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The roundness-measure of a natural number

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Abstract. *The paper is devoted to the definition and investigation of arithmetic functions which measure the roundness of a natural number. Two main results are proved. Furthermore, some results of algebraic and computational character are provided.*

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1. Introduction

The purpose of the present paper is the definition and the investigation of two arithmetic functions which measure the roundness of a positive integer and allow the establishment of an ordering in the set of natural numbers according to their roundness.

A round number is usually thought of as one having a large number of divisors compared with its size. We recall that the function $d(n)$ is defined as the number of divisors of n. The function $\omega(n)$ (number of different prime factors of n) gives an index of its roundness (see [1, p.476 and foll.]).

However, as for every positive integer n there are infinitely many positive integers N such that $d(N) = 2^n$ and $\omega(N) = n$, for a given pair of them it is not clear which one should be considered the roundest. This means that the estimate of roundness given by ω is too coarse and needs a refinement. The functions d^* and d^{**}, which we are going to define, are closely related to the class of superior highly composite numbers, introduced by S. Ramanujan in [4]. In Proposition 2.2 we show that a number n is superior highly composite if and only if, for every $m < n$, $d^*(m) > d^*(n)$ or, equivalently, $d^{**}(m) < d^{**}(n)$. As Ramanujan's numbers are the incarnation of roundness, our characterization may suggest that round numbers have low d^* and high d^{**}. Furthermore, it leads to the investigation, for $r > 0$, of the set D_r^+ of all integers n such that $d^*(n) \geq r$. We prove in Theorem 2.4 that for every value (arbitrarily high) of r almost every integer (in the sense of natural density) belongs to D_r^+.

The relationship between d^* and ω is examined in Theorem 2.6. It is shown there that, for sufficiently small r, all numbers with d^* less than r have ω greater than any prefixed value: sufficiently round numbers are divided by all primes up to a prefixed P.

In Section 3, among some other miscellaneous results, we prove a theorem concerning the quotient of two consecutive superior highly composite numbers. Some matters related to the order of magnitude and the transcendency character of d^* are also discussed.

2. Preliminaries and main results

In the above quoted paper, S. Ramanujan gave the following definition: a positive integer N is said to be superior highly composite (s.h.c.) if there is a positive real number ϵ such that

$$\frac{d(N)}{N^\epsilon} \geq \frac{d(N^{'})}{N'^\epsilon} \text{ for all } N^{'} < N$$

and

$$\frac{d(N)}{N^\epsilon} > \frac{d(N^{'})}{N'^\epsilon} \text{ for all } N^{'} > N,$$

where d denotes the number of divisors. For fixed ϵ, $0 < \epsilon \leq 1$ there is a unique s.h.c. number associated with it:

$$N_\epsilon = 2^{[(2^\epsilon-1)^{-1}]} \cdot 3^{[(3^\epsilon-1)^{-1}]} \cdot p_1^{[(p_1^\epsilon-1)^{-1}]},$$

where p_1 is the greatest prime not exceeding $2^{\frac{1}{\epsilon}}$ and square brackets designate the integral part. N_ϵ turns out to be the greatest of the integers which maximize the function $\frac{d(n)}{n^\epsilon}$.

The first s.h.c. numbers are: $2, 6, 12, 60, 120$. A longer list (about 50) is included in [4]. In the section devoted to miscellaneous results we give an effective criterium which can be applied to establish whether a given number is s.h.c.

We make use of the following notation: if m, n are positive integers, $m > n$, we pose

$$\langle m, n\rangle = (\log \frac{d(m)}{d(n)}) \cdot (\log \frac{m}{n})^{-1}.$$

Let N be s.h.c., $N > 2, N_1 < N < N_2$, where N_1, N_2 are respectively the s.h.c. numbers which precede or follow N (the predecessor and the successor).

By inspection of the functions $F_{p,\epsilon}(x)$, for p prime, $\epsilon > 0$, x real: $F_{p,\epsilon}(x) = \frac{x+1}{p^{\epsilon x}}$, which are related to the map $\frac{d(n)}{n^\epsilon}$ for x integer, it can be shown that the set of all ϵ such that $N = N_\epsilon$ is the interval $\bar{\epsilon}_N < \epsilon \leq \epsilon_N$, where $\bar{\epsilon}_N = \langle N_2, N\rangle$ and $\epsilon_N = \langle N, N_1\rangle$ while $\bar{\epsilon}_2 = 0,6309\ldots$, $\epsilon_2 = 1$.

PROPOSITION 2.1. *Let N be a positive integer. Then there is a unique positive real number $d^*(N)$ such that the following conclusions hold:*

1. *for every integer $n > N$ and every real number $x > d^*(N)$*

$$\frac{d(N)}{N^x} > \frac{d(n)}{n^x};$$

2. *there is an integer $\bar{n} > N$ such that*

$$\frac{d(N)}{N^{d^*(N)}} = \frac{d(\bar{n})}{\bar{n}^{d^*(N)}}.$$

Proof. Let m be an an integer such that $m > N$ and $d(m) > d(N)$. Let $\delta = \langle m, N\rangle$. Since $d(n) = o(n^{\frac{\delta}{2}})$ (see [1, p.343]), there is an n_0 such that $d(n) < n^{\frac{\delta}{2}}$ and

$$\frac{1}{n^{\frac{\delta}{2}}} < \frac{d(N)}{N^{\delta}}$$

for $n > n_0$. Therefore we have

$$\frac{d(N)}{N^{\delta}} > \frac{d(n)}{n^{\delta}},\ n > n_0 \geq m > N.$$

Define $d^*(N) = \max\langle n, N\rangle$, $N < n \leq n_0$. As

$$\frac{d(N)}{N^x} > \frac{d(n)}{n^x}$$

if and only if $x > \langle n, N\rangle$, if $x > d^*(N) \geq \delta$ and $n > N$, the condition 1) is fulfilled. Since $d^*(N) = \langle \bar{n}, N\rangle$ for an $\bar{n}$ with $N < \bar{n} \leq n_0$, the condition 2) is also satisfied. The uniqueness of $d^*(N)$ follows from this argument: if $r > d^*(N)$, the condition 2) cannot be satisfied for any $\bar{n} > N$; if $r < d^*(N)$, then 1) is not satisfied for $n = \bar{n}$ and $r < x < d^*(N)$. This completes the proof. □

Furthermore, let's pose

$$d^{**}(N) = \frac{d(N)}{N^{d^*(N)}}.$$

From the above proposition it follows that $d^*(N) = max\langle n, N\rangle$, $n > N$. This formula holds as an alternative definition for d^*; it is possible to show that n can be taken less or equal to the smallest s.h.c. number greater than N.

PROPOSITION 2.2. *For a positive integer N the following assertions are equivalent:*

1. *N is s.h.c.;*

2. *$m < N$ implies $d^*(m) > d^*(N)$;*

3. *$m < N$ implies $d^{**}(m) < d^{**}(N)$.*

Proof. We prove firstly that 1) is equivalent to 2). According to our notation, remark that, if N_1 is s.h.c. and N_2 its successor, then $d^*(N_1) = \bar{\epsilon}_{N_1} = \epsilon_{N_2}$. Therefore $d^*(2) = 0,6309\ldots$, $d^*(6) = 0,5849\ldots$ and so on in a strictly decreasing sequence. Suppose that N verifies 2). If by contradiction it is not s.h.c., there are two consecutive s.h.c. numbers N_1 and N_2 such that $N_1 < N < N_2$. By the definitions, if $\beta = \langle N_2, N\rangle$ we have $d^*(N) \geq \beta > \epsilon_{N_2} = d^*(N_1)$ against 2). Conversely, suppose that N satisfies 1) and $n < N$. Let N_1 be the greatest s.h.c. number which does not exceed n. By the previous argument, $d^*(n) \geq d^*(N_1) > d^*(N)$.

The equivalence between 1) and 3) follows immediately from the preceding one. This completes the proof. □

Highly composite numbers, also introduced in [4], are defined by a maximality condition on the number d of their divisors; the two Propositions we have just proved show that s.h.c. numbers can be characterized by means of a minimality condition on d^* and a maximality one on d^{**}. Since these numbers are "round par excellence", the terms of their characterization suggest that the rounder a number is, the smaller its d^* and the bigger its d^{**}. Therefore, the following order appears to be motivated: given two positive integers m and n, we say that m is rounder than n if one of the following conditions holds:

1. $d^*(m) < d^*(n)$;

2. $d^*(m) = d^*(n)$ and $d^{**}(m) > d^{**}(n)$;

3. $d^*(m) = d^*(n)$, $d^{**}(m) = d^{**}(n)$ and $d(m) > d(n)$.

It's clear that these three conditions are exhaustive. Therefore the roundness-measure defines a total order in the set of positive integers. If p is a prime number, N a positive integer, the p-adic valuation $v_p(N)$ is the greatest integer h such that p^h divides N.

Lemma 2.3. *Let N be a positive integer, $d^*(N) < r$, r integer. If $\bar{s}$ is a prime, $v_{\bar{s}}(N) = h$, there are positive constants $A(\bar{s}, h, r)$ and $B(\bar{s}, h, r)$ depending only on $\bar{s}, h, r$ such that the following conditions hold:*

1. *if p is the greatest prime that divides N, then $p < A(\bar{s}, h, r)$;*

2. *if s is a prime and $v_s(N) = \alpha_s$, then $\alpha_s < B(\bar{s}, h, r)$.*

Proof. Suppose firstly $\bar{s} < p$. We can write: $N = 2^{\alpha_2} \cdot \bar{s}^{\alpha_{\bar{s}}} \cdot p^{\alpha_p}$, where $\alpha_i \geq 0$ for every prime $i < p$, $\alpha_p \geq 1$, $\alpha_{\bar{s}} = h$. Let k be the integer such that $\bar{s}^{k-1} < p < \bar{s}^k$. Define n as follows: $n = N \cdot \frac{\bar{s}^k}{p}$. Then $\frac{n}{N} < \bar{s}$ and we have:

$$\frac{d(n)}{d(N)} = \frac{(h+k+1)\alpha_p}{(h+1)(\alpha_p+1)} \geq \frac{(h+k+1)}{2(h+1)}.$$

If $k \geq (2\bar{s}^r - 1)(h+1)$, then

$$\frac{d(n)}{d(N)} \geq \bar{s}^r > (\frac{n}{N})^r$$

against the hypothesis that $d^*(N) < r$. Therefore $k \leq (2\bar{s}^r - 1)h + 2\bar{s}^r - 2$ and 1) holds with $A(\bar{s}, h, r) = \bar{s}^{(2\bar{s}^r - 1)h + 2\bar{s}^r - 2}$ for $\bar{s} < p$. If $\bar{s} > p$, then $h = 0$ and $\bar{s} \geq 3$ while $h \geq 1$ if $\bar{s} = p$: in both cases the formula holds. Consequently 1) is shown. To prove 2), consider any prime s which divides N and its exponent α_s. Define the following quantities Q, k_s and m_s.

1. $Q = p_1 \cdot p_2 \ldots p_r$ where the p_i are primes, p_1 is the successor of p, p_{i+1} the successor of p_i $1 \leq i \leq r-1$. Owing to the so-called Bertrand's Postulate, $Q < (2^{\frac{r(r+1)}{2}}) \cdot p^r$;
2. k_s is defined as the integer such that $s^{k_s} < (3s+1)Q \leq s^{(k_s+1)}$;
3. m_s is the integer satisfying $(m_s - 1)Q \leq s^{k_s} < m_s Q$.

From these inequalities it follows that

$$\frac{m_s Q}{s^{k_s}} < \frac{4}{3}.$$

We are going to prove that $\alpha_s < (2^{r+1} \cdot k_s) - 1$. Suppose by contradiction that this is false; then we can write equivalently:

$$2^r(\alpha_s - k_s + 1) \geq \frac{(2^{r+1} - 1)}{2}(\alpha_s + 1).$$

Define the following number:

$$n = N \cdot \frac{m_s Q}{s^{k_s}}.$$

Then

$$\frac{d(n)}{d(N)} \geq \frac{2^r(\alpha_s - k_s + 1)}{(\alpha_s + 1)} \geq \frac{(2^{r+1} - 1)}{2}.$$

As

$$\frac{n}{N} < \frac{4}{3},$$

we have eventually

$$\frac{d(n)}{d(N)} > \frac{3(2^{r+1}-1)}{8} \cdot \frac{n}{N}.$$

As the inequality

$$(2^{x+1}-1) > 2 \cdot (\frac{4}{3})^x$$

holds for $x \geq 1$,

$$\log[(2^{r+1}-1) \cdot \frac{3}{8}] \cdot (\log \frac{4}{3})^{-1} > r-1.$$

Choose y such that

$$r-1 < y < \log[(2^{r+1}-1) \cdot \frac{3}{8}] \cdot (\log \frac{4}{3})^{-1}.$$

As

$$\frac{n}{N} < \frac{4}{3},$$

we get

$$\frac{d(n)}{d(N)} > \frac{3(2^{r+1}-1)}{8} \cdot \frac{n}{N} > (\frac{n}{N})^{y+1} > (\frac{n}{N})^r$$

against the hypothesis that $d^*(N) < r$. Therefore $\alpha_s < (2^{r+1} \cdot k_s) - 1$. From the definition of k_s given in b), the majoration of Q in a) and the above inequality, as $2 \leq s \leq p$ we obtain the required majoration of α_s in terms of $\bar{s}, h, r$; we can take

$$B = 2^{r+1}\{\frac{\log(2^{\frac{r(r+1)+4}{2}})}{\log \bar{s}} + (r+1)[(2\bar{s}^r - 1)h + 2\bar{s}^r - 2]\}\frac{\log \bar{s}}{\log 2} - 1.$$

This completes the proof of the lemma. □

Theorem 2.4. *For any $r > 0$, the set D_r^+ of all integers N such that $d^*(N) \geq r$ has natural density one.*

Proof. We can suppose that r is an integer and prove that the set D_r^- of the integers N such that $d^*(N) < r$ has natural density zero. To this purpose, if $A(n)$ is the number of integers less than n that belong to D_r^-, we have to show that

$$\lim_{n\to\infty} \frac{A(n)}{n} = 0.$$

From Lemma 2.3 it follows that for any choice of $\bar{s}$ prime and h nonnegative integer, there are at most finitely many N in D_r^- such that $\bar{s}^{h+1}$ does not divide N. In fact, none of them can be greater than $A^{B\cdot\pi(A)}$, where $\pi(A)$ is the number of primes not exceeding A. If m is any positive integer, applying the lemma to its prime factors and their exponents it follows that all but finitely many integers in D_r^- are multiple of m. As m can be chosen arbitrarily big, it follows that the density of D_r^- must be zero and that of D_r^+ is one. $\square$

As a direct consequence of this theorem we state here without proof the following Corollary which shows the preponderance of the numbers with high d^*.

Corollary 2.5. *Let S be a set of positive integers whose natural density is not zero. Then the arithmetic mean of the d^* of the first n elements of S tends to infinity with n.*

Theorem 2.6. *For any prime P there is an $r_P > 0$ such that every n in $D_{r_P}^-$ is divided by all primes less or equal P. Thus $\omega(n) \geq \pi(P)$ if $d^*(n) < r_P$.*

Proof. Apply Lemma 2.3 for $\bar{s} = P$, $h = 0$, $r = 1$. Then there is $M_P > 0$ such that every $n \geq M_P$ with $d^*(n) < 1$ is divided by all primes $p \leq P$. Let N_1 and N_2 be consecutive s.h.c. numbers, with $N_1 < M_P \leq N_2$, $d^*(N_1) = \bar{\epsilon}_1$. We have shown in Proposition 2.2 that $d^*(n) < \bar{\epsilon}_1$ implies $n \geq N_2 \geq M_p$. As $\bar{\epsilon}_1 < 1$, the theorem is proved taking $r_P = \bar{\epsilon}_1$. $\square$

3. Miscellaneous topics

In the paper [4], at p.392, it is claimed that the quotient of two consecutive s.h.c. numbers is a prime number. This assertion is supported by a faulty argument, unless the Four Exponential Conjecture, which we are going to quote together with the Six Exponential Theorem, is assumed.

Six Exponential Theorem (Lang [2]). *Let β_1, β_2 be a couple of complex numbers, linearly independent over Q, and z_1, z_2, z_3 a triple, likewise complex and linearly independent over Q. Then at least one of the six numbers $e^{\beta_i z_j}$, $i \leq 2$, $j \leq 3$, is transcendental.*

It has been conjectured that the triple can be reduced to a couple (Four Exponential Conjecture). We prove a (weaker) result which makes use of the Six Exponential Theorem.

Theorem 3.1. *The quotient of two consecutive s.h.c. numbers is either a prime or the product of two different primes.*

Proof. Let $N = 2^{\alpha_2} \cdots p^{\alpha_p}$ be a s.h.c. number. For well known properties (see [4]) of these numbers every prime less or equal p divides N, i.e. $\alpha_q \geq 1$ for every $q \leq p$. Let P be the prime successor of p; we pose $\alpha_P = 0$. Moreover we know that $N = N_{\epsilon'} = 2^{[(2^{\epsilon'}-1)^{-1}]} \cdot p^{[(p^{\epsilon'}-1)^{-1}]}$, with ϵ' arbitrarily chosen in the interval $]\bar{\epsilon}_N, \epsilon_N]$. For q prime, $q \leq P$, define

$$\epsilon_q = \frac{\log \frac{\alpha_q+2}{\alpha_q+1}}{\log q}.$$

As $\alpha_q = [(q^{\epsilon'} - 1)^{-1}]$ we have for every ϵ such that $\epsilon_q < \epsilon \leq \epsilon'$ the two equations: $[(q^{\epsilon} - 1)^{-1}] = \alpha_q$, $[(q^{\epsilon_q} - 1)^{-1}] = \alpha_q + 1$. Let $\bar{\epsilon} = \max \epsilon_q$. We have two cases:

1. $\bar{\epsilon} = \epsilon_{\bar{q}}$ for a unique $\bar{q} \leq P$. As $\epsilon_q < \bar{\epsilon}$ for $q \neq \bar{q}$, we have $[(q^{\bar{\epsilon}} - 1)^{-1}] = \alpha_q$ for $q \neq \bar{q}$, $[(\bar{q}^{\bar{\epsilon}} - 1)^{-1}] = \alpha_{\bar{q}} + 1$. Therefore

$$N_{\bar{\epsilon}} = \prod_{q \leq P} q^{[(q^{\bar{\epsilon}}-1)^{-1}]} = (\prod_{q \neq \bar{q}} q^{\alpha_q}) \cdot \bar{q}^{\alpha_{\bar{q}}+1} = N \cdot \bar{q} = \bar{N}.$$

   To prove that $\bar{N}$ is the s.h.c. successor of N it is enough to show that the intervals $]\bar{\epsilon}_{\bar{N}}, \epsilon_{\bar{N}}]$ and $]\bar{\epsilon}_N, \epsilon_N]$ are contiguous. From the above we have, for every ϵ such that $\bar{\epsilon} < \epsilon \leq \epsilon'$, the equality $N = N_\epsilon$ while $\bar{N} = N_{\bar{\epsilon}}$. Thus $\bar{\epsilon} = \epsilon_{\bar{N}} = \bar{\epsilon}_N = d^*(N)$ and we have proved that N and $\bar{N}$ are consecutive.

2. $\bar{\epsilon} = \epsilon_{q_1} = \epsilon_{q_2}$ for two distinct primes. Arguing like in 1), as $\epsilon_q < \bar{\epsilon}$ for $q \neq q_1, q_2$, we have that $[(q^{\bar{\epsilon}} - 1)^{-1}] = \alpha_q$ while $[(q_1^{\bar{\epsilon}} - 1)^{-1}] = \alpha_{q_1} + 1$, $[(q_2^{\bar{\epsilon}} - 1)^{-1}] = \alpha_{q_2} + 1$. Therefore

$$\bar{N} = N_{\bar{\epsilon}} = (\prod_{q \neq q_1, q_2} q^{\alpha_q}) \cdot q_1^{\alpha_{q_1}+1} \cdot q_2^{\alpha_{q_2}+1} = N \cdot q_1 \cdot q_2.$$

The proof that $N, \bar{N}$ are consecutive s.h.c. is identical with that at point 1). In this case their quotient is the product of two distinct primes. We prove that there are no more cases. Firstly we show that, for fixed $\epsilon > 0$, the number $(p^\epsilon - 1)^{-1}$, p prime, is an integer for at most two distinct values, q_1, q_2. Suppose by contradiction that there are three different primes p, q, r, such that $(p^\epsilon - 1)^{-1} = l$, $(q^\epsilon - 1)^{-1} = m$, $(r^\epsilon - 1)^{-1} = n$, with m, l, n integers. It is easy to check that ϵ is irrational. Apply the Six Exponential Theorem with $\beta_1 = 1$, $\beta_2 = \epsilon$, $z_1 = \log p$, $z_2 = \log q$, $z_3 = \log r$. The conditions of linear independence over Q are clearly fulfilled; therefore at least one of the numbers $e^{\beta_i z_j}$ should be transcendental, whereas they are either integers or rational numbers. Thus our claim is proved. Therefore the chain of equalities $\bar{\epsilon} = \epsilon_{q_1} = \cdots = \epsilon_{q_n}$ holds for $n \leq 2$ and there are no more cases to be handled. This completes the proof. □

Remark 3.2: In case 2) there are four numbers (instead of two) which maximize the map $\frac{d(n)}{n^{\bar{\epsilon}}}$, namely N, $\bar{N}$, $N_1 = N \cdot q_1$, $N_2 = N \cdot q_2$. As $N < N_1 < \bar{N}$,

$N < N_2 < \bar{N}$, N_1 and N_2 are not s.h.c. since N and $\bar{N}$ are consecutive. In the paper [3], Nicolas and Robin introduced a class of numbers which they called *hautement composés supérieurs.* They quote Lang's result and are aware of the possibility of the occurence of case 2). However, if this case takes place, their class of numbers turns out to be strictly wider than that of s.h.c.: in fact N_1 and N_2 are *hautement composés supérieurs* because they maximize the map $\frac{d(n)}{n^\epsilon}$.

REMARK 3.3: The reason of Ramanujan's default in absence of the assumption of the four exponential conjecture can be explained as follows: in order to generate the set of s.h.c. numbers, he considers the numbers of the type $\frac{\log p}{\log \frac{m+1}{m}}$, p prime, $m \geq 1$ integer, ordered in an ascending chain

$$x_1 = \frac{\log \pi_1}{\log r_1} < \cdots < x_n = \frac{\log \pi_n}{\log r_n} < \cdots .$$

Starting from $\pi_1 = 2$ (the first s.h.c.) the $(n+1)$-th is obtained multiplying the n-th by π_{n+1}, $n \geq 1$. At the end of Par. 36 (see p.392–3) of [4] it is claimed that "π_n is **the** prime corresponding to x_n". This is true in case 1) of Theorem 3.1; in case 2), instead, x_n does not determine π_n univocally as

$$x_n = \frac{\log q_1}{\log \frac{m_1+1}{m_1}} = \frac{\log q_2}{\log \frac{m_2+1}{m_2}} \ , \ q_1 \neq q_2$$

and the procedure breaks off. For a given positive integer n, the question whether $d^*(n)$ is algebraic arises. There are examples of integer values of d^*: for instance $d^*(18) = d^*(90) = 1$. If N is s.h.c. we can show the transcendency of $d^*(N)$ and give an assessment of its order of magnitude.

THEOREM 3.4. *The function $d^*(N)$ for N s.h.c. is asymptotic to $\frac{\log 2}{\log\log N}$ and its values are transcendental.*

Proof. We must show that, for any $\delta > 0$ and $N > N_\delta$, N s.h.c.

$$\frac{(1-\delta)\log 2}{\log\log N} < d^*(N) < \frac{(1+\delta)\log 2}{\log\log N}.$$

Fix $\delta > 0$. There exists n_0 such that for every $N > n_0$, we have:

$$\frac{(1-\delta)\log 2}{\log\log N} < \frac{\log 2}{\log\log N + \log \frac{8}{\log 2}}.$$

Moreover, for $0 < \epsilon < \epsilon_0$,

$$2^{\left(\frac{1}{\epsilon}\right)^{\frac{\delta}{2}}} > \frac{1}{2^\epsilon - 1}.$$

Furthermore, there is n_1 such that, for every $N > n_1$, N s.h.c., $\epsilon_N < \epsilon_0$. Finally, for $N > n_2$:

$$\frac{(1+\frac{\delta}{2})\log 2}{\log\log N - \log\log 3} < \frac{(1+\delta)\log 2}{\log\log N}.$$

Set $N_\delta = \max(n_0, n_1, n_2)$. Let N be s.h.c. Then

$$N = 2^{[(2^{\epsilon'_N}-1)^{-1}]} \cdot p_N^{[(p^{\epsilon'_N}-1)^{-1}]}$$

with $\bar{\epsilon}_N < \epsilon'_N \leq \epsilon_N$, $p_N \leq 2^{\frac{1}{\epsilon'_N}} < 2p_N$ by Bertrand's Postulate. Let

$$\theta(x) = \sum_{p \leq x} \log p.$$

It is known that

$$x\frac{\log 2}{4} \leq \theta(x) \leq x \log 3$$

for $x \geq 2$. As $2 \cdots p_N \leq N$, taking the logarithms from the inequalities of the above we have:

$$\epsilon'_N \geq \frac{\log 2}{\log\log N + \log\frac{8}{\log 2}}.$$

Since $d^*(N) = \inf \epsilon'_N$, the same inequality holds for $d^*(N)$. On the other hand, since

$$2^{[(2^{\epsilon'_N}-1)^{-1}]} \cdot p_N^{[(2^{\epsilon'_N}-1)^{-1}]} > N,$$

we get

$$2^{\frac{1}{\epsilon'_N}}(2^{\epsilon'_N}-1)^{-1} > \frac{1}{\log 3}\log N.$$

For $N > N_\delta$ s.h.c. as $\epsilon'_N \leq \epsilon_N < \epsilon_0$ we obtain

$$2^{(\frac{1}{\epsilon'_N})^{1+\frac{\delta}{2}}} > \frac{1}{\log 3}\log N,$$

namely

$$d^*(N) < \epsilon'_N < \frac{(1+\frac{\delta}{2})\log 2}{\log\log N - \log\log 3} < \frac{(1+\delta)\log 2}{\log\log N}.$$

As $N_\delta \geq n_0$, we find

$$\frac{(1-\delta)\log 2}{\log\log N} < \frac{\log 2}{\log\log N + \log\frac{8}{\log 2}} \leq d^*(N).$$

For the second part, recall Gelfond-Schneider's Theorem (see [2]): if α, β are algebraic, $\alpha \neq 0, 1$ and β irrational, then α^β is transcendental. By proving Theorem 3.1 we observed that

$$d^*(N) = \bar{\epsilon}_N = \frac{\log \frac{m+1}{m}}{\log p}$$

for suitable p prime and $m \geq 1$ integer, $\bar{\epsilon}_N < 1$. It's easily seen that $\bar{\epsilon}_N$ is irrational. If by contradiction it were algebraic, setting $\alpha = p$, $\beta = \bar{\epsilon}_N$ in Gelfond-Schneider's Theorem, we would get that $p^{\bar{\epsilon}_N}$ is transcendental while $p^{\bar{\epsilon}_N} = \frac{m+1}{m}$. Therefore $\bar{\epsilon}_N$ is transcendental. $\square$

In the general case, the question about the maximum order of $d^*(n)$ as well as the problem of comparing the sizes of n and $d^*(n)$ lead to the

Theorem 3.5. *The following assertions hold:*

1. *for every* $n \geq 2$, $d^*(n) < 2n \log \frac{n+1}{2}$*;*
2. *for every real number* $M \geq 1$*, there are infinitely many integers* n *such that* $d^*(n) > M \cdot n$*.*

Proof.

1. The formula holds for $n = 2$. If $n > 2$, from calculus it follows that, for every $k > 1$:
$$\frac{\log \frac{n+k}{2}}{\log \frac{n+k}{n}} < \frac{\log \frac{n+1}{2}}{\log \frac{n+1}{n}};$$
moreover
$$\log \frac{n+1}{n} > \frac{1}{2n}.$$
As $d(n) \geq 2$, from the definition of $d^*(n)$ we have, for a suitable integer $k \geq 1$:
$$d^*(n) \leq \frac{\log \frac{d(n+k)}{2}}{\log \frac{n+k}{n}} < \frac{\log \frac{n+k}{2}}{\log \frac{n+k}{n}} \leq \frac{\log \frac{n+1}{2}}{\log \frac{n+1}{n}} < 2n \log \frac{n+1}{2}.$$
2. Take an integer $h > 2 \cdot e^M$. As 2^{h-1} and $2^{h-1} - 1$ are relatively prime, for a theorem of Dirichlet, [1, p.16], there are infinitely many primes p such that p is congruent to $2^{h-1} - 1 \pmod{2^{h-1}}$. It follows that $p + 1$ is a multiple of 2^{h-1} and therefore $d(p + 1) \geq h$. Since $d(p) = 2$ and
$$\log \frac{p+1}{p} < \frac{1}{p}, d^*(p) \geq \frac{\log \frac{d(p+1)}{2}}{\log \frac{p+1}{p}} \geq \frac{\log \frac{h}{2}}{\log \frac{p+1}{p}} \geq M \cdot p.$$

This completes the proof. □

Finally, we tackle some computational problems concerning the function d^{**}. If S is an infinite set of positive integers, n_S its smallest element, $n_0 \geq n_S$, define $S^{**}_{n_0} = \sum_{n \in S}^{n \leq n_0} d^{**}(n)$ and $S^{**} = \lim_{n_0 \to \infty} S^{**}_{n_0}$. The value of S^{**} can be considered as an index of the occurence in S of numbers with low d^*: if the series diverges, their effectiveness prevails. Denote respectively by N, O, P the sets of natural, odd and prime numbers.

Proposition 3.6. *The following assertions hold:*

1. $N^{**} = +\infty$;
2. $P^{**} = 1,7211\dots$;
3. *there is an* N_0 *such that, for* $\bar{n} > N_0$ *the following holds:* $O^{**}_{\bar{n}} < O^{**} < O^{**}_{\bar{n}} + \frac{5}{[\sqrt{\bar{n}}]}$, *where* $[\sqrt{\bar{n}}]$ *denotes the integral part of* $\sqrt{\bar{n}}$.

Proof.

1. If N_1 and N_2 are consecutive s.h.c. numbers, $N_1 < N_2$, as we know that $d^*(N_1) = \bar{\epsilon}_{N_1} = \epsilon_{N_2} = \langle N_2, N_1 \rangle$, then we get the equalities:
$$d^{**}(N_1) = \frac{d(N_1)}{N_1^{d^*(N_1)}} = \frac{d(N_1)}{N_1^{\bar{\epsilon}_{N_1}}} = \frac{d(N_2)}{N_2^{\epsilon_{N_2}}},$$
as well as the inequality
$$\frac{d(N_2)}{N_2^{\epsilon_{N_2}}} < \frac{d(N_2)}{N_2^{\bar{\epsilon}_{N_2}}} = d^{**}(N_2).$$
Therefore $d^{**}(2) < d^{**}(6) < d^{**}(12) \cdots$ and $S^{**} = +\infty$ for every S which contains the set of s.h.c. numbers.

2. With the usual technique, it's possible to check that, for every N, $d^*(N) = \max\langle n, N \rangle, N < n \leq \bar{N}$, where $\bar{N}$ is the smallest s.h.c. number greater than N. The proof is left to the reader. As 6 and 12 are s.h.c., $d^*(p)$ for $p = 2, 3, 5, 7, 11$ is easily computed. If p is prime, $p \geq 13$, since $d(p+1) \geq 4$, then $d^*(p) \geq \langle p+1, p \rangle \geq \frac{\log 2}{\log \frac{14}{13}} \geq 9,35$. Therefore
$$\sum_{p \geq 13} d^{**}(p) < 2 \sum_{n \geq 13} n^{-9,35} < (2,5) \cdot 10^{-10}.$$
Thus $P^{**} = d^{**}(2) + d^{**}(3) + d^{**}(5) + d^{**}(7) + d^{**}(11) + \epsilon, \epsilon < 10^{-9}$, namely $P** = 1,7211\dots$.

3. Following Lemma 2.3, applied for $\bar{s} = 2$, $h = 0$, $r = 2$, if $N = 3^{\alpha_3} \cdot p^{\alpha_p}$ is an odd number such that $d^*(N) < 2$, then $p \leq 61$. Moreover, according to the points 1) and 2) of Lemma 2.3, for s prime, $3 \leq s \leq 61$, set $Q = 67 \cdot 71$ and k_s equal to the integral part of $\frac{\log[(3s+1)Q]}{\log s}$. Then $\alpha_s < 8k_s - 1$. Define $N_0 = 3^{\bar{\alpha}_3} \cdots 61^{\bar{\alpha}_{61}}$, where $\bar{\alpha}_s = 8k_s - 1$. Thus, for n odd, $n > N_0$ we have $d^*(n) \geq 2$. Now fix an integer $n_0 > 1$ and consider $A_{n_0} = \sum\limits_{n \leq n_0} n^{-2}$. By comparison with the integral, $\zeta(2) < A_{n_0} + \frac{1}{n_0}$. As $\frac{1}{n_0} < A_{n_0} < 1,645$, it follows that $\zeta^2(2) < A_{n_0}^2 + \frac{5}{n_0}$. Since the series $\sum\limits_n n^{-2}$ is absolutely convergent, the equality $\zeta^2(2) = (\sum\limits_n n^{-2})^2 = \sum\limits_{h,k} (h \cdot k)^{-2}$ holds. For a positive integer n, let $k_{1,n}, \ldots k_{d(n),n}$ be its divisors. Then the terms of the series $\sum\limits_{h,k} (h \cdot k)^{-2}$ which are equal to n^{-2} are:

$$(k_{1,n} \cdot \frac{n}{k_{1,n}})^{-2}, \ldots (k_{d(n),n} \cdot \frac{n}{k_{d(n),n}})^{-2},$$

namely their number is $d(n)$. Therefore, after an arrangement of the terms, we get

$$\sum_{h,k} (h \cdot k)^{-2} = \sum_n \frac{d(n)}{n^2}$$

(see also [1, p.327] for a general treatment of Dirichlet series). Analogously $(\sum\limits_{n \leq n_0} n^{-2})^2 = \sum\limits_{n \leq n_0^2} \frac{m_n}{n^2}$, $0 \leq m_n \leq d(n)$. For every n, $1 \leq n \leq n_0^2$, m_n is the number of indices i, $1 \leq i \leq d(n)$ such that both $k_{i,n}$ and $\frac{n}{k_{i,n}}$ are less or equal n_0. Therefore $m_n = d(n)$ if and only if $n \leq n_0$. This implies the following inequalities:

$$\sum_{n \leq n_0} \frac{d(n)}{n^2} < (\sum_{n \leq n_0} n^{-2})^2 < \sum_{n \leq n_0^2} \frac{d(n)}{n^2} < \zeta^2(2).$$

Consequently,

$$\sum_{n > n_0^2} \frac{d(n)}{n^2} < \frac{5}{n_0}.$$

Since we have shown that every odd number greater than N_0 belongs to D_2^+, if $\bar{n} > N_0$, posing $n_0 = [\sqrt{\bar{n}}]$ we have eventually:

$$\sum_{n \in O}^{n > \bar{n}} d^{**}(n) < \sum_{n > \bar{n}} \frac{d(n)}{n^2} \leq \sum_{n > n_0^2} \frac{d(n)}{n^2} < \frac{5}{n_0} = \frac{5}{[\sqrt{\bar{n}}]}.$$

The assertion at point 3) follows from the fact that

$$O^{**} = O^{**}_{\bar{n}} + \sum_{n\in O}^{n>\bar{n}} d^{**}(n).$$

The proof is complete. □

COROLLARY 3.7. *The function $d^*(n)$ tends to infinity over the sequence of odd integers.*

Proof. Arguing as in part 3) of the Proposition 3.6, if Lemma 2.3 is applied for $\bar{s} = 2$, $h = 0$ and r arbitrarily big, it follows that $d^*(n) > r$ for n odd and $n > N_0(r)$. □

The following criterion can be used to detect whether a given number is s.h.c. We state it without proof.

CRITERION. *Let N be a positive integer. Define $\rho = \min\langle N, n\rangle, n < N$. Then N is s.h.c. if and only if for every prime p, $2 \leq p \leq 2^{\frac{1}{\rho}}$, the p-adic valuation of N equals $[(p^\rho - 1)^{-1}]$. Moreover, N is highly composite if and only if $\rho > 0$.*

EXAMPLES. For $N = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$, a standard computational program shows that the minimum ρ of the quantities $\langle N, n\rangle$, $n < N$ is attained for $n = 360$, $\rho = 0,3562\ldots$.
Besides, $2^{\frac{1}{\rho}} = 7$, $(2^\rho - 1)^{-1} = 3,57\ldots$, $(3^\rho - 1)^{-1} = 2,08\ldots$, $(5^\rho - 1)^{-1} = 1,29\ldots$, $(7^\rho - 1)^{-1} = 1$. Therefore $[(p^\rho - 1)^{-1}] = v_p(N)$, p prime and $2 \leq p \leq 7$. Thus N turns out to be s.h.c. .
For $N = 2^3 \cdot 3^3 \cdot 5 \cdot 7 = 7560$, the same procedure yields $\rho = \langle 7560, 5040\rangle = 0,159\ldots$. As in this case $(2^\rho - 1)^{-1} = 8,58\ldots$ whereas $v_2(N) = 3$, N is not s.h.c. .
If we take $N = 2^5 \cdot 3^2 = 288$, since $\langle 288, 240\rangle = -0,57\ldots$, the number is not even highly composite.

4. Concluding remarks

From the above Proposition it follows that the series O^{**} is convergent. Theoretically we should be able to compute its sum within any preassigned degree of accuracy by choosing $\bar{n} > N_0$ sufficiently big. However, since $N_0 = 3^{71}\cdot 5^{47}\cdot(7\cdot 11)^{39}\cdot(13\cdot 17\cdot 19\cdot 23)^{31}\cdot(29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\cdot 59\cdot 61)^{23} > (2,5)\cdot 10^{663}$, the computation of $O^{**}_{\bar{n}}$ for $\bar{n} > N_0$ is far beyond the reach of any existing machine. In this computational context a main problem is the numerical value of O^{**}, whose size is completely unknown at the present state of affairs. Another remark concerns highly composite numbers. In their definition, only the **predecessors** of a given number are involved while, on the contrary, the function

$d^*(n)$ is defined uniquely in terms of the **successors** of n. This may account for the difficulty in proving theorems which relate the two concepts. But the numerical tests that we have performed show that highly composite numbers have a d^* much lower than the average; for instance there is only one highly composite number $n < 10^{13}$ whose d^* is greater than 1 (incidentally, $n = 50400$ and $d^*(n) > 1,05$). In other words, the fact of having more divisors than all predecessors has an impact on the value of d^*, which is defined by looking uniquely at the successors.

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