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Tensor decompositions in rank $+1$

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Abstract. *We prove (without exceptions) the existence of irredundant tensor decompositions with the number of addenda equal to rank* $+1$. *We also discuss the existence of decompositions with more than the tensor rank terms, which are concise, while the original tensor is not concise.*

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1. Introduction

Take $k \geq 2$ finite dimensional vector spaces $V_1, \dots, V_k$. A $(\dim V_1) \times \cdots \times (\dim V_k)$ tensor is an element of $V_1 \otimes \cdots \otimes V_k$. An element $T \in V_1 \otimes \cdots \otimes V_k \setminus \{0\}$ is said to have tensor rank 1 if there are $v_i \in V_i$ such that $T = v_1 \otimes \cdots \otimes v_k$. The tensor rank of $T \in V_1 \otimes \cdots \otimes V_k \setminus \{0\}$ is the minimal integer r such that $T = T_1 + \cdots + T_r$ with each T_i a rank 1 tensor. Now suppose that T has rank r. In this paper we prove that there are $r + 1$ rank 1 tensors $U_1, \dots, U_{r+1}$ such that $T = U_1 + \cdots + U_{r+1}$, but T is not in the linear span of any r of the tensors $U_1, \dots, U_{r+1}$. If c is a non-zero scalar, T and cT have the same tensor rank. Thus it is natural to work with the projective space associated to $V_1 \otimes \cdots \otimes V_k$. This projective space is the projective space generated by the Segre embedding of the multiprojective space $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_k)$.

Fix $q \in \mathbf{P}^N$ and a finite subset $A \subset \mathbf{P}^N$. As in [1] and most references we say that A *irredundantly spans* q if $q \in \langle A \rangle$, where $\langle \ \ \rangle$ denote the linear span, and $q \notin \langle A' \rangle$ for any $A' \subset A$, $A' \neq A$.

Now take an integral and non-degenerate variety $X \subset \mathbf{P}^N$. For any $q \in \mathbf{P}^N$ the X-rank $r_X(q)$ of X is the minimal cardinality of a set $A \subset X$ such that $q \in \langle A \rangle$. The minimality assumption implies that A irredundantly spans q. Let $\mathcal{S}(X, q)$ be the set of all $A \subset X$ such that $\#(A) = r_X(q)$ and $q \in \langle A \rangle$. For any positive integer t let $\mathcal{S}(X, q, t)$ denote the set of all $A \subset X$ such that $\#(A) = t$ and A irredundantly spans q. Obviously $\mathcal{S}(X, q, t) = \emptyset$ for all $t \geq N + 2$ and, since X is non-degenerate, $\mathcal{S}(X, q, N + 1) \neq \emptyset$. Obviously $\mathcal{S}(X, q, t) = \emptyset$ for $t < r_X(q)$ and $\mathcal{S}(X, q, r_X(q)) = \mathcal{S}(X, q) \neq \emptyset$. For many X and q there are gaps, i.e. there are integers q and t such that $r_X(q) < t \leq N$ and $\mathcal{S}(X, q, t) = \emptyset$. This is not pathological, it is well-known that it may occur even when X is a

Veronese variety (see Remarks 2.2 and 2.3 for explicit examples). One of the main results of [1] was that "for not too special q" this is not the case when X is the Segre variety, i.e. [1, Theorem 3.8] may be restated in the following equivalent way.

THEOREM 1.1. *Let $X \subset \mathbf{P}^N$ be a Segre variety with $\dim X > 0$. Fix a linearly independent set $S \subset X$ such that $\#(S) < N$. For a general $q \in \langle S \rangle$ we have $\mathcal{S}(X, q, t) \neq \emptyset$ for all integers t such that $\#(S) < t \leq N + 1$.*

Note that for any linearly independent finite set $S \subset X$ the set S irredundantly spans a general $q \in \langle S \rangle$ (we may take any q in the complement of $\#(S)$ hyperplanes of $\langle S \rangle$). In this paper when $t = r_X(q) + 1$ we show that there is no exception. We prove the following result.

THEOREM 1.2. *Let $X \subset \mathbf{P}^N$ be a Segre variety with $\dim X > 0$. For any $q \in \mathbf{P}^N$ we have $\mathcal{S}(X, q, r_X(q) + 1) \neq \emptyset$.*

We also prove that concision fails at the level of the tensor rank $+1$, again with no exceptions on q. We prove the following result.

THEOREM 1.3. *Fix multiprojective spaces $Y \subset W$, $Y \neq W$, and let $\nu : W \to \mathbf{P}^N$ be the Segre embedding of W. Fix $q \in \langle \nu(Y) \rangle$. Then there is $B \subset W$ such that $B \cap Y \neq B$ and $\nu(B) \in \mathcal{S}(\nu(W), r_{\nu(Y)}(q) + 1)$.*

Let $X \subset \mathbf{P}^N = \langle X \rangle$ be a Segre variety. A tensor $q \in \mathbf{P}^N$ is said to be concise if there is no smaller Segre variety $X' \subset X$ such that $q \in \langle X' \rangle$. By concision ([3, Proposition 3.1.3.1]), Theorem 1.3 is false for all tensors $q \in \langle \nu(Y) \rangle$ if we look at decompositions of the tensor q with at most $r_{\nu(Y)}(q)$ terms. Note that concision holds also for Veronese embeddings ([3, Ex. 3.2.2.2]), hence the difference between Segre varieties (i.e. tensors and tensor decompositions) and Veronese varieties (i.e. additive decompositions of homogeneous polynomials) comes only for decompositions with number of components not minimal. Indeed, the result corresponding to Theorem 1.3 for Veronese varieties fails for all q (Remarks 2.2 and 2.3 and Proposition 2.4).

In Section 3 we look at the following problem.

Suppose you have a Segre variety $X \subset \mathbf{P}^N$ and a smaller Segre variety $X' \subset X$. Take a tensor $q \in \langle X' \rangle$ which is concise for X'. Take any tensor decomposition $A \in \mathcal{S}(X, q)$. By concision we have $r_X(q) = r_X(q')$ and $A \subset X'$ ([3, Proposition 3.1.3.1]). A finite set $S \subset X$ is said to be *concise for* X if there is no smaller Segre variety $X'' \subset X$ such that $S \subset X''$. Given any finite set $S \subset X$ it is easy to determine the minimal Segre variety $X'' \subseteq X$ containing S and this Segre variety $X'' \subseteq X$ is the only Segre variety $X' \subseteq X$ such that $S \subset X'$ and S is concise for X' (Remark 3.1).

OPEN PROBLEM 1.4: Let $X \subset \mathbf{P}^N$ be a Segre variety and $X' \subset X$ a smaller Segre variety. Fix $q \in \langle X' \rangle$ which is concise for X'. Compute the minimal integer t such that there is $B \in \mathcal{S}(X, q, t)$ which is concise for X.

If $X \cong X' \times \mathbf{P}^m$ for some $m > 0$, then this question is the content of Proposition 3.1.

In Section 4 we give a few remarks on the Segre-Veronese varieties obtained gluing together ideas and proofs given for the Segre varieties and the Veronese varieties.

We give the following motivation for the results and problems considered in this paper. Suppose you have a finite set $S \subset X \subset \mathbf{P}^N$ with S linearly independent. Write $\mathbf{P}^N$ as a proiectivization of a vector space V. For each $p \in S$ chose some $v_p \in V \setminus \{0\}$. Call $\mathbf{K}$ your algebraically closed base field. The vector space $W \subset V$ corresponding to the projective space $\langle S\rangle$ is the set of all $v_q = \sum_{p\in S} c_p v_p$ with $c_p \in \mathbf{K}$. The point q associated to v_q is irredundantly spanned by S if and only if $c_p \neq 0$ for all $p \in S$. Now assume that X is a Segre variety, so that v_q is a tensor and $v_q = \sum_{p\in S} c_p v_p$. Having S it is very easy, effective and cheap to find the minimal Segre $X' \subseteq X$ containing S. Is it possible to measure how far is q from being certified to be concise, i.e. to give an upper bound on the integer $\dim X - \dim X'$, for instance as a function of the integer $\#(S) - r_X(q)$? If $\#(S) > r_X(q)$ we show this in some cases (see Proposition 3.1). We also point out that for a Segre with at least 4 factors it is very time consuming to insert in a computer all entries in fixed bases. Thus tensor decompositions may be used to define the tensor in a cheap way, especially if we need many tensors associated to the same set S. It is sufficient to precompute each v_p and then for each tensor it is sufficient to give $\#(S)$ elements of the field.

2. Proofs of theorems and examples on the Veronese variety

PROPOSITION 2.1. *Let $Y = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$, $k \geq 1$, $n_i > 0$ for all i, be a multiprojective space and let $\nu : Y \to \mathbf{P}^N$ denote its Segre embedding. Set $X := \nu(Y)$. Fix $i \in \{1, \dots, k\}$ and $o_j \in \mathbf{P}^{n_j}$, $j \neq i$. Let $F \subset Y$ be the multiprojective subspace (isomorphic to $\mathbf{P}^{n_i}$) with $\mathbf{P}^{n_i}$ as its i-th factor and $\{o_j\}$ as its j-th factor. Fix $q \in \mathbf{P}^N$ and take $A \subset Y$ such that $\nu(A) \in \mathcal{S}(X, q)$.*

Then, we have $\#(A \cap F) \leq 1$.

Proof. Assume $\#(A \cap F) \geq 2$ and take $u, v \in F$ such that $u \neq v$, say $u = (u_1, \dots, u_k)$, $v = (v_1, \dots, v_k)$ with $u_j = v_j = o_j$ for all $j \neq i$ and $v_i \neq u_i$. Set $A' := A\setminus\{u, v\}$. Let $D \subseteq \mathbf{P}^{n_i}$ be the line spanned by $\{u_i, v_i\}$. Let $L \subset Y$ be the set of all $(x_1, \dots, x_k) \in Y$ such that $x_j = o_j$ for all $j \neq i$ and $x_i \in D$. The set $\nu(L) \subset \mathbf{P}^N$ is a line containing 2 points of A. Thus $\langle \nu(L\cup A')\rangle = \langle \nu(A)\rangle$. Hence $q \in \langle \nu(L \cup A')\rangle$. Since $\nu(L)$ is a line, there is $a \in L$ such that $q \in \langle \nu(A' \cup \{a\})\rangle$. Thus $r_X(q) < \#(A)$, a contradiction. □

In the last proposition we used in an essential way that $\nu(A) \in \mathcal{S}(X, q)$, not just that $\nu(A)$ irredundantly span q (see for instance the proof of Theorem 1.1).

Proof of Theorem 1.2. Set $b := r_X(q)$. Since $\dim X > 0$, we have $b \le N$ ([4, Proposition 5.1]). Fix $A \in \mathcal{S}(X,q)$ and $a \in A$. Set $A' := A \setminus \{a\}$. Write $X = \nu(Y)$ with Y a multiprojective space, say $Y = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$ with $n_i > 0$ for all i, and ν the Segre embedding of Y. Write $a = (a_1, \dots, a_k)$. Set $U := \langle \nu(A)\rangle$, $E := \mathbf{P}^{n_1} \times \{a_2\} \times \cdots \times \{a_k\}$ and $F := \nu(E)$. Note that F is a linear subspace of $\mathbf{P}^N$. By construction we have $\nu(a) \in F$. Thus $F \cap U$ is a non-empty linear subspace of U. By Proposition 2.1 we have $A \cap E = \{a\}$, i.e. $\{\nu(a)\} = F \cap \nu(A)$.

(a) Assume that F is not contained in U. Since $A \cap E = \{a\}$, the set $\langle \nu(A')\rangle$ is a linear subspace of F with codimension at least 2. Take a general line $L \subseteq \mathbf{P}^{n_1}$ containing a_1. Fix a general $(u_1, v_1) \in L \times L$. In particular $\#(\{u_1, v_1, a_1\}) = 3$. Set $u = (u_1, \dots, u_k)$ and $v = (v_1, \dots, v_k)$ with $u_i = v_i = a_i$ for $i = 2, \dots, k$. Set $B := A' \cup \{u, v\}$. Since $a_1 \in \langle \{u_1, v_1\}\rangle$ and $\nu(L)$ is a line of $\mathbf{P}^N$, we have $q \in \langle \nu(B)\rangle$. We may take u_1 and v_1 with the additional restriction that none of them is the first coordinate of a point of A'. With this restriction we have $\#(B) = b + 1$. Since $\langle \nu(A')\rangle$ is a linear subspace of F with codimension at least 2, $a \notin A'$ and L is general, we have $\dim(\langle \nu(B)\rangle \cap F) = \dim(\langle \nu(A')\rangle \cap F) + 2$. Since $\#(B) = \#(A') + 2$, $\nu(B)$ is linearly independent. Since $a_1 \in \langle \{u_1, v_1\}\rangle$ and $\nu(L)$ is a line, we have $U \subseteq \langle \nu(B)\rangle$ and in particular $q \in \langle \nu(B)\rangle$. To conclude the proof it is sufficient to prove that $\nu(B)$ irredundantly spans q. Assume that this is not true and take a minimal $K \subset B$, $K \ne B$, such that $q \in \langle \nu(K)\rangle$. Since $r_X(q) = b$ and $\#(B) = b+1$, we have $\#(K) = b$. Thus $\nu(K) \in \mathcal{S}(X,q)$. Proposition 2.1 gives that $\{u, v\}$ is not contained in K. Thus $\#(K \cap \{u, v\}) = 1$, say $K = A' \cup \{u\}$. Since $q \in U \cap \langle \nu(K)\rangle$, $K \cap A = A'$ and $q \notin \langle \nu(A')\rangle$, we have $\langle K\rangle = U$. Since $u \in K$, we get $\nu(u) \in U$, contradicting our choice of u_1.

(b) By step (a) we may assume $F \subseteq U$ for any choice of the point $a = (a_1, \dots, a_k) \in A$ and any choice of the index $i \in \{1, \dots, k\}$ (in step (a) we chose $i = 1$). Fix a general $o_1 \in E$ and write $o := (o_1, a_2, \dots, a_k)$ and $A_1 := A' \cup \{o\}$. Since $E \cap A = \{a\}$ by Proposition 2.1, we get $U = \langle A_1\rangle$. Using $i = 2$ and o instead of a we get $U = \langle \nu(A_2)\rangle$, where $A_2 = A' \cup \{w\}$ with $w = (o_1, o_2, a_3, \dots, k)$ and o_2 a general element of $\mathbf{P}^{n_2}$. In $k-2$ steps using $i = 3, \dots, k$ we get that U contains a general point of X, contradicting the inequality $b \le N$. □

Proof of Theorem 1.3. Write $W = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$ with $n_i > 0$ for all i. Up to a permutation of the factors of W we may assume $Y = \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_k}$ with $0 \le m_i \le n_i$ for all i and that there is an integer $s \in \{1, \dots, k\}$ such that $m_i = 0$ if and only if $i > s$. Thus $\dim W - \dim Y = n_1 + \cdots + n_k - m_1 - \cdots - m_s$. Taking a smaller multiprojective space if necessary we may assume $\dim W = \dim Y + 1$. Thus either $k = s+1$, $n_k = 1$ and $m_i = n_i$ for all i or $k = s$ and $n_i \ne m_i$ for a unique i. In the latter case up to permuting the factors of W we may assume $n_k = m_k - 1$. Thus, setting $T := \prod_{i=1}^{k-1} \mathbf{P}^{n_i}$, we may unify the cases saying that in both cases there is a hyperplane $H \subset \mathbf{P}^{n_k}$ such that $Y = T \times H$ and $W = T \times \mathbf{P}^{n_k}$. Set $b := r_{\nu(Y)}(q)$ and take $A \subset Y$ such that $\nu(A) \in \mathcal{S}(\nu(Y), q)$. Concision gives $b = r_{\nu(W)}(q)$ ([3, Proposition 3.1.3.1]). Set $U := \langle \nu(A) \rangle$. We modify the proof of Theorem 1.2 in the following way. Fix $a \in A$ and write $a = (a_1, \dots, a_k)$ with $a_k \in H$. Set $A' := A \setminus \{a\}$. Since $A \subset Y$, we have $\langle \nu(A) \rangle \subseteq \langle \nu(Y) \rangle$ Fix $u_k, v_k \in \mathbf{P}^{n_k} \setminus H$ such that $u_k \neq v_k$ and $a_k \in \langle \{u_k, v_k\} \rangle$. Write $u = (u_1, \dots, u_k) \in W$ and $v = (v_1, \dots, v_k) \in W$ with $u_i = v_i = a_i$ for all $i < k$. Set $B := A' \cup \{u, v\}$. Since $A' \cap \{u, v\} = \emptyset$, we have $\#(B) = r_{\nu(Y)}(q) + 1$. Note that $\nu(a) \in \langle \{\nu(u), \nu(v)\} \rangle$. Set $D := \{a_1\} \times \cdots \times \{a_{k-1}\} \times \mathbf{P}^{n_k}$. Since $A \subset Y$, we have $\langle \nu(A) \rangle \subseteq \langle \nu(Y) \rangle$. Thus $\langle \nu(A) \rangle \cap \nu(D) \subseteq \langle \nu(Y) \rangle \cap D = \{a_1\} \times \cdots \times \{a_{k-1}\} \times H$. Part (a) of the proof of Theorem 1.2 shows that $\nu(B)$ irredundantly spans q. By construction B is not contained in Y. □

Remark 2.2: Let $X \subset \mathbf{P}^N$, $N = -1 + \binom{n+d}{n}$, be a Veronese variety which is an order d embedding of $\mathbf{P}^n$, $n \geq 1$, $d \geq 3$. Fix $q \in \mathbf{P}^N$ with $r_X(q) \leq \lfloor (d+1)/2 \rfloor$. Since any $d+1$ points of X are linearly independent, it is easy to check that $\mathcal{S}(X, q, t) = \emptyset$ for all t such that $r_X(q) < t \leq d + 1 - r_X(q)$.

Remark 2.3: Let $X \subset \mathbf{P}^N$, $N = -1 + \binom{n+d}{n}$, be a Veronese variety which is an order d embedding of $\mathbf{P}^n$, $n \geq 2$, $d \geq 5$. Fix $q \in \mathbf{P}^N$ with border rank 2 and $r_X(q) > 2$. By [2, Theorem 32] we have $r_X(q) = d$. Using [2, Lemma 34] it is easy to check that $\mathcal{S}(X, q, t) = \emptyset$ for all t such that $d + 1 \leq t \leq 2d - 2$.

Proposition 2.4. *Fix integers $d > 0$ and $n > m > 0$. Let $\nu : \mathbf{P}^n \to \mathbf{P}^N$, $N = \binom{n+d}{n} - 1$, be the order d Veronese embedding. Fix an m-dimensional linear subspace $M \subset \mathbf{P}^n$ and $q \in \langle \nu(M) \rangle$. Take any positive t such that there is $S = \nu(A) \in \mathcal{S}(\nu(Y), q, t)$ such that $M = \langle A \rangle$. Then there is $E = \nu(B) \in \mathcal{S}(X, q, t + d(n-m))$ such that $\langle B \rangle = \mathbf{P}^n$.*

Proof. By induction on the integer $n - m$ we reduce to the case $n - m = 1$. Fix $a \in A$ and take any line $L \subset \mathbf{P}^n$ such that $L \cap M = \{a\}$. Fix a general $G \subset L \setminus \{o\}$ such that $\#(G) = d + 1$ and take $B := (A \setminus \{a\}) \cup G$. We have $\#(B) = t + d$. Since any $D \in |\mathcal{I}_{G, \mathbf{P}^n}(d)|$ contains L, we have $\nu(a) \in \langle \nu(B) \rangle$. Since $A \setminus \{a\} \subset B$, we get $q \in \langle \nu(B) \rangle$. Thus to conclude the proof it is sufficient to prove that $\nu(B)$ irredundantly spans q. Assume that is is not the case, i.e. assume the existence of $B' \subset B$ such that $\#(B') = \#(B) - 1$ and $q \in \langle \nu(B') \rangle$. Set $\{p\} := B \setminus B'$.

(a) Assume $p \in G$. Let $\mathcal{V} \subseteq |\mathcal{O}_L(d)|$ be the projectivization of the image of $H^0(\mathcal{I}_{A \setminus \{a\}}(d))$ by the restriction map $H^0(\mathcal{O}_{\mathbf{P}^n}(d)) \to H^0(\mathcal{O}_L(d))$.

Claim: $\mathcal{V} = |\mathcal{O}_L(d)|$.

Proof of the claim. $\mathcal{V}$ contains all divisors $a + D$ with D effective of degree $d - 1$ (take the image of all degree d forms on $\mathbf{P}^n$ with an equation of M as one of their forms). Thus $\mathcal{V}$ has at most codimension 1 in $|\mathcal{O}_L(d)|$ and to prove that $\mathcal{V} = |\mathcal{O}_L(d)|$ it is sufficient to prove that a is not a base point of $\mathcal{V}$. Since A irredundantly spans q, there is $T \in |\mathcal{O}_M(d)|$ containing $A \setminus \{a\}$, but not containing a. For any $o \in \mathbf{P}^n \setminus M$ the cone with vertex o and T as its base does not contain a, concluding the proof of the claim.

By the claim there is $K \in |\mathcal{O}_{\mathbf{P}^n}(d)|$ containing E, but not containing a. Thus K is not contained in M. Thus $K_{|M}$ vanishes on $A \setminus \{a\}$, but not a. Since A irredundantly spans $q \in \langle \nu(M)\rangle$, we have $\langle \nu(A)\rangle = \langle \nu(A') \cup \{q\}\rangle$. Thus $K_{|M}$ shows that $q \notin \langle \nu(E)\rangle$, a contradiction.

(b) Assume $p \in A \setminus \{a\}$. The curve $\nu(L)$ is a degree d rational normal curve in its linear span and $\langle \nu(L)\rangle \cap \langle \nu(M)\rangle = \{\nu(a)\}$. Since $\{a\} \cup G \subset L$ and $q \in \langle \nu(M)\rangle$, we get $q \notin \langle \nu(E)\rangle$, a contradiction. □

3. Concision for tensor decompositions

Let $W := \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$, $n_i > 0$ for all i, be a multiprojective space and $\nu : W \to \mathbf{P}^N$, $N = (n_1+1)\cdots(n_k+1)-1$, its Segre embedding. Set $X := \nu(W)$. Let $\pi_i : W \to \mathbf{P}^{n_i}$ denote the projection of W onto its i-th factor. Take a finite set $S \subset X$ and write $S = \nu(A)$ with $A \subset W$. The minimal Segre subvariety of X containing S is the Segre variety $\nu(\prod_{i=1}^{k}\langle \pi_i(A)\rangle)$. Thus for any finite S it is easy, quick and very cheap to determine the minimal Segre variety containing S.

PROPOSITION 3.1. *Let Y' be a multiprojective space. Fix an integer $m \ge 2$ and fix $o \in \mathbf{P}^m$. Set $W := Y' \times \mathbf{P}^m$ and $Y := Y' \times \{o\}$. Let $\nu : W \to \mathbf{P}^N$ be the Segre embedding of W. Set $X := \nu(W)$. Take $q \in \langle \nu(Y)\rangle$.*

- *(a) For any integer $t \le m - 1 + r_X(q)$ no $B \in \mathcal{S}(X, q, t)$ is concise for X.*
- *(b) If q is concise for $\nu(Y)$, then there is $B \in \mathcal{S}(X, q, r_X(q) + m)$ concise for X.*

Proof. Let $\pi : W \to \mathbf{P}^m$ denote the projection onto the last factor of W. Assume the existence of $S \in \mathcal{S}(X, q, t)$ which is concise for X and write $S = \nu(A)$ with $A \subset W$. Since S is concise for X, we have $\langle \pi(A)\rangle = \mathbf{P}^m$. Thus there is a hyperplane $H \subset \mathbf{P}^m$ such that $\#(H \cap \pi(A)) \ge m$. Set $D := \pi^{-1}(H)$ and $A' := A \setminus A \cap D$. D is a hypersurface of W isomorphic to $Y' \times \mathbf{P}^{m-1}$ and $\#(D \cap A) \ge m$. Thus concision gives $\#(A') \le t - m < r_{\nu(Y)}(q)$.

(a) Set $U := \langle \nu(D)\rangle$ and let $\ell : \mathbf{P}^N \setminus U \to \mathbf{P}^r$, $r = N - \dim U - 1$, denote the linear projection from U. By definition of Segre embedding we have $N + 1 = (m+1)(\dim\langle \nu(Y)\rangle)$. Since $H \cong Y' \times \mathbf{P}^{m-1}$, we have $\dim U + 1 = m(\dim\langle \nu(Y)\rangle)$. Hence $r = \dim\langle \nu(Y)\rangle$. By concision for rank 1 tensors we have $U \cap X = \nu(D)$ and $\langle \nu(Y)\rangle \cap U = \emptyset$. Thus $\ell_{|X \setminus \nu(D)} : X \setminus \nu(D) \to \mathbf{P}^r$ is a morphism. Note that for each $(u, v) \in Y' \times \mathbf{P}^m \setminus D$ we have $\ell(\nu(u, v)) = \nu(u, o)$. Since $q \in \langle \nu(Y)\rangle$ and $\langle \nu(Y)\rangle \cap U = \emptyset$, we may identify $\mathbf{P}^r$ with $\langle \nu(Y)\rangle$ and say that, up to this identification, we have $\ell(q) = q$; alternatively we may say that $\ell(q)$ and q have the same $\nu(Y)$-rank and that $\mathcal{S}(\nu(Y), q, t) = \mathcal{S}(\nu(Y), \ell(q), t)$ for all t. Since $q \notin U$, $\ell(\langle \nu(B)\rangle \setminus \langle \nu(D)\rangle)$ is a linear space spanning $\ell(q)$. Since $\langle \nu(B)\rangle$ is spanned by the linearly independent set $\nu(B)$, $\ell(\langle \nu(B)\rangle \setminus \langle \nu(D)\rangle)$ is spanned by the set $\ell(\nu(B \setminus B\cap D))$ with cardinality at most $r_X(q) - 1$, a contradiction.

(b) Take $E \subset Y$ such that $\nu(E) \in \mathcal{S}(\nu(Y), q)$. By concision we have $\#(E) = r_X(q)$. Write $W = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$ with $n_k = m$ and $Y = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_{k-1}} \times \{o\}$. Fix $a = (a_1, \dots, a_k) \in A$. Obviously $a_k = o$. Fix $m+1$ general points $c_0, \dots, c_m \in \mathbf{P}^m$ and set $u_i = (a_1, \dots, a_{k-1}, c_i)$. Set $E := (A \setminus \{a\}) \cup \{u_0, \dots, u_m\}$. We have $\#(E) = r_X(q) + m$ and W is the minimal multiprojective space containing E. Thus to conclude the proof it is sufficient to prove that $\nu(E)$ minimally spans q. Note that $\langle \nu(A)\rangle = \langle \nu(A') \cup \{o\}\rangle$. For general $c_0, \dots, c_m$ the point o is not contained in the linear span of any proper subset of $\{u_0, \dots, u_m\}$. Thus for any proper subset E' of E containing $A \setminus \{o\}$ we have $q \notin \langle \nu(E')\rangle$. Note that $\langle \nu(A)\rangle = \langle \nu(A') \cup \{o\}\rangle$. Assume $q \in \langle \nu(J \cup \{u_0, \dots, u_m\}\rangle$ with $J \subset A \setminus \{a\}$, $J \neq A \setminus \{a\}$. Let $H \subset \mathbf{P}^m$ be the hyperplane spanned by $c_1, \dots, c_m$. Take ℓ as in step (a). Since $\#(J) \le r_X(q) - 2$, we would get that q has rank at most $r_X(q) - 1$. □

4. Segre-Veronese varieties

For any multiprojective space $Y = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$ and all positive integers $d_1, \dots, d_k$ let $\nu_{d_1,\dots,d_k} : Y \to \mathbf{P}^N$, $N = -1 + \prod_{i=1}^k \binom{d_i+k_i}{n_i}$, denote the Segre-Veronese embedding of Y with multidegree $(d_1, \dots, d_k)$. Since concision holds for Segre-Veronese embeddings, it is natural to consider if there are irredundantly spanning sets with cardinality rank $+1$ or rank $+d_i$.

Remark 4.1: Fix integers $m > 0$, $d > 0$ and take $o \in \mathbf{P}^m$. Set $W = Y \times \mathbf{P}^m$ and use the Segre-Veronese embedding $\nu_{d_1,\dots,d_k,d}$ of W. Fix $q \in \langle \nu_{d_1,\dots,d_k,d}(Y \times \{o\})\rangle$ and call ρ its rank. If $d = 1$ there is $S \subset W$ such that $\#(S) = \rho + 1$ and $\nu_{d_1,\dots,d_k,d}(S)$ irredundantly spans q (just use the proof of Theorem 1.3). Moreover, if $m \ge 2$ there is no concise S irredundantly spanning q until $\#(S) = \rho + m$. Now assume $d > 1$. We may repeat the proof of Proposition 2.4 with $M = Y \times \{o\}$ and get $\mathcal{S}(\nu_{d_1,\dots,d_k,m}(W), q, \rho + dm) \neq \emptyset$.

Remark 4.2: Proposition 2.4 may be extended with minimal modifications to an inclusion $Y \subset W$ of multiprojective spaces with the same number of non-trivial factors.

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