Rend. Istit. Mat. Univ. Trieste
Vol. 53 (2021), Art. No. 2, pp. 1–17 (electronic)
DOI: 10.13137/2464-8728/31215

Nonlocal constants of motion and first integrals in higher-order Lagrangian Dynamics

Mattia Scomparin

"Dedicated to Professor Pietro Greco, who changed my life"

Abstract. *We generalize the theory of nonlocal constants of motion to higher-order Lagrangian Dynamics. Novel first integrals are discovered, standard noetherian results are recovered. Applications include Pais-Uhlenbeck oscillator and models of modified gravity.*

Keywords: higher-order Lagrangian, nonlocal constant, first integral, Pais-Uhlenbeck oscillator, modified gravity.
MS Classification 2020: 34A05, 70H03, 70H06, 70H50.

1. Introduction

Within the framework of higher-order *Analytical Dynamics* in one independent real variable t, a higher-order Lagrangian of order $N \in \mathbb{N}$ is a scalar smooth function of the form

$$L(t, q, q^{(1)}, q^{(2)}, \dots, q^{(N)}), \tag{1}$$

with $q, q^{(1)}, \dots, q^{(N)} \in \mathbb{R}^m$ depending on time t. In accordance with conventional notation, we use $q^{(n)} := d^n q/dt^n$ for the n-th derivative of q with respect to t.

By standard arguments in *Calculus of Variations* a fixed-endpoint, stationarizing motion for the action functional $S_{a,b} := \int_a^b L\, dt$, called a "natural" motion, is a solution $t \mapsto q(t)$ to the *Euler-Lagrange* equation

$$\sum_{i=0}^{N} (-1)^i \frac{d^i}{dt^i} \frac{\partial L(t, q, \dots, q^{(N)})}{\partial q^{(i)}} = 0 \qquad \text{for all } t. \tag{2}$$

Here $\partial/\partial q^{(i)}$ is the partial derivative with respect to the vector $q^{(i)}$.

A first integral for equation (2) is a smooth function of the form

$$K(t, q, q^{(1)}, q^{(2)}, \dots), \tag{3}$$

that is constant along the natural motion for all t. The celebrated *Noether's Theorem* establishes a relation between invariance proprieties of a Lagrangian function and first integrals associated to the related natural motions [2, 6].

A paper by Gorni and Zampieri [9] proposed a rethinking of Noether's Theorem for Lagrangians of order $N = 1$. They revisited Noether's Theorem introducing the concept of *nonlocal constants* of motion, i.e. functions that are constant along natural motions but whose value at time t also depends on the past history of the motion itself. Their nonlocal constants of motion look like this:

$$Q = \frac{\partial L(t, q, q^{(1)})}{\partial q^{(1)}} \cdot \frac{\partial q_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon=0} - \int_{t_0}^{t} \frac{\partial L(s, q_\varepsilon, q_\varepsilon^{(1)})}{\partial \varepsilon}\bigg|_{\varepsilon=0} ds\,, \tag{4}$$

with q_ε perturbed motions. Throughout this work we use the notation $a \cdot b$ for the Euclidean scalar product of vectors in $\mathbb{R}^m$, and $\|a\|$ for the norm. The result (4) was applied to some neat standard classical mechanics systems [10, 11] where well-conceived nonlocal constants of motion gave new results in dynamics. In particular for: (i) the homogeneous potentials of degree -2, (ii) the mechanical systems with viscous fluid resistance, (iii) the mechanical system with hydraulic (quadratic) fluid resistance, and (iv) the conservative and dissipative Maxwell-Bloch equations of laser dynamics.

With reference to a more general higher-order framework than $N = 1$, a question which arises is if the machinery designed in [9] holds for every Lagrangian order N. The issue has not been addressed under this nonlocal perspective until now. Since higher-order Lagrangians provide a very large class of models for modified gravity theories [18], quantum-loop cosmologies [17], and string theories [13], an in-depth examination is strongly motivated. Furthermore, approaching higher-order mechanics from a new nonlocal point of view could provide new perspectives to identify novel first integrals without necessarily requiring invariance proprieties on the already difficult to investigate structure of higher-order Lagrangians.

In the present work we extend the theory of nonlocal constants of motion [9] to higher-order Lagrangian Dynamics. The main, very simple result of our work is presented in Section 2, where we deduce the revisited nonlocal Noether's Theorem 2.2 for higher-order Lagrangians (1).

Generally, higher-order nonlocal constants of motion are trivial or of no apparent practical value. However, in Section 3 we exhibit that in particular cases they can be used to obtain first integrals. In this respect, we derive first integrals by employing time-shift families (Subsection 3.1), nonlocal space-shift families (Subsection 3.2), and finite invariances (Subsection 3.3) as perturbed motions. Standard noetherian results are recovered by Theorem 3.3 and Theorem 3.7, whereas the first integral of Theorem 3.4 is actually new for Lagrangians such that $\partial L/\partial q^{(i)} \propto d^i(\partial L/\partial q)/dt^i$, for all $i = 1, \ldots, N$. This latter theorem generalizes the method employed in [9] to get energy conservation for the canonical harmonic oscillator.

In the rest of the paper, Section 4, we deal with some neat applications of our theorems to higher-order Lagrangian systems. In Subsection 4.1 we derive nonlocal constants of motion and first integrals for the Pais and Uhlenbeck oscillator [20]. Such system is well-known in many branches of physics like quantum mechanics and field theory. Theorem 3.3 and Theorem 3.4 result in two first integrals that, opportunely combined, exactly recover from a nonlocal perspective the recent results of [14]. Despite this, Theorem 3.7 leads to a new angular-momentum-like first integral that never seems obtained before.

In Subsection 4.2 we use our new nonlocal theorems to analyze a higher order generalization of the Pais-Uhlenbeck oscillator [1, 3]. The obtained first integrals seem completely new, and generalize the results of Subsection 4.1. For computational convenience, a revisited version of Theorem 3.4 is employed.

Finally, Subsection 4.3 is dedicated to apply our theorems to a simple Degenerate Higher-Order model of Scalar-Tensor (DHOST) theory that, in these last years, inspired many modified gravity theories [18]. In such case a scalar particle is coupled to n degrees of freedom. Again, the full consistency of our machinery is confirmed.

We remark that higher-order nonlocal constants of motion and related first integrals are a precious instrument also for studying models that, like the Pais and Uhlenbeck oscillator, easily exhibit a general solution without requiring a conservation law. Under this sense, as already analyzed in recent works for Lagrangians of order $N = 1$ by [11, 15, 14], we believe our results could provide a valuable tool to give a novel insight into stability proprieties of higher-order models and boundedness of related solutions. However, the issue seems not so easy to address as in the $N = 1$ case. We leave a complete analysis of these points for future investigation, we now just focus on the definition and implementation of the nonlocal theory.

2. Nonlocal constants of motion

In this section, we introduce the key concept of *perturbed motions* and outline how nonlocal constants of motion in higher-order Lagrangian Dynamics can be obtained.

Definition 2.1. *Given a natural motion $t \mapsto q(t)$, a one parameter family of perturbed motions, or simply a perturbed motion, associated to $q(t)$ will be a smooth function $(\varepsilon, t) \mapsto q_\varepsilon(t)$ with $\varepsilon \in \mathbb{R}$ in a neighbourhood of 0, and such that $q_0(t) = q(t)$.*

Among all, general perturbed motions are the *time-shift family* $q_\varepsilon(t) = q(t + \varepsilon f(t))$, and the *space-shift family* $q_\varepsilon(t) = q(t) + \varepsilon g(t)$, with f, g free smooth functions of t.

The following Theorem 2.2 gives us a simple tool that takes a perturbed motion and computes the related nonlocal constant.

THEOREM 2.2. *Let $t \mapsto q(t)$ be a natural motion associated to the Lagrangian* (1)*, and $q_\varepsilon(t)$ a perturbed motion associated to $q(t)$. Then the function Q is constant along $q(t)$ for all t, with*

$$Q := \sum_{i=1}^{N}\sum_{k=0}^{i-1}(-1)^k \frac{d^k}{dt^k}\frac{\partial L(t,q,\dots,q^{(N)})}{\partial q^{(i)}}\cdot \frac{\partial q_\varepsilon^{(i-k-1)}}{\partial \varepsilon}\bigg|_{\varepsilon=0} - \\ - \int_{t_0}^{t} \frac{\partial L(s,q_\varepsilon,\dots,q_\varepsilon^{(N)})}{\partial \varepsilon}\bigg|_{\varepsilon=0} ds\,. \quad (5)$$

Proof. Define $L_\varepsilon := L(t,q_\varepsilon,\dots,q_\varepsilon^{(N)})$ with $L_0 = L$. Take the derivative of L_ε with respect to ε at $\varepsilon = 0$ and use $\partial q_\varepsilon^{(i)}/\partial\varepsilon|_{\varepsilon=0} = d^i(\partial q_\varepsilon/\partial\varepsilon)/dt^i|_{\varepsilon=0}$:

$$\frac{\partial L_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \sum_{i=0}^{N}\frac{\partial L}{\partial q^{(i)}}\cdot\frac{d^i}{dt^i}\frac{\partial q_\varepsilon}{\partial\varepsilon}\bigg|_{\varepsilon=0}. \quad (6)$$

Note that given two vectors $A(t)\in\mathbb{R}^m$ and $B(t)\in\mathbb{R}^m$, the following identity holds

$$A\cdot\frac{d^iB}{dt^i} = \frac{d}{dt}\sum_{k=0}^{i-1}(-1)^k\frac{d^kA}{dt^k}\cdot\frac{d^\alpha B}{dt^\alpha} + (-1)^i\frac{d^iA}{dt^i}\cdot B\,, \quad (7)$$

with $\alpha = i-k-1$ and $i \geq 1$. The choices $A := \partial L/\partial q^{(i)}$ and $B := \partial q_\varepsilon/\partial\varepsilon|_{\varepsilon=0}$ lead to

$$\frac{\partial L_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \frac{d}{dt}\sum_{i=1}^{N}\sum_{k=0}^{i-1}(-1)^k\frac{d^k}{dt^k}\frac{\partial L}{\partial q^{(i)}}\cdot\frac{\partial q_\varepsilon^{(\alpha)}}{\partial\varepsilon}\bigg|_{\varepsilon=0} + \\ + \left[\sum_{i=0}^{N}(-1)^i\frac{d^i}{dt^i}\frac{\partial L}{\partial q^{(i)}}\right]\cdot\frac{\partial q_\varepsilon}{\partial\varepsilon}\bigg|_{\varepsilon=0}. \quad (8)$$

Using equation (2), the second term on the right-hand-side of (8) disappears. The final result (5) is obtained by integrating, putting the integration constant to zero. □

Following the nomenclature recommended by Gorni and Zampieri [9], expression (5) will be called *nonlocal constant of motion* associated to q_ε, as its value at t also depends on the past history of the motion. It should be noticed that when $N = 1$, expression (5) exactly recovers the result (4) for canonical Lagrangians.

In Theorem 2.2, the one parameter family of perturbed motions can be chosen randomly, giving generally trivial or of no apparent practical interest nonlocal constants of motion. However, there are cases when expression (5) becomes useful, as we will see in Section 3.

3. First Integrals

The guiding idea of the following section is to describe particular cases for which Theorem 2.2 yields a true first integral for equation (2), in the sense of a point function of $t, q, q^{(1)}, \dots$ that is constant along the motions. This purpose justifies the following terminology:

DEFINITION 3.1. *Let q_ε be a perturbed motion associated to a natural motion $t \mapsto q(t)$. We say that q_ε satisfies the total derivative condition with the smooth scalar function $\psi(t, q, \dots, q^{(2N-1)})$ if*

$$\left.\frac{\partial L(t, q_\varepsilon, \dots, q_\varepsilon^{(N)})}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{d\psi(t, q, \dots, q^{(2N-1)})}{dt} \quad \text{for all } t. \tag{9}$$

Definition 3.1 is less general than the one introduced by [9], as our focus are natural motions only. In principle one could define a total derivative condition considering a general smooth path $q(t)$, whether it solves Euler-Lagrange equations (2) or not.

The condition (9) becomes interesting when applied to Theorem 2.2.

THEOREM 3.2. *Let $t \mapsto q(t)$ be a natural motion, and let q_ε be a perturbed motion associated to $q(t)$ satisfying the total derivative condition* (9) *for some $\psi(t, q, \dots, q^{(2N-1)})$. Then the point function $K(t, q, \dots, q^{(2N-1)})$ is a first integral for equation* (2)*, with*

$$K(t, q, \dots, q^{(2N-1)}) := \sum_{i=1}^{N} \sum_{k=0}^{i-1} (-1)^k \frac{d^k}{dt^k} \frac{\partial L(t, q, \dots, q^{(N)})}{\partial q^{(i)}} \cdot \left.\frac{\partial q_\varepsilon^{(i-k-1)}}{\partial \varepsilon}\right|_{\varepsilon=0} - \\ - \psi(t, q, \dots, q^{(2N-1)})\,. \tag{10}$$

Proof. Add condition (9) inside expression (5) and compute the integral. □

By inspection, Theorem 3.2 works without necessarily requiring a general invariance theory on the Lagrangian. After having sought a $\psi(t, q, \dots, q^{(2N-1)})$ satisfying the total derivative condition, we are naturally led by Theorem 3.2 to consider (10) as a first integral. Generally we cannot expect to find such a $\psi(t, q, \dots, q^{(2N-1)})$ for a random choice of q_ε. However there are few and precious Lagrangians that make the research easier.

3.1. Time-shift families

As already noted in [10, 11], time-independent Lagrangians are a classical and simple prototype to find first integrals.

THEOREM 3.3. *Let $t \mapsto q(t)$ be a natural motion for the time-independent Lagrangian $L = L(q, \dots, q^{(N)})$. Then, the point function*

$$K_1(q, \dots, q^{(2N-1)}) = \\ = \sum_{i=1}^{N} \sum_{k=0}^{i-1} (-1)^k \frac{d^k}{dt^k} \frac{\partial L(q, \dots, q^{(N)})}{\partial q^{(i)}} \cdot q^{(i-k)} - L(q, \dots, q^{(N)}) \,, \quad (11)$$

is a first integral for the equation of motion (2).

Proof. Expression (5) can be written as:

$$Q = \sum_{i=1}^{N} \sum_{k=0}^{i-1} (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial q^{(i)}} \cdot \frac{\partial q_\varepsilon^{(i-k-1)}}{\partial \varepsilon}\bigg|_{\varepsilon=0} - \int_{t_0}^{t} \sum_{i=0}^{N} \frac{\partial L}{\partial q^{(i)}} \cdot \frac{\partial q_\varepsilon^{(i)}}{\partial \varepsilon}\bigg|_{\varepsilon=0} ds \,. \quad (12)$$

Consider the time-shift perturbed motion $q_\varepsilon(t) = q(t+\varepsilon)$. It follows that $\partial q_\varepsilon^{(j)}/\partial\varepsilon|_{\varepsilon=0} = q^{(j+1)}$ for all $j \in 0, \dots, N$, that combined in expression (12) yields

$$Q = \sum_{i=1}^{N} \sum_{k=0}^{i-1} (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial q^{(i)}} \cdot q^{(i-k)} - \int_{t_0}^{t} \frac{dL}{ds} \, ds \,. \quad (13)$$

Observe that our time-shift q_ε satisfies the total derivative condition with the scalar function $\psi = L$. Computing the integral in expression (13) the final result (11) is obtained, putting the integration constant to zero. □

Interestingly, expression (11) with $N = 2$ recovers from the nonlocal point of view the noetherian result employed by [19] for second-order variational problems. This fact should not surprise, since Theorem 3.3 assumes a time-shift symmetric Lagrangians.

3.2. Nonlocal space-shift families

Nonlocal noetherian constants of motion starting from nonlocal transformations were introduced by [12]. We noticed that in [9] the authors proved energy conservation for the 1-dimensional canonical harmonic oscillator $L = \frac{1}{2}{q^{(1)}}^2 - \frac{1}{2}q^2$ taking inspiration from the nonlocal space-change $q_\varepsilon = q + \varepsilon \int q \, d\tau$ and then replacing it with a local one using the equations of motion. Inspired by this example, we deduce that generally the total derivative condition seems to be too easy to satisfy starting from nonlocal space-changes if $\partial L/\partial q^{(i)} \propto d^i(\partial L/\partial q)/dt^i$ for all $i \in 1, \dots, N$.

THEOREM 3.4. *Consider a Lagrangian $L = L(t, q, \dots, q^{(N)})$ such that there exists a set of constant parameters $\rho_1 \dots \rho_N \in \mathbb{R}$ such that for all motions,* *whether natural or not,*

$$\frac{\partial L}{\partial q^{(i)}} = \rho_i \frac{d^i}{dt^i}\frac{\partial L}{\partial q} \qquad \text{for all } i \in 1,\dots,N\,. \tag{14}$$

Let $t \mapsto q(t)$ *be a natural motion associated to* L*, and define the function*

$$F^{(0)} := \sum_{j=1}^{N} (-1)^{j+1} \frac{d^{j-1}}{dt^{j-1}} \frac{\partial L}{\partial q^{(j)}}\,. \tag{15}$$

Then,

$$F^{(\ell)} = \frac{d^{\ell-1}}{dt^{\ell-1}}\frac{\partial L}{\partial q} \qquad \ell \in 1,\dots,2N\,, \tag{16}$$

and the point function

$$K_2(t,q,\dots,q^{(3N-1)}) = \\ = \sum_{i=1}^{N} \rho_i \left[\sum_{k=0}^{i-1} (-1)^k F^{(i+k+1)} \cdot F^{(i-k-1)} - \frac{1}{2}\|F^{(i)}\|^2 \right] - \frac{1}{2}\|F^{(0)}\|^2, \tag{17}$$

is a first integral for the equation of motion (2)*.*

Proof. Expression (5) can be written as:

$$Q = \sum_{i=1}^{N}\sum_{k=0}^{i-1} (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial q^{(i)}} \cdot \left.\frac{\partial q_\varepsilon^{(i-k-1)}}{\partial \varepsilon}\right|_{\varepsilon=0} - \int_{t_0}^{t} \sum_{i=0}^{N} \frac{\partial L}{\partial q^{(i)}} \cdot \left.\frac{\partial q_\varepsilon^{(i)}}{\partial \varepsilon}\right|_{\varepsilon=0} ds\,. \tag{18}$$

Consider the space-shift perturbed motion $q_\varepsilon(t) = q(t) + \varepsilon F(t)$, with F a free function. It follows that $\partial q_\varepsilon^{(j)}/\partial\varepsilon|_{\varepsilon=0} = d^j F/dt^j := F^{(j)}$ for all $j \in 0,\dots,N$, that combined in expression (18) yields

$$Q = \sum_{i=1}^{N}\sum_{k=0}^{i-1} (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial q^{(i)}} \cdot F^{(i-k-1)} - \int_{t_0}^{t} \sum_{i=0}^{N} \frac{\partial L}{\partial q^{(i)}} \cdot F^{(i)}\, ds\,. \tag{19}$$

Fixing $F = F^{(0)} := \int_{t_0}^{t} \partial L/\partial q\, d\tau$ and using condition (14), expression (19) becomes

$$Q = \sum_{i=1}^{N} \rho_i \sum_{k=0}^{i-1} (-1)^k F^{(k+i+1)} \cdot F^{(i-k-1)} - \\ - \int_{t_0}^{t} \left[F^{(1)} \cdot F^{(0)} + \sum_{i=1}^{N} \rho_i\, F^{(i+1)} \cdot F^{(i)} \right] ds. \tag{20}$$

Clearly, the integrand inside expression (20) can be rewritten:

$$Q = \sum_{i=1}^{N} \rho_i \sum_{k=0}^{i-1} (-1)^k F^{(k+i+1)} \cdot F^{(i-k-1)} - \\ - \int_{t_0}^{t} \frac{d}{ds} \left[\frac{1}{2} \|F^{(0)}\|^2 + \sum_{i=1}^{N} \frac{\rho_i}{2} \|F^{(i)}\|^2 \right] ds \,. \quad (21)$$

Observe that our space-shift q_ε satisfies the total derivative condition with the scalar function $\psi = \frac{1}{2}(\|F^{(0)}\|^2 + \sum_{i=1}^{N} \rho_i \|F^{(i)}\|^2)$. Computing the integral in expression (21) the final result (17) is obtained, putting the integration constant to zero. To be precise, the result is not a true first integral since $F^{(0)} = \int_{t_0}^{t} \partial L/\partial q \, d\tau$ is nonlocal. To make $F^{(0)}$ a point function, isolate $\partial L/\partial q$ from the equation of motion (2)

$$\frac{\partial L}{\partial q} = \sum_{j=1}^{N} (-1)^{j+1} \frac{d^j}{dt^j} \frac{\partial L}{\partial q^{(j)}} \,, \quad (22)$$

and replace expression (22) in $F^{(0)}$. Integrating, we obtain definition (15). □

From formula (16), notice that the highest order derivative $F^{(2N)}$ formally depends on $q^{(3N-1)}$, hence also K_2 in formula (17) depends on $q^{(3N-1)}$.

We leave it to the reader to verify the energy conservation of the 1-dimensional canonical harmonic oscillator is a trivial consequence of Theorem 3.4, with $q \in \mathbb{R}$ and $N = 1$. Our result (17) is actually new and seems to provide a powerful perspective to get undiscovered first-integrals, as we show in Subsection 4.1 and Subsection 4.2.

Remark 3.5: Conditions (14) can be read as functional Partial Differential Equations which restrict the form of the Lagrangians to which apply Theorem 3.4. In this respect, a relevant class of solutions is $L = \sum_{n=0}^{N} \rho_n \{a \|q^{(n)}\|^2 + b^{(n)}(t) \cdot q^{(n)}\} + c(t)$, with $\rho_0 = 1$, $a \in \mathbb{R}$ constant, and $b \in \mathbb{R}^m, c \in \mathbb{R}$ free functions of t. The related first integral is expression (17) with $F^{(0)} = \sum_{n=1}^{N} (-1)^{n+1} \rho_n X^{(2n-1)}$ and $F^{(\ell)} = X^{(\ell-1)}$, where $X := 2aq + b(t)$. We leave for future investigation an analysis of the prospects and a complete search of further general solutions.

When $k = i - 1$, formula (17) is not easily manageable since summation contains many $F^{(0)}$ terms (15) that, generally, have a long expression for large values of N. Supposing $N \geq 2$, here is an equivalent reformulation of expression (17) that bypasses the presence of $F^{(0)}$ terms in the summation. Such result will be useful in Subsection 4.2.

LEMMA 3.6. *If $N \geq 2$, expression* (17) *is equivalent to*

$$2K_2(t, q, \dots, q^{(2N-1)}) = \|F^{(0)}\|^2 - \sum_{i=1}^{N} \rho_i \|F^{(i)}\|^2 + \\ + 2 \sum_{i=2}^{N} \rho_i \sum_{k=0}^{i-2} (-1)^k F^{(i+k+1)} \cdot F^{(i-k-1)} . \quad (23)$$

Proof. Let $x = i + k + 1$ and $y = i - k - 1$. Let us separate in (17) the addends inside the summation with respect to the k index, and multiply by two:

$$2K_2 = -\|F^{(0)}\|^2 - \sum_{i=1}^{N} \rho_i \|F^{(i)}\|^2 + 2 \sum_{i=1}^{N} \rho_i \sum_{k=0}^{i-1} (-1)^k F^{(x)} \cdot F^{(y)} . \quad (24)$$

Notice that, supposing $N \geq 2$, the double summation on the right-hand-side of expression (24) can be rewritten as

$$\sum_{i=1}^{N} \rho_i \sum_{k=0}^{i-1} (-1)^k F^{(x)} \cdot F^{(y)} = \\ = \sum_{i=1}^{N} \rho_i (-1)^{i+1} F^{(2i)} \cdot F^{(0)} + \sum_{i=2}^{N} \rho_i \sum_{k=0}^{i-2} (-1)^k F^{(x)} \cdot F^{(y)} , \quad (25)$$

where the $F^{(0)}$ contributes have been isolated from the summation.

By simple calculations we see that from (15) and (16) it follows

$$F^{(0)} = \sum_{i=1}^{N} \rho_i (-1)^{i+1} F^{(2i)} , \quad (26)$$

that added in (25) yields

$$\sum_{i=1}^{N} \rho_i \sum_{k=0}^{i-1} (-1)^k F^{(x)} \cdot F^{(y)} = \|F^{(0)}\|^2 + \sum_{i=2}^{N} \rho_i \sum_{k=0}^{i-2} (-1)^k F^{(x)} \cdot F^{(y)} . \quad (27)$$

Combined with result (27), expression (24) returns our final expression (23). □

3.3. Finite invariances

Another interesting situation generating first integrals arises when the Lagrangian, evaluated on a perturbed motion q_ε, has constant derivative at $\varepsilon = 0$. Following the nomenclature of Gorni and Zampieri [9], this condition will be called *finite invariance*.

THEOREM 3.7. *Let $t \mapsto q(t)$ be a natural motion associated to the Lagrangian $L = L(t, q, \ldots, q^{(N)})$, and suppose that for a given perturbed motion q_ε there exist a constant $\mu \in \mathbb{R}$ such that*

$$\left.\frac{\partial L(t, q_\varepsilon, \ldots, q_\varepsilon^{(N)})}{\partial \varepsilon}\right|_{\varepsilon=0} = \mu \,. \tag{28}$$

Then, the point function

$$K_3(t, q, \ldots, q^{(2N-1)}) = \\ = \sum_{i=1}^{N} \sum_{k=0}^{i-1} (-1)^k \frac{d^k}{dt^k} \frac{\partial L(t, q, \ldots, q^{(N)})}{\partial q^{(i)}} \cdot \left.\frac{\partial q_\varepsilon^{(i-k-1)}}{\partial \varepsilon}\right|_{\varepsilon=0} - \mu t \,, \tag{29}$$

is a first integral for the equation of motion (2).

Proof. Add condition (28) inside expression (5) and compute the integral. Observe that our perturbed motion q_ε satisfies the total derivative condition with the scalar function $\psi = \mu t$. □

Standard cases for Theorem 3.7 are Lagrangians such that $L(t, q_\varepsilon, \ldots, q_\varepsilon^{(N)})$ does not depends on ε, i.e. are invariant under the perturbed motion considered.

4. Applications

In the present section we deal with some neat applications of our theorems to higher-order Lagrangian systems, where novel first integrals are discovered, standard noetherian results are recovered.

4.1. The Pais-Uhlenbeck oscillator

The *Pais-Uhlenbeck* (P-U) *oscillator* is one of the simplest higher-order mechanical systems to test our formal results. The P-U oscillator has the following $N = 2$ order Lagrangian:

$$L = \frac{1}{2} q^{(2)2} - \frac{1}{2}(w_1^2 + w_2^2)\, q^{(1)2} + \frac{1}{2} w_1^2 w_2^2 q^2 \,, \tag{30}$$

where $q \in \mathbb{R}^m$, and $w_1, w_2 \in \mathbb{R}$ are positive constants. The Euclidean scalar product will remain implicit in the entire section, hence $q^{(i)} q^{(j)} := q^{(i)} \cdot q^{(j)}$ with $i, j = 0, 1, 2$.

The equation of motion (2) contains terms up to the fourth time derivative:

$$q^{(4)} + (w_1^2 + w_2^2) q^{(2)} + w_1^2 w_2^2 q = 0 \,. \tag{31}$$

This oscillator was proposed by Pais and Uhlenbeck [20] for solving the ultraviolet behavior of field theories involving higher derivatives. The P-U oscillator has been applied in many branches of physics, e.g. its quantum behavior was discussed by Simon [13], Smilga [22], Chen et al. [5]. In addition, Pulgar et al. [21] introduced some scalar-tensor field cosmologies inspired by the P-U oscillator. In recent time, the P-U oscillator has also attracted much attention within the context of dynamical realizations of nonrelativistic conformal groups [8].

Let us try some random perturbed motions for Theorem 2.2. The first family is $q_\varepsilon = q - \varepsilon$. We can compute $\partial q_\varepsilon / \partial \varepsilon|_{\varepsilon=0} = -1$, so the related nonlocal constant of motion (5) is

$$Q = q^{(3)} + (w_1^2 + w_2^2) q^{(1)} + w_1^2 w_2^2 \int_{t_0}^{t} q \, ds \,. \tag{32}$$

Clearly, q can not be a total derivative of some function ψ being the same for all smooth paths. However, since q is a natural motion, equation (31) leads to

$$Q = q^{(3)} + (w_1^2 + w_2^2) q^{(1)} - \int_{t_0}^{t} \frac{d}{ds} \Big[q^{(3)} + (w_1^2 + w_2^2) q^{(1)} \Big] \, ds \,. \tag{33}$$

that returns a first integral which is trivially 0.

Let us search for a second nonlocal constant of motion starting from the space-change family $q_\varepsilon = q - \varepsilon q^{(2)}$, that gives $\partial q_\varepsilon / \partial \varepsilon|_{\varepsilon=0} = -q^{(2)}$. Using Theorem 2.2 we can compute the nonlocal constant of motion

$$Q = (w_1^2 + w_2^2) q^{(2)} q^{(1)} + \int_{t_0}^{t} \Big[w_1^2 w_2^2 q^{(2)} q - (w_1^2 + w_2^2) q^{(3)} q^{(1)} + q^{(4)} q^{(2)} \Big] \, ds \,. \tag{34}$$

where, even in this case, the integrand never seems satisfying the total derivative condition. However, multiplying equation (31) by $q^{(2)}$ we get

$$q^{(4)} q^{(2)} + w_1^2 w_2^2 q^{(2)} q = -(w_1^2 + w_2^2) \, q^{(2)2} \,, \tag{35}$$

that simplifies expression (34) to

$$Q = (w_1^2 + w_2^2) \left[q^{(2)} q^{(1)} - \int_{t_0}^{t} \frac{d(q^{(2)} q^{(1)})}{ds} \, ds \right] \,. \tag{36}$$

The resulting first integral is again trivially 0.

What matters is that Theorem 2.2 is not powerful enough to yield non trivial first integrals for the P-U oscillator. However, we come to Theorem 3.3, which is the most natural to consider as our Lagrangian (30) is time-independent. After a couple of calculations, we finally obtain our first non trivial first integral:

$$2K_1 = q^{(2)2} - (w_1^2 + w_2^2) \, q^{(1)2} - 2 q^{(3)} q^{(1)} - w_1^2 w_2^2 q^2 \,. \tag{37}$$

It is also easy to prove that our Lagrangian (30) also satisfies the hypothesis of validity of the Theorem 3.4 with

$$\rho_1 = -\frac{w_1^2 + w_2^2}{w_1^2 w_2^2} \qquad \rho_2 = \frac{1}{w_1^2 w_2^2}\,. \tag{38}$$

Hence, from formulas (15) and (16) we get

$$F^{(\ell)} = \begin{cases} -(w_1^2 + w_2^2) q^{(1)} - q^{(3)} & \text{if} \quad \ell = 0 \\ w_1^2 w_2^2 q^{(n-1)} & \text{if} \quad \ell \neq 0 \end{cases} \qquad \ell = 0, 1, 2\,, \tag{39}$$

which added with (38) in (17) yields our second non trivial first integral

$$2K_2 = \big[w_1^4 + w_1^2 w_2^2 + w_2^4\big] q^{(1)2} + q^{(3)2} + 2 w_1^2 w_2^2\, q q^{(2)} + \\ + (w_1^2 + w_2^2)\big[2 q^{(3)} q^{(1)} + w_1^2 w_2^2 q^2\big], \tag{40}$$

Our result (40), opportunely combined with (37), exactly recovers from a non-local perspective the two first integrals recently proposed by [14]:

$$\Sigma_k = (q^{(3)} + \omega_k^2 q^{(1)})^2 + (\omega_1^2 + \omega_2^2 - \omega_k^2)(q^{(2)} + \omega_k^2 q)^2 \qquad k = 1, 2\,, \tag{41}$$

that, again, are constant along the natural motions.

Let $q = (q_1, q_2) \in \mathbb{R}^2$ and define the rotation family

$$q_\varepsilon = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix} q\,. \tag{42}$$

It is clear that, when evaluated on the rotation family (42), the P-U Lagrangian does not depend on ε. Noticing that $q^{(i)} \cdot q_\varepsilon^{(j)}/\partial\varepsilon|_{\varepsilon=0} = \det\big(q^{(j)}, q^{(i)}\big)$, our Theorem 3.7 gives a new angular-momentum-like first integral for P-U natural motions:

$$K_3 = (w_1^2 + w_2^2)\det\big(q^{(1)}, q\big) + \det\big(q^{(1)}, q^{(2)}\big) + \det\big(q^{(3)}, q\big)\,. \tag{43}$$

Generalizations in more than two spatial dimension is straightforward and will be left for future works, new first integrals are naturally expected.

We leave it to the reader to prove that the time derivatives of $K_{1,2,3}$ are identically zero. These first integrals are all independent, and expressed as linear combinations of quadratic functions of $q^{(i)}$ or their components.

4.2. The higher-order Pais-Uhlenbeck oscillator

The research of higher-derivative theories of quantum gravity motivates the definition of Lagrangians with an arbitrary number of higher-order derivative terms [4, 7]. In this respect, physically viable quantum theories are inspired by the so-called *higher-order Pais - Uhlenbeck* (P-U) *oscillator*, which has the following Lagrangian [1, 3]:

$$L = \frac{1}{2} \sum_{i=0}^{N} a_i q^{(i)2} \,, \tag{44}$$

where $q \in \mathbb{R}^m$ with $q^{(i)2} := q^{(i)} \cdot q^{(i)}$, and $a_0, \ldots a_N \in \mathbb{R}$ are constants. In the $N = 2$ limit, the specific choices $a_0 = \omega_1^2 \omega_2^2$, $a_1 = -(\omega_1^2 + \omega_2^2)$ and $a_2 = 1$ lead to the standard P-U oscillator (30) of Subsection 4.1. This makes Lagrangian (44) a well thought-out model to exhibit how our machinery works for a Lagrangian with an arbitrary order N.

The equation of motion (2) for (44) reads

$$\sum_{i=0}^{N} (-1)^i a_i q^{(2i)} = 0 \,, \tag{45}$$

and contains up to $2N$-order derivative terms. Our contribution in this subsection is to deduce three first integrals for equation (45).

For such purpose, let us search for a first result beginning with Theorem 3.3, since Lagrangian (44) does not explicitly depend on time. Using standard calculation rules to evaluate time derivatives, expression (11) leads to the first integral

$$2K_1 = \sum_{i=1}^{N} a_i \left(2 \sum_{k=0}^{i-1} (-1)^k q^{(i+k)} \cdot q^{(i-k)} - q^{(i)2} \right) - a_0 q^2 \,. \tag{46}$$

Notice that the conserved quantity (46) contains up to $2N - 1$ order derivative terms.

As in the previous section, also the higher-order version of the P-U oscillator satisfies the hypothesis of validity (14) of our Theorem 3.4 with

$$\rho_i = \frac{a_i}{a_0} \qquad i = 1, \ldots, N \,. \tag{47}$$

According to formulas (15) and (16), we get

$$F^{(\ell)} = \begin{cases} \displaystyle\sum_{j=1}^{N} (-1)^{j+1} a_j q^{(2j-1)} & \text{if } \ \ell = 0 \\ a_0 q^{(\ell-1)} & \text{if } \ \ell \neq 0 \end{cases} \qquad \ell = 0, \ldots, N \ , \tag{48}$$

that added into expression (23) enables us to calculate the actually new first integral

$$2K_2 = \left(\sum_{i=1}^{N}(-1)^{i+1}a_i q^{(2i-1)}\right)^2 - a_0 \sum_{i=1}^{N} a_i q^{(i-1)2} + \\ + 2a_0 \sum_{i=2}^{N} a_i \sum_{k=0}^{i-2}(-1)^k q^{(k+i)} \cdot q^{(i-k-2)} , \quad (49)$$

that, again, contains up to $2N-1$ order derivative terms. Assuming $N \geq 2$, we used expression (23) instead of expression (17) to effectively get (49). In fact, in expression (23) the $F^{(0)}$ terms are already isolated from the $F^{(i\neq 0)}$ ones. Thanks to this, we substituted expression (48) without expanding the summations of (17) to recognize $F^{(0)}$ and $F^{(i\neq 0)}$ terms one by one.

Let $q = (q_1, q_2) \in \mathbb{R}^2$ and define the rotation family

$$q_\varepsilon = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix} q \,. \quad (50)$$

It is evident that, when evaluated on the rotation family (50), the higher-order P-U Lagrangian (44) does not depend on ε. Remembering as in the previous section that $q^{(i)} \cdot q_\varepsilon^{(j)}/\partial\varepsilon|_{\varepsilon=0} = \det\left(q^{(j)}, q^{(i)}\right)$, our Theorem 3.7 gives a new angular-momentum-like first integral for all higher order P-U motions:

$$K_3 = \sum_{i=1}^{N} a_i \sum_{k=0}^{i-1}(-1)^k \det\left(q^{(i-k-1)}, q^{(i+k)}\right) . \quad (51)$$

Also in the case of the higher-order P-U oscillator, the three first integrals $K_{1,2,3}$ are all independent and expressed as linear combinations of quadratic functions of $q^{(i)}$ or their components.

4.3. Degenerate higher-order theories

Higher-Order Scalar-Tensor (HOST) theories of modified gravity have attracted significant attention in the last years to explain the present cosmic acceleration and exploring alternative theories of gravitation [16]. Such theories generically suffer from ghost instabilities. This fate can be avoided by considering *degenerate* Lagrangians, whose kinetic matrix cannot be inverted [23]. In this regard, an interesting Degenerate HOST (DHOST) model was studied in [18], where the authors described a point particle $\phi = \phi(t)$ with higher derivatives, coupled to $i = 1, \ldots, n$ degrees of freedom $q_i = q_i(t)$. Using Einstein's convention on dummy indices, the Lagrangian looks as follows:

$$L = \frac{1}{2}a\phi^{(2)2} + \frac{1}{2}k_0\phi^{(1)2} + \frac{1}{2}k_{ij}q_i^{(1)}q_j^{(1)} + b_i\phi^{(2)}q_i^{(1)} + c_i\phi^{(1)}q_i^{(1)} - V(\phi, q) \,, \quad (52)$$

with degenerate condition

$$a - b_i b_j k_{ij}^{-1} = 0\,. \tag{53}$$

Here a, b_i, c_i, k_0, and $k_{ij} = k_{ji}$ have constant values.

We retain Lagrangian (52) a valid case to exhibit how our results work for many-particle systems. The equations of motion for (52) with respect to ϕ and q_i read respectively

$$a\phi^{(4)} - k_0\phi^{(2)} + b_i q_i^{(3)} - c_i q_i^{(2)} - V_\phi = 0\,, \tag{54}$$

$$k_{ij} q_j^{(2)} + b_i\phi^{(3)} + c_i\phi^{(2)} + V_i = 0 \qquad i = 1, \ldots, n\,, \tag{55}$$

where $V_i := \partial V/\partial q_i$ and $V_\phi := \partial V/\partial \phi$.

It is worthwhile to note that our Lagrangian is explicitly time-independent, so Theorem 3.3 yields the first integral

$$\begin{aligned} 2K_1 = k_0\phi^{(1)2} + a\left(a\phi^{(2)2} - 2\phi^{(3)}\phi^{(1)}\right) + k_{ij} q_i^{(1)} q_j^{(1)} + \\ + 2b_i\left(\phi^{(2)} q_i^{(1)} - \phi^{(1)} q_i^{(2)}\right) + 2c_i\phi^{(1)} q_i^{(1)} + 2V(\phi, q)\,. \end{aligned} \tag{56}$$

By inspection we see that if $V(\phi, q) = 0$, then our Lagrangian depends on derivatives only, i.e. it is invariant under the space-changes $q_{i,\varepsilon} = q_i + \varepsilon\vartheta_i$ and $\phi_\varepsilon = \phi + \varepsilon\psi$. Here ϑ_i and ψ are constants, with $i = 1, \ldots, n$. Being simply $\partial q_{i,\varepsilon}/\partial\varepsilon|_{\varepsilon=0} = \vartheta_i$ and $\phi_\varepsilon/\partial\varepsilon|_{\varepsilon=0} = \psi$, the first integral of Theorem 3.7 takes the following form

$$K_3 = \left(k_{ij} q_j^{(1)} + b_i\phi^{(2)} + c_i\phi^{(1)}\right)\vartheta_i - \left(a\phi^{(3)} - k_0\phi^{(1)} + b_i q_i^{(2)} - c_i q_i^{(1)}\right)\psi\,. \tag{57}$$

This result should not surprise, since taking $V(\phi, q) = 0$ both equations (54) and (55) become total derivatives. Interestingly, in turn, also K_3 is a total derivative, which yields the new first integral

$$\Xi = \left(k_{ij} q_j + b_i\phi^{(1)} + c_i\phi\right)\vartheta_i - \left(a\phi^{(2)} - k_0\phi + b_i q_i^{(1)} - c_i q_i\right)\psi - K_3 t\,. \tag{58}$$

that contains up to second derivative terms in ϕ.

Acknowledgments

The author would like to thank the anonymous referee for the careful reading of the manuscript and the constructive comments and perspectives about Theorem 3.4 and Remark 3.5.

References

[1] K. Andrzejewski, *Hamiltonian formalisms and symmetries of the Pais–Uhlenbeck oscillator*, Nuclear Phys. B **889** (2014), 333–350.

[2] M. Borneas, *Principle of action with higher derivatives*, Phys. Rev. **186** (1969), 1299–1303.

[3] N. Boulanger, F. Buisseret, F. Dierick, and O. White, *Higher-derivative harmonic oscillators: stability of classical dynamics and adiabatic invariants*, Eur. Phys. J. C. **79** (2019), 60–68.

[4] A. R. R. Castellanos, F. Sobreira, I. L. Shapiro, and A. A. Starobinsky, *On higher derivative corrections to the $r + r^2$ inflationary model*, J. Cosmol. Astropart. Phys. **2018 (12)** (2018), 7–26.

[5] T. Chen, M. Fasiello, E. A. Lim, and Tolley A. J., *Higher derivative theories with constraints: exorcising Ostrogradski's ghost*, J. Cosmol. Astropart. Phys. **2013 (02)** (2013), 42–59.

[6] G.C. Constantelos, *Integrals of motion for Lagrangians including higher-order derivatives*, Il Nuovo Cimento B **21** (1974), 279–288.

[7] S. Ferrara, A. Kehagias, and D. Lust, *Aspects of Weyl supergravity*, J. High Energy Phys. **08** (2018), 197–220.

[8] A. Galajinsky and I. Masterov, *Dynamical realizations of l-conformal Newton-Hooke group*, Phys. Lett. B **723** (2013), 190–195.

[9] G. Gorni and G. Zampieri, *Revisiting Noether's theorem on constants of motion*, J. Nonlinear Math. Phys. **21 (1)** (2014), 43–73.

[10] G. Gorni and G. Zampieri, *Nonlocal variational constants of motion in dissipative dynamics*, Differential Integral Equations **30 (7/8)** (2017), 631–640.

[11] G. Gorni and G. Zampieri, *Lagrangian dynamics by nonlocal constants of motion*, Discrete Contin. Dyn. Syst. Ser. S **13 (10)** (2020), 2751–2759.

[12] K. S. Govinder and P. G. L. Leach, *Noetherian integrals via nonlocal transformation*, Phys. Lett. A **201 (2)** (1995), 91–94.

[13] Z. S. Jonathan, *Higher derivative Lagrangians, nonlocality, problems and solutions*, Phys. Rev. D **41** (1990), 3720–3733.

[14] D. S. Kaparulin, *Conservation laws and stability of field theories of derived type*, Symmetry **11 (5)** (2019), 642–660.

[15] D. S. Kaparulin and S. L. Lyakhovich, *Energy and stability of the Pais-Uhlenbeck oscillator geometric methods in physics*, Geom. Methods in Phys. **10** (2015), 127–134.

[16] K. Koyama, *Cosmological tests of modified gravity*, Rep. Progr. Phys. **79 (4)** (2016), 69–151.

[17] D. Langlois, H. Liu, K. Noui, and E. Wilson-Ewing, *Effective loop quantum cosmology as a higher-derivative scalar-tensor theory*, Classical Quantum Gravity **34 (22)** (2017), 1361–6382.

[18] D. Langlois and K. Noui, *Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability*, J. Cosmol. Astropart. Phys. **2016 (02)** (2016), 34–56.

[19] J. D. Logan and J. S. Blakeslee, *An invariance theory for second-order variational problems*, J. Math. Phys. **16** (1975), 1374–1379.

[20] A. Pais and G. E. Uhlenbeck, *On field theories with non-localized action*, Phys. Rev. **79** (1950), 145–165.

[21] G. Pulgar, J. Saavedra, G. Leon, and Y. Leyva, *Higher order Lagrangians inspired by the Pais-Uhlenbeck oscillator and their cosmological applications*, J.

Cosmol. Astropart. Phys. **2015 (5)** (2015), 46–76.

[22] A. Smilga, *Comments on the dynamics of the Pais-Uhlenbeck oscillator*, Symmetry, Integrability and Geometry: Methods and Applications **5** (2009), 17–30.

[23] R. P. Woodard, *Ostrogradsky's theorem on Hamiltonian instability*, Scholarpedia **10 (8)** (2015), 322–343.

Author's address:

Mattia Scomparin
Mogliano Veneto 31021, Italy
E-mail: mattia.scompa@gmail.com

Received October 06, 2020
Revised December 18, 2020
Accepted December 23, 2020