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Fractional integral operators on Borel-Morrey spaces with $q \leq p$

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Abstract. *This paper establishes the mapping properties of the fractional integral operators from Morrey spaces to Borel-Morrey spaces for the case $q \leq p$.*

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1. Introduction

The mapping property of the fractional integral operators on Lebesgue spaces is one of the important topics in harmonic analysis, theory of function spaces and partial differential equations. It is related to the embedding properties of Sobolev spaces and homogeneous Sobolev spaces [25]. It is also a major component in fractional calculus [37]. Furthermore, it is used to establish the restriction theorem of Fourier transform [41, Chapter VIII, Section 4].

The study of the fractional integral operators (Riesz potential) was initialized by Riesz in [36]. The mapping properties of the fractional integral operators were extended to Morrey spaces by Spanne (result published in [35]) and Adams [1]. They show that the fractional integral operator I_α is bounded from M^p_r to M^q_r when $0 < p < \frac{n}{\alpha}$ and $0 \leq r < n$ satisfy $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ for the Spanne's result and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-r}$ for the Adams's result. Notice that for both results of Spanne and Adams, they require $p < q$.

The mapping properties of the fractional integral operators on Morrey spaces provide some estimates and inequalities for the studies of partial differential equation [17, 30, 34].

Recently, the mapping properties of the fractional integral operators have been extended to a number of Morrey type spaces such as the generalized Morrey spaces [9, 20, 31], the Orlicz-Morrey spaces [33, 38], the Morrey spaces with variable exponent [3, 10, 12, 14, 15, 28, 29, 27], the weak Morrey spaces [19] and local Morrey spaces [5, 4, 11, 18]. Roughly speaking, the above mentioned results are generalizations of the Spanne and the Adams results in different settings. For instance, for those results on Morrey spaces with variable exponent $M^{p(\cdot)}_u$ in [12], we find that $I_\alpha : M^{p(\cdot)}_u \to M^{q(\cdot)}_u$ is bounded when $p(\cdot)$ and $q(\cdot)$

satisfy $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n}$ and some other conditions. Therefore, we see that we also have $p(\cdot) \leq q(\cdot)$.

To obtain a complete description on the mapping properties of the fractional integral operators on Morrey spaces, it is natural to investigate the mapping properties for I_α acting from M_r^p to M_r^q when $0 < q \leq p < \frac{n}{\alpha}$.

From the study of the fractional integral operator on Lebesgue spaces, we find that $\frac{1}{p} - \frac{\alpha}{n} = \frac{1}{q}$ is a necessary condition for the boundedness of $I_\alpha : L^p \to L^q$ [40, Chapter V, Section 1.2]. Hence, the condition $p < q$ is also a necessary condition.

On the other hand, when we consider Lebesgue space defined on general Borel measure instead of Lebesgue measure, we have positive results.

The results for the mapping properties of I_α with Lebesgue spaces on Borel measure as the target function space [25, Section 8] give us direction for the study on Morrey spaces. In [25], we see that for the case $q \leq p$ on Lebesgue spaces, we have positive results when we consider I_α as a mapping from Lebesgue space to Lebesgue space on a Borel measure when this Borel measure satisfies some conditions involving capacity.

Therefore, the main theme of this paper is to extend this result to Morrey spaces. Roughly speaking, when the Borel measure satisfies the requirements for the boundedness properties of I_α on Lebesgue spaces, then I_α is also a bounded operator from the Morrey spaces to the Morrey space built on the given Borel measure. As applications of our main results, we have the Poincaré and the Sobolev inequalities on Borel-Morrey spaces.

This paper is organized as follows. Some definitions and the mapping properties for the fractional integral operators on Lebesgue spaces for the case $q \leq p$ are presented in Section 2. The boundedness properties for the fractional integral operators from Morrey spaces to Borel-Morrey spaces are established in Section 3.

2. Fractional integral operator on Lebesgue spaces

This section aims to present the mapping properties of the fractional integral operators from Lebesgue spaces L^p to Lebesgue spaces on Borel measure $L^q(\mu)$ with $q \leq p$.

For any $x \in \mathbb{R}^n$ and $r > 0$, write $B(x,r) = \{y : |y - x| < r\}$. Define $\mathcal{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$.

For any positive locally finite Borel measure ω on $\mathbb{R}^n$ and E being a ω measurable set, we write $\omega(E) = \int_E d\omega$. Let $\mathcal{M}(\omega)$ be the collection of ω measurable functions. For any $1 \leq p < \infty$, define

$$L^p(\omega) = \left\{ f \in \mathcal{M}(\omega) : \|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^p d\omega \right)^{1/p} < \infty \right\}.$$

When ω is the Lebesgue measure, for brevity, we write $\mathcal{M}(\omega) = \mathcal{M}$ and $L^p(\omega) = L^p$.

For any $0 < \alpha < n$ and Lebesgue measurable function f, the fractional integral operator (Riesz potentials) I_α is defined as

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Let $0 < \alpha < n$ and ω be a locally finite Borel measure. We define

$$I_\alpha \omega(x) = \int_{\mathbb{R}^n} \frac{d\omega(y)}{|x-y|^{n-\alpha}}.$$

As the condition that guarantees the mapping properties for the fractional integral operator involves the notion of capacity, we recall the definition of capacity from [25, Section 7.2.1].

Let $1 < p < \infty$, $\alpha > 0$ and Δ be the Laplacian. The Riesz potential space h^α_p consists of those Lebesgue measurable function g satisfying

$$\|g\|_{h^\alpha_p} = \|(-\Delta)^{\alpha/2} g\|_{L^p} < \infty.$$

For any compact set E, the capacity associated with the Riesz potential space h^α_p is defined as

$$cap_{p,\alpha}(E) = \inf \left\{ \|g\|^p_{h^\alpha_p} : g \in C^\infty_0,\, g \geq 1 \,\mathrm{on}\, E \right\}.$$

Definition 2.1. *Let $0 < \alpha < n$ and $1 \leq q < p < \infty$. For any positive locally finite Borel measure ω, we write $\omega \in C_{p,q,\alpha}$ if there is a constant $C > 0$ such that for any collection of open sets $\{E_k\}_{k\in\mathbb{Z}}$ with $\bar{E}_{k+1} \subset E_k$, $k \in \mathbb{Z}$, we have*

$$\sum_{k=-\infty}^{\infty} \left(\frac{\omega(E_k) - \omega(E_{k+1})}{cap_{p,\alpha}(E_k)^{q/p}} \right)^{\frac{p}{p-q}} < C.$$

A sufficient condition for $\omega \in C_{p,q,\alpha}$ is given in [25, Section 8.4.3]. When $q = 1$, [25, Section 8.4.4, Theorem 2] assures that $\omega \in C_{p,1,\alpha}$ if $\|T\omega\|_{L^{p'}} < \infty$ where T is either the Riesz potential or the Bessel potential.

We now present the mapping properties for the fractional integral operator on Lebesgue spaces for the case $q < p$.

Theorem 2.2. *Let $0 < \alpha < n$, $1 \leq q < p < \frac{n}{\alpha}$ and ω be a positive locally finite Borel measure on $\mathbb{R}^n$. If $\omega \in C_{p,q,\alpha}$, then there exists a constant $C > 0$ such that for any $f \in L^p$, we have*

$$\|I_\alpha f\|_{L^q(\omega)} \leq C \|f\|_{L^p}. \tag{1}$$

The above result is obtained by Maz'ja [25]. For the case $p = q$, we have the following result [2] and [26, Section 11.5, Theorem 2].

THEOREM 2.3. *Let* $0 < \alpha < n$, $1 < p < \infty$ *and* ω *be a positive locally finite Borel measure on* $\mathbb{R}^n$. *There exists a constant* $C_1 > 0$ *such that for any* $f \in L^p$, *we have*

$$\|I_\alpha f\|_{L^p(\omega)} \leq C_1 \|f\|_{L^p} \tag{2}$$

if and only if there exists constant $C_0 > 0$ *such that*

$$I_\alpha((I_\alpha \omega)^{p'})(x) \leq C_0 I_\alpha \omega(x), \quad a.e. \tag{3}$$

The reader is referred to [25, Sections 8.3 and 8.4.2] and [2] for the proofs of Theorems 2.2 and 2.3, respectively. The result presented in [25, Sections 8.3 and 8.4.2] is in term of the embedding from the Riesz potential spaces to Lebesgue spaces. In view of the definition of Riesz potential spaces, it is easy to see that it is equivalent with the boundedness of the fractional integral operator presented in (1).

The reader is referred to [13, 16, 24, 32] for the mapping properties of the fractional integral operators on rearrangement-invariant quasi-Banach function spaces, modular spaces, Herz spaces and some other function spaces appeared in analysis.

The above theorem gives us the direction and foundation for the study of the fractional integral operators on Morrey spaces with $q < p$ in the next section.

3. Main result

The main result of this paper is presented and established in this section. We begin with the definitions of Morrey spaces.

DEFINITION 3.1. *Let* $1 < p < \infty$ *and* ω *be a positive locally finite Borel measure on* $\mathbb{R}^n$. *Let* $u : \mathcal{B} \to (0, \infty)$. *The Borel-Morrey space* $M^p_u(\omega)$ *consists of* $f \in \mathcal{M}(\omega)$ *satisfying*

$$\|f\|_{M^p_u(\omega)} = \sup_{B \in \mathcal{B}} \frac{1}{u(B)} \|\chi_B f\|_{L^p(\omega)} < \infty.$$

In particular, the Morrey space M^p_u consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{M^p_u} = \sup_{B \in \mathcal{B}} \frac{1}{u(B)} \|\chi_B f\|_{L^p} < \infty.$$

When $1 \leq p \leq r < \infty$ and $u(B) = |B|^{\frac{1}{p} - \frac{1}{r}}$, M^p_u reduces to the classical Morrey space introduced by Morrey in [30]. Furthermore, the family of Morrey spaces defined in Definition 3.1 also covers the generalized Morrey spaces studied in [31].

The main result of this paper is presented in the following theorems.

THEOREM 3.2. *Let* $0 < \alpha < n$, $1 \leq q < p < \frac{n}{\alpha}$, ω *be a positive locally finite Borel measure on* $\mathbb{R}^n$ *and* $u : \mathcal{B} \to (0, \infty)$. *If* $\omega \in C_{p,q,\alpha}$ *and there is a constant* $C > 0$ *such that for any* $x \in \mathbb{R}^n$ *and* $r > 0$, u *satisfies*

$$u(B(x, 2r)) \leq C u(B(x, r)) \,, \tag{4}$$

$$\sum_{j=0}^{\infty} \left(\frac{\omega(B(x, r))}{\omega(B(x, 2^{j+1}r))} \right)^{1/q} u(B(x, 2^{j+1}r)) \leq C u(B(x, r)), \tag{5}$$

then there exists a constant $K > 0$ *such that for any* $f \in M^p_u$, *we have*

$$\|I_\alpha f\|_{M^q_{u}(\omega)} \leq K \|f\|_{M^p_u}.$$

We give an example for Theorem 3.2. Since $p < \frac{n}{\alpha}$, we have $\frac{\alpha}{n} + \frac{1}{p'} < 1$. Thus $(\frac{\alpha}{n} + \frac{1}{p'}, 1) \neq \emptyset$.

Let $\epsilon \in (\frac{\alpha}{n} + \frac{1}{p'}, 1)$, $\theta = \frac{\epsilon}{\frac{\alpha}{n} + \frac{1}{p'}}$. We have $\theta > 1$ and $\frac{\epsilon}{\theta} - \frac{1}{p'} = \frac{\alpha}{n}$. Let $f \in L^\theta(\mathbb{R}^n)$. Define $g = (Mf)^\epsilon$ where M is the Hardy-Littlewood maximal function on $\mathbb{R}^n$. In view of [8, Theorem 6.1.3], we have

$$\|I_\alpha g\|_{L^{p'}} \leq C\|g\|_{L^{\theta/\epsilon}} = C\|(Mf)^\epsilon\|_{L^{\theta/\epsilon}} = C\|Mf\|^\epsilon_{L^\theta} \leq C\|f\|^\epsilon_{L^\theta} < \infty \,,$$

where we use the boundedness of the Hardy-Littlewood maximal function on L^θ because $\theta > 1$. Let $d\omega = g dx$. According to [25, Section 8.4.4, Theorem 2], we have $\omega \in C_{p,1,\alpha}$.

In view of [8, Theorem 9.2.8], $\omega \in A_1$ where A_1 is the Muckenhoupt class of weight functions. For simplicity, we refer the reader to [8, Chapter 9] for the definition and properties of the Muckenhoupt class of weight functions.

According to [8, (9.3.3) and Theorem 9.3.3 (d)], we have a $\epsilon_0 > 0$ such that for any $r > 0$, $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$\frac{\omega(B(x, r))}{\omega(B(x, 2^j r))} \leq C_2 2^{-jn\epsilon_0} \tag{6}$$

for some $C_2 > 0$.

Let $s \in [0, 1)$ and $u(B) = \omega(B)^s$, $B \in \mathcal{B}$. By using (6), we have

$$\sum_{j=0}^{\infty} \frac{\omega(B(x, r))}{\omega(B(x, 2^{j+1}r))} \frac{u(B(x, 2^{j+1}r))}{u(B(x, r))} = \sum_{j=0}^{\infty} \left(\frac{\omega(B(x, r))}{\omega(B(x, 2^{j+1}r))} \right)^{1-s} < C$$

for some $C > 0$. Thus, (5) is fulfilled for u and ω.

In view of [8, Proposition 9.1.5 (9)], we have

$$u(B(x,2r)) = \omega(B(x,2r))^s \le C\omega(B(x,r))^s = Cu(B(x,r)).$$

Therefore, (4) is satisfied for u.

Theorem 3.2 gives the mapping properties of I_α from Morrey spaces to Borel-Morrey spaces for the case $q < p$. The subsequent result gives the corresponding result for the case $p = q$.

THEOREM 3.3. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, ω be a positive locally finite Borel measure on $\mathbb{R}^n$ and $u : \mathcal{B} \to (0,\infty)$. If ω satisfies (3) and there is a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, u satisfies*

$$u(B(x,2r)) \le Cu(B(x,r))\,, \tag{7}$$

$$\sum_{j=0}^{\infty} \left(\frac{\omega(B(x,r))}{\omega(B(x,2^{j+1}r))}\right)^{1/p} u(B(x,2^{j+1}r)) \le Cu(B(x,r)), \tag{8}$$

then there exists a constant $C_0 > 0$ such that for any $f \in M^p_u$, we have

$$\|I_\alpha f\|_{M^p_u(\omega)} \le C_0 \|f\|_{M^p_u}.$$

Since the proof for Theorem 3.3 follows from the proof of Theorem 3.2, for brevity, we just present the proof for Theorem 3.2 in the following.

Let $v \in A_\infty$ where A_∞ is the Muckenhoupt class of weight functions. For any Lebesgue measurable set E, we write $v(E) = \int_E v(x)dx$. In view of [39, Theorem 2], we find that (2) holds, if there exists a constant $C_1 > 0$ such that for any $B \in \mathcal{B}$

$$|B|^{\frac{\alpha}{n}-\frac{1}{p}} \left(\int_B v(x)dx\right)^{1/p} < C_1.$$

Notice that the conditions given in [39] are presented in terms of cubes while it is easy to see that the conditions for cubes and conditions for balls are equivalent. Moreover, [39, Theorem 2] assumes that v belongs to A^β_∞ which includes A_∞ as a special case, see [39, p.817-818].

According to Theorem 2.3, the measure vdx satisfies (3).

In particular, as $v \in A_\infty$, when $0 < s < 1/p$ and $u(B) = (v(B))^s$, $B \in \mathcal{B}$, [8, Theorem 9.3.3 (d)] guarantees that (8) is fulfilled with $d\omega = vdx$ and [8, Proposition 9.1.5 (9)] shows that (7) is also fulfilled.

Conditions that similar to (5) had been used in [12, 15, 17] for the mapping properties of the fractional integral operators on some Morrey type spaces.

We now present some applications of Theorems 3.2 and 3.3.

We start with the Poincaré inequality. Let $n > 1$. For any $D \in \mathcal{B}$, if $\int_D f(x)dx = 0$ or $\mathrm{supp} f \subset D$, then

$$|f(x)| \le CI_1(\chi_D|\nabla f|) = C\int_D \frac{|\nabla f(y)|}{|x-y|^{n-1}}dy, \quad \forall x \in D, \tag{9}$$

see [7, (4.3.4) and (4.3.5)].

The above inequality and our main results yield the Poincaré inequality on the Borel-Morrey spaces.

THEOREM 3.4 (The Poincaré inequality). *Let $n > 1$, $1 < q \le p < \infty$, ω be a positive locally finite Borel measure on $\mathbb{R}^n$ and $u : \mathcal{B} \to (0, \infty)$. If*

1. *$\omega \in C_{p,q,1}$ and u satisfies (4)-(5) when $q < p$,*
2. *ω satisfies (3) with $\alpha = 1$ and u satisfies (7)-(8) when $q = p$,*

then, for any $D \in \mathcal{B}$ and for any once continuously differentiable function f, if either $\int_D f(x)dx = 0$ or supp$f \subset D$, we have

$$\sup_{B\in\mathcal{B}} \frac{1}{u(B)}\Big(\int_{D\cap B} |f(x)|^q d\omega\Big)^{1/q} \le C \sup_{B\in\mathcal{B}} \frac{1}{u(B)}\Big(\int_B |\nabla f(x)|^p dx\Big)^{1/p}$$

for some $C > 0$.

For the studies and the use of Poincaré inequality on function spaces, the reader is referred to [7].

We also have the Sobolev inequality for the Borel-Morrey spaces. Let $n > 2$ and Δ denote the Laplacian. Since $f = I_2(\Delta f)$, Theorems 3.2 and 3.3 give the following Sobolev inequality on the Borel-Morrey spaces.

THEOREM 3.5 (The Sobolev inequality). *Let $n > 2$, $1 < q \le p < \infty$, ω be a positive locally finite Borel measure on $\mathbb{R}^n$ and $u : \mathcal{B} \to (0, \infty)$. If*

1. *$\omega \in C_{p,q,2}$ and u satisfies (4)-(5) when $q < p$,*
2. *ω satisfies (3) with $\alpha = 2$ and u satisfies (7)-(8) when $q = p$,*

then, for any $D \in \mathbb{B}$ and for any twice continuously differentiable function f with supp$f \subseteq D$, we have

$$\sup_{B\in\mathcal{B}} \frac{1}{u(B)}\Big(\int_{B\cap D} |f(x)|^q d\omega\Big)^{1/q} \le C \sup_{B\in\mathcal{B}} \frac{1}{u(B)}\Big(\int_B |\Delta f(x)|^p dx\Big)^{1/p}$$

for some $C > 0$.

The reader is referred to [17] for more results on the Poincaré and the Sobolev inequalities on Morrey spaces. We now correct two typos in [17]. The conditions [17, (4.2) and (5.3)] should be

$$\sup_{B\in\mathcal{B}} \frac{1}{|B|^{1-\frac{\alpha p}{n}}} \int_B u(y)dy\Big(\frac{1}{|B|}\int_B v^{1-p'}(y)dy\Big)^{p-1} < \infty$$

with $(v, u) \in \mathbb{F}_{p,p,\alpha}$ and $\frac{1}{|B|^{1-\frac{2}{n}}} \int_B v(y)dy < C$, respectively.

We now going to prove our main results. We need some estimates from the fractional maximal operator to prove our main result. For any $0 < \alpha < n$ and locally integrable function f, the fractional maximal operator is defined as

$$M_\alpha f(x) = \sup_{x \ni B} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)| dy \,,$$

where the supremum is taken over all $B \in \mathcal{B}$ containing x.

It is well known that for any locally integrable function, we have

$$M_\alpha(f) \leq C I_\alpha(|f|) \tag{10}$$

for some $C > 0$ independent of f and $x \in \mathbb{R}^n$. For the proof of (10), the reader is referred to [23, (3.2.3)-(3.2.4)].

LEMMA 3.6. *Let $0 < \alpha < n$, $1 \leq q \leq p < \frac{n}{\alpha}$ and ω be a positive locally finite Borel measure on $\mathbb{R}^n$. If $I_\alpha : L^p \to L^q(\omega)$ is bounded, then there is a constant $C > 0$ such that for any $B \in \mathcal{B}$*

$$\omega(B)^{\frac{1}{q}} \leq C |B|^{\frac{1}{p} - \frac{\alpha}{n}}. \tag{11}$$

Proof. For any $B \in \mathcal{B}$ and locally integrable function g, define

$$P_{\alpha,B} g(x) = \chi_B(x) \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B g(y) dy.$$

According to the definition of M_α, there is a constant $C > 0$ such that

$$|P_{\alpha,B} g(x)| \leq C M_\alpha g(x).$$

Consequently,

$$|P_{\alpha,B} g(x)| \leq C M_\alpha g(x) \leq C I_\alpha(|g|)(x) \,.$$

Since $\omega \in C_{p,q,\alpha}$, Theorem 2.2 assures that $I_\alpha : L^p \to L^q(\omega)$ is bounded. Therefore,

$$\|P_{\alpha,B} g\|_{L^q(\omega)} \leq C \|g\|_{L^p}.$$

Consequently, for any $g \in L^p$ with $\|g\|_{L^p} \leq 1$, we have

$$\omega(B)^{1/q} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \left| \int_B g(y) dy \right| = \|P_{\alpha,B} g\|_{L^q(\omega)} \leq C \|g\|_{L^p} \leq C.$$

That is,

$$\left| \int_B g(y) dy \right| \leq C \frac{|B|^{1-\frac{\alpha}{n}}}{\omega(B)^{1/q}}.$$

By taking supremum over $g \in L^p$ with $\|g\|_{L^p} \leq 1$, the definition of associate space yields

$$|B|^{1/p'} = \|\chi_B\|_{L^{p'}} \leq C \frac{|B|^{1-\frac{\alpha}{n}}}{\omega(B)^{1/q}}.$$

Hence, we have $\omega(B)^{1/q} \leq C |B|^{\frac{1}{p} - \frac{\alpha}{n}}$. $\square$

When ω is the Lebesgue measure, (11) is valid if and only if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. This gives $p < q$ which reassures that the Morrey spaces defined on the Lebesgue measure cannot be used to study the case $q \leq p$.

The preceding lemma is crucial for the proof of our main result. Moreover, it also has its own independent interest. It gives a necessary condition for the boundedness of the fractional integral operator I_α and the fractional maximal operator M_α. The reader may consult [44] for more necessary or sufficient conditions on the boundedness of the fractional maximal operators.

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Let $f \in M^p_u$. For any $y \in \mathbb{R}^n$ and $r > 0$, define $f_0 = \chi_{B(y,2r)} f$ and $f_j = \chi_{B(y,2^{j+1}r)\setminus B(y,2^j r)} f$, $j \in \mathbb{N}\setminus\{0\}$. Thus, we have $f = \sum_{j=0}^{\infty} f_j$.

Theorem 2.2 guarantees that

$$\|\chi_{B(y,r)} I_\alpha f_0\|_{L^q(\omega)} \leq C\|f_0\|_{L^p} = C\|f\chi_{B(y,2r)}\|_{L^p}. \tag{12}$$

Consequently, (4) and (12) gives

$$\begin{aligned}\frac{1}{u(y,r)}\|\chi_{B(y,r)} I_\alpha f_0\|_{L^q(\omega)} &\leq C\frac{1}{u(y,r)}\|f\chi_{B(y,2r)}\|_{L^p} \\ &\leq C\frac{1}{u(y,2r)}\|f\chi_{B(y,2r)}\|_{L^p} \leq C\|f\|_{M^p_u}.\end{aligned} \tag{13}$$

We consider $I_\alpha f_j$ for $j \geq 1$. For any $x \in B(y,r)$, we have

$$|I_\alpha f_j(x)| \leq C 2^{-j(n-\alpha)} r^{-n+\alpha} \int_{B(y,2^{j+1}r)} |f(z)|dz.$$

The Hölder inequality yields

$$\begin{aligned}\chi_{B(y,r)}(x)|I_\alpha f_j(x)| \leq& C2^{-j(n-\alpha)} r^{-n+\alpha}\chi_{B(y,r)}(x) \\ &\times \|\chi_{B(y,2^{j+1}r)} f\|_{L^p}\|\chi_{B(y,2^{j+1}r)}\|_{L^{p'}} \\ \leq& C2^{-j(n-\alpha)} r^{-n+\alpha}\chi_{B(y,r)}(x) \\ &\times \|\chi_{B(y,2^{j+1}r)} f\|_{L^p}\frac{|B(y,2^{j+1}r)|}{|B(y,2^{j+1}r)|^{\frac{1}{p}}}.\end{aligned}$$

Since $\omega \in C_{p,q,\alpha}$, Theorem 2.2 assures that $I_\alpha : L^p \to L^q(\omega)$ is bounded. Consequently, (11) guarantees that

$$\begin{aligned}&\chi_{B(y,r)}(x)|I_\alpha f_j(x)| \\ &\qquad \leq C2^{-j(n-\alpha)} r^{-n+\alpha}|B(y,2^{j+1}r)|^{1-\frac{\alpha}{n}}\frac{\|\chi_{B(y,2^{j+1}r)} f\|_{L^p}}{\omega(B(y,2^{j+1}r))^{1/q}} \\ &\qquad \leq C\frac{\|\chi_{B(y,2^{j+1}r)} f\|_{L^p}}{\omega(B(y,2^{j+1}r))^{1/q}}.\end{aligned}$$

As a result of the above inequalities, we obtain

$$\chi_{B(y,r)}(x)\sum_{j=1}^{\infty}|I_\alpha f_j(x)| \leq C\chi_{B(y,r)}(x)\sum_{j=1}^{\infty}\frac{\|\chi_{B(y,2^{j+1}r)}f\|_{L^p}}{\omega(B(y,2^{j+1}r))^{1/q}}.$$

By applying the norm $\|\cdot\|_{L^q(\omega)}$ on both sides of the above inequality, we find that

$$\begin{aligned}\left\|\chi_{B(y,r)}\sum_{j=1}^{\infty}|I_\alpha f_j|\right\|_{L^q(\omega)} &\leq C\omega(B(y,r))^{1/q}\sum_{j=1}^{\infty}\frac{\|\chi_{B(y,2^{j+1}r)}f\|_{L^p}}{\omega(B(y,2^{j+1}r))^{1/q}} \\ &\leq C\sum_{j=1}^{\infty}\left(\frac{\omega(B(y,r))}{\omega(B(y,2^{j+1}r))}\right)^{1/q}u(y,2^{j+1}r)\|f\|_{M_u^p}.\end{aligned}$$

In view of (5), we have

$$\frac{1}{u(y,r)}\left\|\chi_{B(y,r)}\sum_{j=1}^{\infty}|I_\alpha f_j|\right\|_{L^q(\omega)} \leq C\|f\|_{M_u^p}. \tag{14}$$

Consequently, (13) and (14) yield

$$\begin{aligned}&\frac{1}{u(y,r)}\|\chi_{B(y,r)}(x)I_\alpha f\|_{L^q(\omega)} \\ &\quad\leq C\left(\frac{1}{u(y,r)}\|\chi_{B(y,r)}I_\alpha f_0\|_{L^q(\omega)}+\frac{1}{u(y,r)}\left\|\chi_{B(y,r)}\sum_{j=1}^{\infty}|I_\alpha f_j|\right\|_{L^q(\omega)}\right) \\ &\quad\leq C\|f\|_{M_u^p}\end{aligned}$$

for some $C>0$ independent of $B(y,r)\in\mathcal{B}$. By taking supremum over $B(y,r)\in\mathcal{B}$, we obtain

$$\|I_\alpha f\|_{M_u^q(\omega)} \leq C\|f\|_{M_u^p}.$$

The proof for Theorem 3.3 follows from the proof of Theorem 3.2 with the modification that we use Theorem 2.3 instead of Theorem 2.2. Therefore, for brevity, we omit the details.

In view of (10), whenever ω and u satisfy the conditions in Theorems 3.2 and 3.3, we also have

$$\begin{aligned}\|M_\alpha f\|_{M_u^q(\omega)} &\leq C\|f\|_{M_u^p}, \\ \|M_\alpha f\|_{M_u^p(\omega)} &\leq C\|f\|_{M_u^p},\end{aligned}$$

respectively. $\square$

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