proof of Proposition 1.4. The sufficiency is trivial so we prove for the necessity. Assume that X is infinite-dimensional, then by Dvoretsky's Theorem [3, Theorem 19.1], there exist for each $N \in \mathbb{N}$ an N-dimensional subspace E_N of X and a linear isomorphism $J_N : \ell^2_N \to E_N$ such that $\|J_N\| \le 2$ and $\|J_N^{-1}\| = 1$. Let $e_k^{(N)}$ be the kth standard unit vector of ℓ^2_N and set $x_k^{(N)} = J_N e_k^{(N)}$. We also let $\{S_N\}$ be a disjoint sequence of compact sets such that both $|S_N|$ and $\mathrm{Cap}_{\alpha,s}(S_N)$ are fixed non-zero constants. Define

$$f_{k,N}(t) = x_k^{(N)} \chi_{S_k}(t), \qquad t \in \mathbb{R}^n, \qquad k = 1, 2, ..., N.$$

We compute that

$$\|f_{k,N}(t)\|_X \ge \|f_{k,N}(t)\|_{E_N} \ge \left\| e_k^{(N)} \chi_{S_k}(t) \right\|_{\ell^2_N} = \chi_{S_k}(t),$$

and hence

$$\|f_{k,N}\|_{M(X)} \ge (\mathrm{Cap}_{\alpha,s}(S_k))^{-1/p} \left(\int_{S_k} \|f_{k,N}(t)\|_X^p dt \right)^{1/p} \gtrsim 1.$$

On the other hand, assuming the equivalence that $\|\cdot\|_{wM(X)} \approx \|\cdot\|_{M(X)}$, we choose an $x^* \in X^*$ such that $\|x^*\| \le 1$ and

$$\|(x^* \circ f_{k,N})\|_{M(\mathbb{C})} \gtrsim 1.$$

Let $y_N^* \in E_N^*$ be the restriction of x^* to E_N and $J_N^* y_N^* \in (\ell^2_N)^*$ with

$$\|J_N^* y_N^*\|_{(\ell^2_N)^*} \le \|J_N\| \|x^*\|_{X^*} \le 2,$$

where J_N^* is the adjoint of J_N. Choose $c_1^{(N)}, ..., c_N^{(N)}$ such that

$$(J_N^* y_N^*)(u_1, ..., u_N) = \sum_{l=1}^{N} c_l^{(N)} u_l.$$

Hence,

$$(x^* \circ f_{k,N})(t) = y_N^*(f_{k,N}(t)) = (J_N^* y_N^*)(e_k^{(N)}) \chi_{S_k}(t) = c_k^{(N)} \chi_{S_k}(t).$$

For any compact set K, by the standard Sobolev's Embedding

$$|K| \lesssim \mathrm{Cap}_{\alpha,s}(K),$$

we have

$$(\mathrm{Cap}_{\alpha,s}(K))^{-1/p}\left(\int_K |(x^*\circ f_{k,N})(t)|^p dt\right)^{1/p} \lesssim |c_k^{(N)}|,$$

we deduce that $\|(x^*\circ f_{k,N})\|_{M(\mathbb{C})} \lesssim |c_k^{(N)}|$. Keep in mind that

$$\|\{c_k^{(N)}\}\|_{\ell^2_N} = \|J_N^* y_N^*\|_{(\ell^2_N)^*},$$

so

$$|c_k^{(N)}| \le \left(\sum_{l=1}^N |c_l^{(N)}|^2\right)^{1/2} \le 2, \qquad k=1,...,N, \qquad N=1,2,...$$

We extend $c_l^{(N)}=0$ for all $l\ge N+1$. By a standard diagonalization argument, we let $c_l\ge 0$ and $N_1<N_2<\cdots$ be such that $\lim_{k\to\infty}|c_l^{(N_k)}|=c_l$ for $l=1,2,...$ As a result,

$$\left(\sum_{l=1}^\infty c_l^2\right)^{1/2} \le \liminf_{k\to\infty}\left(\sum_{l=1}^\infty |c_l^{(N_k)}|^2\right)^{1/2} < \infty,$$

the sequence $\{c_l\}$ tends to zero. Given any arbitrarily small $\epsilon>0$, choose an l such that $c_l<\epsilon$, then we can further pick an N_l such that $|c_l^{N_l}|<\epsilon$. This shows that $\|(x^*\circ f_{l,N_l})\|_{M(\mathbb{C})} \lesssim |c_l^{N_l}|$ can be controlled as arbitrarily small, which contradicts the strictly positive lower bound that $\|(x^*\circ f_{l,N_l})\|\gtrsim 1$. In sum, the norms $\|\cdot\|_{wM(X)}$ and $\|\cdot\|_{M(X)}$ are not equivalent. □

Proof of Proposition 1.5. We prove the necessity. The construction goes with Dvoretsky's Theorem and is similar to that of Proposition 1.4 which we will retain certain notation. This time we let

$$f_N(t)=\sum_{k=1}^N \frac{x_k^{(N)}}{k}\chi_{S_k}(t),$$

where $\{S_k\}$ is disjoint, bounded, both $|S_k|$ and $\mathrm{Cap}_{\alpha,s}(S_k)$ are fixed constants. We see at once that $\|f_N\|_{L^1_X}\ge\sum_{k=1}^N|S_k|\cdot k^{-1}\approx\sum_{k=1}^N k^{-1}$. By Lemma 1.4, we have $\sup_N\|f_N\|_{B(X)}=\infty$. However, since $|S_k|$ and $\mathrm{Cap}_{\alpha,s}(S_k)$ are fixed non-zero constants, we see that

$$\|x^*\circ f_N\|_{B(\mathbb{C})} \lesssim \sum_{k=1}^N \frac{|c_k|}{k} \le \|\{c_k\}\|_{\ell^2_N}\left(\sum_{k=1}^N \frac{1}{k^2}\right)^{1/2},$$

therefore $\sup_N\|f_N\|_{wB(X)}\lesssim 1$. □

Proof of Proposition 1.6. We first claim that $B(\mathbb{C}) \otimes X \hookrightarrow B(X)$. Consider the tensor

$$\sum_{k=1}^{N} \lambda_k f_k \otimes x_k,$$

$f_k \in B(\mathbb{C})$ and $x_k \in X$ and λ_k a scalar, write $f_k = \sum_{j=1} c_{j,k} a_{j,k}$ as in the definition of $B(\mathbb{C})$, then $f_k \otimes x_k = \sum_j c_{j,k}(a_{j,k} \otimes x_k)$. As

$$\|a_{j,k} \otimes x_k\|_{L^q_X} = \|a_{j,k}\|_{L^q} \|x_k\|_X,$$

it is easy to see that $\|f_k \otimes x_k\|_{B(X)} \leq \sum_j |c_{j,k}| \|x_k\|_X$ and hence

$$\|f_k \otimes x_k\|_{B(X)} \leq \|f_k\|_{B(\mathbb{C})} \|x_k\|_X.$$

We deduce that

$$\left\| \sum_{k=1}^{N} \lambda_k f_k \otimes x_k \right\|_{B(X)} \leq \sum_{k=1}^{N} |\lambda_k| \|f_k\|_{B(\mathbb{C})} \|x_k\|_X,$$

the claim now follows by the definition of projective tensor norm.

Now the density can be seen as the following way. Let $f \in B(X)$ be represented by $f = \sum_j c_j a_j$ as in the definition of $B(X)$. The finite sum $\sum_{j=1}^{N} c_j a_j$ approximates f in $B(X)$. Since one can approximate a_j by continuous functions with compact support in L^q_X, it is easy to see that the finite linear combination of tensors $f_k \otimes x_k \in B(\mathbb{C}) \otimes X$ approximates f in $B(X)$, where f_k are complex-valued continuous functions with compact support, and $x_k \in X$, cf. [5, Corollary 1.6], now the proposition follows. □

Proof of Theorem 1.7. The only difficult part is the necessity. Assume the contrary that X is infinite-dimensional. Suppose that $B(\mathbb{C}) \widehat{\otimes}_\pi X$ is isomorphic to $B(X)$, one obtains the isomorphism between $(B(\mathbb{C}) \widehat{\otimes}_\pi X)^*$ and $(B(X))^*$. On the other hand, it is standard by the theory of projective tensor products that $(B(\mathbb{C}) \widehat{\otimes}_\pi X)^*$ is isometrically isomorphic to $\mathcal{L}(X^*, (B(\mathbb{C}))^*)$, which in turn isomorphic to $\mathcal{L}(X^*, M(\mathbb{C}))$, see [8, p. 24].

We now observe that $wM(X^*) \hookrightarrow \mathcal{L}(X^*, M(\mathbb{C}))$. In fact, this is merely by definition. Indeed, for $f \in wM(X^*)$, it induces the linear map defined by

$$X^* \to M(\mathbb{C}), \quad x^* \to x^* \circ f,$$

and the norm of the induced map is exactly $\|f\|_{wM(X)}$.

Since X^* has the Radon-Nikodym property, we obtain by Theorem 1.2 that both $B(X)^*$ and $M(X^*)$ are isomorphic. Since $M(X^*) \hookrightarrow wM(X^*)$, the isomorphism between $(B(\mathbb{C}) \widehat{\otimes}_\pi X)^*$ and $(B(X))^*$ will imply the isomorphism between $M(X^*)$ and $wM(X^*)$, this contradicts Proposition 1.4 as X^* is also infinite-dimensional. □

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References

[1] O. Blasco, *On the dual space of* $H^{1,\infty}$, Colloq. Math. **55** (1988), no. 2, 253–259.
[2] O. Blasco, A. Ruiz, and L. Vega, *Non-interpolation in Morrey-Campanato and Block spaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **(4) 28** (1999), no. 1, 31–40.
[3] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, no. 43, Cambridge University Press, Cambridge, 1995.
[4] J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys Monogr., no. 15, Amer. Math. Soc., Providence, R.I., 1977.
[5] N. Dinculeanu, *Integration on locally compact spaces*, Noordhoff International Publishing, Leiden, 1974.
[6] V. G. Maz'ya and T. O. Shaposhnikova, *Theory of Sobolev multipliers with applications to differential and integral operators*, Grundlehren Math. Wiss., no. 337, Springer, Berlin, 2009.
[7] K. H. Ooi and N. C. Phuc, *Characterizations of predual spaces to a class of Sobolev multiplier type spaces*, preprint `http://arxiv.org/abs/2005.04349`.
[8] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monogr. Math., Springer, London, 2002.

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