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Tetrahedral Coxeter groups, large group-actions on 3-manifolds and equivariant Heegaard splittings

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Abstract. *We consider finite group-actions on closed, orientable and nonorientable 3-manifolds M which preserve the two handlebodies of a Heegaard splitting of M of some genus $g > 1$ (maybe interchanging the two handlebodies). The maximal possible order of a finite group-action on a handlebody of genus $g > 1$ is $12(g-1)$ in the orientation-preserving case and $24(g-1)$ in general, and the maximal order of a finite group preserving the Heegaard surface of a Heegaard splitting of genus g is $48(g-1)$. This defines a hierarchy for finite group-actions on 3-manifolds which we discuss in the present paper; we present various manifolds with an action of type $48(g-1)$ for small values of g, and in particular the unique hyperbolic 3-manifold with such an action of smallest possible genus $g = 6$ (in strong analogy with the Euclidean case of the 3-torus which has such actions for $g = 3$).*

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1. A hierarchy for large finite group-actions on 3-manifolds

The maximal possible order of a finite group G of orientation-preserving diffeomorphisms of an orientable handlebody of genus $g > 1$ is $12(g-1)$ ([11], [7, Theorem 7.2]); for orientation-reversing finite group-actions on an orientable handlebody and for actions on a nonorientable handlebody, the maximal possible order is $24(g-1)$; we will always assume $g > 1$ in the present paper.

Let G be a finite group of diffeomorphisms of a closed, orientable or nonorientable 3-manifold M. We consider Heegaard splittings of M into two handlebodies of genus g (nonorientable if M is nonorientable) such that each element of G either preserves both handlebodies or interchanges them. The maximal possible orders are $12(g-1)$, $24(g-1)$ or $48(g-1)$, and we distinguish four types of G-actions:

Definition 1.1. *i.* The G-action is of type $12(g-1)$: *M is orientable, G is orientation-preserving and non-interchanging (i.e., preserves both handlebodies of the Heegaard splitting), and G is of maximal possible order $12(g-1)$ for such a situation (these actions are called maximally symmetric in various papers, see [15] and its references).*

ii. The G-action is of non-interchanging type $24(g-1)$: *either M is orientable and G orientation-preserving or M is non-orientable, and G is of maximal possible order $24(g-1)$ for such a situation (these actions are called strong maximally symmetric in [15]).*

iii. The G-action is of interchanging type $24(g-1)$: *M is orientable, the subgroup G_0 of index 2 preserving both handlebodies is orientation-preserving, and G is of maximal possible order $24(g-1)$ for such a situation.*

iv. The G-action is of type $48(g-1)$: *G is interchanging, either M is orientable and the subgroup G_0 preserving both handlebodies is orientation-reversing, or M is non-orientable, and G is of maximal possible order $48(g-1)$ for such a situation.*

The second largest orders in the four cases are $8(g-1)$, $16(g-1)$ and $32(g-1)$, then $20(g-1)/3$, $40(g-1)/3$ and $80(g-1)/3$, next $6(g-1)$, $12(g-1)$ and $24(g-1)$ etc.; in the present paper, we consider only the cases of largest possible orders $12(g-1)$, $24(g-1)$ and $48(g-1)$.

In Section 2, we present examples of 3-manifolds for various of these types, and in particular the unique hyperbolic 3-manifold of type $48(g-1)$ of smallest possible genus $g=6$. The situation for the orientation-preserving actions of types described in Definitions 1.1-i and 1.1-iii is quite flexible and has been considered in various papers, so in the present paper we concentrate mainly on the orientation-reversing actions and actions on non-orientable manifolds of cases described in Definitions 1.1-ii and 1.1-iv where the situation is more rigid. We finish this section with a short discussion of 3-manifolds of type $12(g-1)$, for small values of g. The following is proved in [14].

Theorem 1.2. *The closed orientable 3-manifolds with a G-action of type $12(g-1)$ and of genus $g=2$ are exactly the 3-fold cyclic branched coverings of the 2-bridge links, the group G is isomorphic to the dihedral group $\mathbb{D}_6$ of order 12 and obtained as the lift of a symmetry group $\mathbb{Z}_2\times\mathbb{Z}_2$ of each such 2-bridge link.*

Examples of such 3-manifolds are the spherical Poincaré homology sphere (the 3-fold branched covering of the torus knot of type (2,5)), the Euclidean Hantzsche-Wendt manifold (the 3-fold branched covering of the figure-8 knot, see [12]) and the hyperbolic Matveev-Fomenko-Weeks manifold of smallest volume among all closed hyperbolic 3-manifolds (the 3-fold branched covering of the 2-bridge knot 5_2).

Examples of 3-manifolds of type $12(g-1)$ and genus 3 are the 3-torus, again the Euclidean Hantzsche-Wendt manifold, the spherical Poincaré homology 3-sphere and the hyperbolic Seifert-Weber dodecahedral space, see Propositions 2.3, 2.4 and 2.5.

The finite groups G which admit an action of type $12(g-1)$ and genus $g \leq 6$ are $\mathbb{D}_6$, $\mathbb{S}_4$, $\mathbb{D}_3 \times \mathbb{D}_3$, $\mathbb{S}_4 \times \mathbb{Z}_2$ and $\mathbb{A}_5$ ([13]).

2. Tetrahedral Coxeter groups and large group-actions

2.1. Non-interchanging actions of type $24(g-1)$ and actions of type $48(g-1)$

The following is proved in [15].

Theorem 2.1. *i) Let M be a closed, irreducible 3-manifold with a non-interchanging G-action of type $24(g-1)$. Then M is spherical, Euclidean or hyperbolic and obtained as a quotient of the 3-sphere, Euclidean or hyperbolic 3-space by a normal subgroup K of finite index, acting freely, in a spherical, Euclidean or hyperbolic Coxeter group $C(n,m;2,2;2,3)$ or in a twisted Coxeter group $C_\tau(n,m;2,2;3,3)$; the G-action is the projection of the Coxeter or twisted Coxeter group C to M, so $G \cong C/K$. Conversely, each such subgroup K gives a G-action of type $24(g-1)$ on the 3-manifold $M = \mathbb{S}^3/K$, $\mathbb{E}^3/K$ or $\mathbb{H}^3/K$.*

ii) The G-action of type $24(g-1)$ on M extends to a G-action of type $48(g-1)$ if and only $n = m$ and the universal covering group K of M is a normal subgroup also of the twisted Coxeter group $C_\mu(n,n;2,2;2,3)$ or in the doubly-twisted Coxeter group $C_{\tau\mu}(n,n;2,2;3,3)$.

In Theorem 2.1, we use the notation in [15] which we now explain. A *Coxeter tetrahedron* is a tetrahedron in the 3-sphere, Euclidean or hyperbolic 3-space all of whose dihedral angles are of the form π/n (denoted by a label n of the edge, for some integer $n \geq 2$) and, moreover, such that at each of the four vertices the three angles of the adjacent edges define a spherical triangle (i.e., $1/n_1 + 1/n_2 + 1/n_3 > 1$). We will denote such a Coxeter tetrahedron by $\mathcal{C}(n,m;a,b;c,d)$ where $(n,m), (a,b)$ and (c,d) are the labels of pairs of opposite edges, and we denote by $C(n,m;a,b;c,d)$ the *Coxeter group* generated by the reflections in the four faces of the tetrahedron $\mathcal{C}(n,m;a,b;c,d)$, a properly discontinuous group of isometries of one of the three geometries. In the following, we list the Coxeter groups of the various types occuring in Theorem 2.1.

2.1.1. The Coxeter groups $C(n,m;2,2;2,3)$

spherical:	$C(2,2;2,2;2,3)$, $C(3,2;2,2;2,3)$, $C(4,2;2,2;2,3)$, $C(5,2;2,2;2,3)$, $C(3,3;2,2;2,3)$, $C(4,3;2,2;2,3)$, $C(5,3;2,2;2,3)$;
Euclidean:	$C(4,4;2,2;2,3)$;
hyperbolic:	$C(5,4;2,2;2,3)$, $C(5,5;2,2;2,3)$.

A Coxeter tetrahedron $\mathcal{C}(n,m;2,2;3,3)$ has a rotational symmetry τ of order two (an isometric involution) which exchanges the opposite edges with labels 2 and 3 and inverts the two edges with lables n and m. The involution τ can be realized by an isometry and defines a *twisted Coxeter group* $C_\tau(n,m;2,2;3,3)$, a group of isometries which contains the Coxeter group $C(n,m;2,2;3,3)$ as a subgroup of index two.

2.1.2. The twisted Coxeter groups $C_\tau(n,m;2,2;3,3)$

spherical:	$C_\tau(2,2;2,2;3,3)$, $C_\tau(3,2;2,2;3,3)$, $C_\tau(4,2;2,2;3,3)$;
Euclidean:	$C_\tau(3,3;2,2;3,3)$;
hyperbolic:	$C_\tau(5,2;2,2;3,3)$, $C_\tau(5,3;2,2;3,3)$, $C_\tau(5,4;2,2;3,3)$, $C_\tau(5,5;2,2;3,3)$, $C_\tau(4,3;2,2;3,3)$, $C_\tau(4,4;2,2;3,3)$.

A Coxeter tetrahedron $\mathcal{C}(n,n;2,2;2,3)$ has a rotational symmetry μ of order two (an isometric involution) which exchanges the opposite edges with labels n and 2 and inverts the two remaining edges with labels 2 and 3. As before, this defines a *twisted Coxeter group* $C_\mu(n,n;2,2;2,3)$ containing $C(n,n;2,2;2,3)$ as a subgroup of index 2.

2.1.3. The twisted Coxeter groups $C_\mu(n,n;2,2;2,3)$

spherical:	$C_\mu(2,2;2,2;2,3)$, $C_\mu(3,3;2,2;2,3)$;
Euclidean:	$C_\mu(4,4;2,2;2,3)$
hyperbolic:	$C_\mu(5,5;2,2;2,3)$.

Finally, a Coxeter tetrahedron $\mathcal{C}(n,n;2,2;3,3)$ has a group $\mathbb{Z}_2 \times \mathbb{Z}_2$ of rotational isometries generated by involutions τ and μ as before, and hence defines a *doubly-twisted Coxeter group* $C_{\tau\mu}(n,n;2,2;3,3)$ containing $C(n,n;2,2;3,3)$ as a subgroup of index 4 (and both $C_\tau(n,n;2,2;3,3)$ and $C_\mu(n,n;2,2;3,3)$ as subgroups of index 2).

2.1.4. The doubly-twisted Coxeter groups $C_{\tau\mu}(n,n;2,2;3,3)$

spherical: $C_{\tau\mu}(2,2;2,2;3,3)$

Euclidean: $C_{\tau\mu}(3,3;2,2;3,3)$

hyperbolic: $C_{\tau\mu}(4,4;2,2;3,3)$, $C_{\tau\mu}(5,5;2,2;3,3)$.

In the following, we will discuss finite-index normal subgroups of small index, acting freely (i.e., torsion-free in the Euclidan and hyperbolic cases) of various Coxeter and tetrahedral groups (their orientation-preserving subgroups). Since this requires computational methods, we need presentations of the various groups.

2.2. Presentations of Coxeter and tetrahedral groups

Denoting by g_1, g_2, g_3 and g_4 the reflections in the four faces of a Coxeter polyhedron $\mathcal{C}(n,m;2,2;c,3)$, the Coxeter group $C(n,m;2,2;c,3)$ has a presentation

$$< g_1, g_2, g_3, g_4 \mid g_1^2 = g_2^2 = g_3^2 = g_4^2 = 1,$$
$$(g_1g_2)^c = (g_2g_3)^2 = (g_3g_4)^3 = (g_4g_1)^2 = (g_1g_3)^n = (g_2g_4)^m = 1 > .$$

A presentation of the twisted group $C_\tau(n,m;2,2;3,3)$ is obtained by adding to this presentation a generator τ and the relations

$$\tau^2 = 1, \ \ \tau g_1 \tau^{-1} = g_3, \ \ \tau g_2 \tau^{-1} = g_4.$$

If $n = m$, for a presentation of $C_\mu(n,n;2,2;c,3)$ one adds a generator μ and the relations

$$\mu^2 = 1, \ \ \mu g_1 \mu^{-1} = g_2, \ \ \mu g_3 \mu^{-1} = g_4,$$

and for a presentation of $C_{\tau\mu}(n,n;2,2;3,3)$ both generators τ and μ with their relations, and also the relation $(\tau\mu)^2 = 1$.

We consider also the orientation-preserving subgroups of index 2 of the Coxeter groups. The generators f_i in their presentations below denote rotations now (products of two reflections), see [13] for computations of the orbifold fundamental groups in some of these cases. Representing the 1-skeleton of a tetrahedron by a square with its two diagonals, a Wirtinger-type representation of the orbifold-fundamental group is obtained; here the two horizontal edges of the square have labels n and m, the two vertical edges labels c and 3 (generators f_1 and f_4), the two diagonals labels 2 (generators f_2 and f_3), and similarly for the quotients of the 1-skeleton of the tetrahedron by the involutions τ and μ (represented by rotations around a vertical and a horizontal axis, so one easily depicts the singular sets of the quotient orbifolds). In this way on obtains the following presentations:

the *tetrahedral group* $T(n,m;2,2;c,3)$ of index 2 in $C(n,m;2,2;c,3)$:

$$< f_1,f_2,f_3,f_4 \mid f_1^c = f_2^2 = f_3^2 = f_4^3 = 1, \quad f_1f_2f_3f_4 = (f_1f_2)^n = (f_2f_4)^m = 1 >;$$

the *twisted tetrahedral group* $T_\tau(n,m;2,2;3,3)$ of index 2 in $C_\tau(n,m;2,2;3,3)$:

$$< f_1,f_2,f_3,f_4 \mid f_1^2 = f_2^2 = f_3^2 = f_4^3 = 1, \quad f_1f_2f_3f_4 = (f_1f_2)^n = (f_2f_3f_2f_4)^m = 1 >;$$

the *twisted tetrahedral group* $T_\mu(n,n;2,2;c,3)$ of index 2 in $C_\mu(n,n;2,2;c,3)$:

$$< f_1,f_2,f_3,f_4,f_5 \mid f_1^c = f_2^2 = f_3^2 = f_4^3 = 1,\ f_1f_2f_3f_4 = (f_1f_2)^n = 1, \quad f_5^2 = 1,\ (f_1f_5)^n = (f_4f_5)^n = f_3(f_4f_5)f_2(f_4f_5) = 1 >;$$

the *doubly-twisted tetrahedral group* $T_{\tau\mu}(n,n;2,2;3,3)$ of index 2 in $C_{\tau\mu}(n,n;2,2;3,3)$:

$$< f_1,f_2,f_3,f_4,f_5 \mid f_1^2 = f_2^2 = f_3^2 = f_4^3 = 1,\ f_1f_2f_3f_4 = (f_1f_2)^n = 1, \quad f_5^2 = 1,\ (f_1f_5)^2 = (f_4f_5)^2 = (f_2f_4f_5)^2 = 1 > .$$

As a typical example, we will consider the doubly-twisted Coxeter group $C_{\tau\mu}(5,5;2,2;3,3)$ in the next section. In the hyperbolic and Euclidean cases, we call an epimorphism of a Coxeter group or tetrahedral group *admissible* if its kernel is torsion-free.

2.3. Manifolds of type $48(g-1)$ and $24(g-1)$

By Theorem 2.1, we are interested in torsion-free normal subgroups of finite index of the Coxeter groups in Sections 2.1.1-2.1.4. Using the presentations in the previous section, all computations in the following are easily verified by GAP ([2]) which classifies epimorphisms of a group C to a finite group G up to isomorphisms of G (using quo:= GQuotients(c,G); AbelianInvariants(Kernel(quo[1])) etc.). As a typical example, we consider the twisted Coxeter group $C_\tau(5,5;2,2;3,3)$.

2.3.1. The Coxeter group $C(5,5;2,2;3,3)$

For this group the following holds:

THEOREM 2.2. *i) There is a unique torsion-free normal subgroup K_0 of smallest possible index 120 in the twisted Coxeter group $C_\tau(5,5;2,2;3,3)$*

which splits as a semidirect product of K_0 *with the extended dodecahedral group* $\mathbb{A}_5 \times \mathbb{Z}_2$. *The quotient manifold* $M_0 = \mathbb{H}^3/K_0$ *is an orientable hyperbolic 3-manifold of type* $48(g-1)$ *and genus* $g = 6$, *for an action of* $\mathbb{S}_5 \times \mathbb{Z}_2$. *Since* $H_1(M_0) \cong \mathbb{Z}^6$, *also the ordinary Heegaard genus of* M_0 *is equal to 6.*

ii) The manifold M_0 *is the unique hyperbolic 3-manifold with an action of type* $48(g-1)$, *and also with an action of non-interchanging type* $24(g-1)$, *for genera* $g \leq 6$.

iii) There is a unique torsion-free normal subgroup K_1 *of smallest possible index 120 in the Coxeter group* $C(5,5;2,2;3,3)$ *which splits as a semidirect product of* K_1 *with* $\mathbb{A}_5 \times \mathbb{Z}_2$. *The quotient manifold* $M_1 = \mathbb{H}^3/K_1$, *a 2-fold covering of* M_0, *is an orientable hyperbolic 3-manifold of type* $48(g-1)$ *and genus* $g = 11$. *Since* $H_1(M_1) \cong \mathbb{Z}^{11}$, *also the ordinary Heegaard genus of* M_1 *is equal to 11.*

Proof. i) Since $C_\tau(5,5;2,2;3,3)$ has a finite subgroup (a vertex group) isomorphic to the extended dodecahedral group $\bar{\mathbb{A}}_5 \cong \mathbb{A}_5 \times \mathbb{Z}_2$ of order 120 (isomorphic to the extended triangle group $[2,3,5]$ generated by the reflections in the sides of a hyperbolic triangle with angles $\pi/2, \pi/3$ and $\pi/5$), a torsion-free subgroup of $C_\tau(5,5;2,2;3,3)$ has index at least 120. Similarly, $T_\tau(5,5;2,2;3,3)$ has a vertex group $\mathbb{A}_5$ and a torsion-free subgroup has index at least 60.

Up to isomorphism of the image (this will always be the convention in the following), there are exactly three epimorphisms of the twisted tetrahedral group $T_\tau(5,5;2,2;3,3)$ to its vertex group $\mathbb{A}_5$; the abelianizations of the kernels of the three epimorphisms are $\mathbb{Z}^6$ and two times $\mathbb{Z}_2^4 \times \mathbb{Z}_4 \times \mathbb{Z}_5^3$. The three epimorphisms are admissible (have torsion-free kernel) since, when killing an element of finite order in a vertex group of the Coxeter tetrahedron, the twisted tetrahedral group becomes trivial or of order two. The kernel K_0 of the unique epimorphism with abelianized kernel $\mathbb{Z}^6$ is normal also in the twisted Coxeter group $C_\tau(5,5;2,2;3,3)$, the doubly-twisted tetrahedral group $T_{\tau\mu}(5,5;2,2;3,3)$, and hence also in the doubly-twisted Coxeter group $C_{\tau\mu}(5,5;2,2;3,3)$. By Theorem 2.1, the quotient manifold $M_0 = \mathbb{H}^3/K_0$ is a closed orientable hyperbolic 3-manifold of type $48(g-1)$ and genus $g = 6$. Since $C_\tau(5,5;2,2;3,3)/K_0 \cong \mathbb{A}_5 \times \mathbb{Z}_2$ and there is no epimorphism of $C_{\tau\mu}(5,5;2,2;3,3)$ to $\mathbb{A}_5$, the only remaining possibility is $C_{\tau\mu}(5,5;2,2;3,3)/K_0 \cong \mathbb{S}_5 \times \mathbb{Z}_2$.

There are three epimorphisms of $C_\tau(5,5;2,2;3,3)$ to its vertex group $\mathbb{A}_5 \times \mathbb{Z}_2$, with abelianizations $\mathbb{Z}^6$, $\mathbb{Z}^{12}$ and $\mathbb{Z}^5 \times \mathbb{Z}_2^2$, but only the epimorphism with kernel $\mathbb{Z}^6$ is admissible: since the rank of the other two kernels is larger than 6, they cannot uniformize a 3-manifold with a Heegaard splitting of genus 6. Hence K_1 is the unique torsion-free subgroup of index 120 in $C_\tau(5,5;2,2;3,3)$.

ii) By the lists in Sections 2.1.3 and 2.1.4 and Theorem 2.1-ii), apart from $C_{\tau\mu}(5,5;2,2;3,3)$ the other two Coxeter groups to obtain a hyperbolic 3-manifold of type $48(g-1)$ are $C_{\tau\mu}(4,4;2,2;3,3)$ and $C_{\mu}(5,5;2,2;2,3)$.

Let K be a torsion-free normal subgroup of $C_{\tau}(4,4;2,2;3,3)$, with factor group of order $24(g-1)$. Since the extended octahedral group $\mathbb{S}_4 \times \mathbb{Z}_2$ of order 48 is a vertex group of $C_{\tau}(4,4;2,2;3,3)$, the cases $g = 2, 4$ and 6 are not possible. Also the case $g = 3$ is not possible since there is no epimorphism of $C_{\tau}(4,4;2,2;3,3)$ to $\mathbb{S}_4 \times \mathbb{Z}_2$.

Considering $g = 5$, suppose that there is an admissible epimorphism of $C_{\tau}(4,4;2,2;3,3)$ to a group G of order 96; since there are no epimorphisms of $C(4,4;2,2;3,3)$ onto its vertex group $\mathbb{S}_4 \times \mathbb{Z}_2$, its restriction to $C(4,4;2,2;3,3)$ also surjects onto G. Now G has $\mathbb{S}_4 \times \mathbb{Z}_2$ as a subgroup of index 2; dividing out $\mathbb{Z}_2$, it surjects onto $\mathbb{S}_4 \times \mathbb{Z}_2$, and then also $C(4,4;2,2;3,3)$ surjects onto $\mathbb{S}_4 \times \mathbb{Z}_2$. Since no such epimorphism exists, this excludes also the case $g = 5$, and hence $g \leq 6$ is not possible.

In the case of $C_{\mu}(5,5;2,2;2,3)$, there is no epimorphism of $C(5,5;2,2;2,3)$ to its vertex group $\mathbb{A}_5 \times \mathbb{Z}_2$, and again $g \leq 6$ is not possible.

This completes the proof of ii) for the case of actions of type $48(g-1)$; for the case of actions of non-interchanging type $24(g-1)$ one excludes all other Coxeter groups in a similar way.

iii) We see that there are exactly three epimorphisms of the tetrahedral group $T(5,5;2,2;3,3)$ to $\mathbb{A}_5$, with abelianized kernels $\mathbb{Z}^{11}$ and two times $\mathbb{Z}_2 \times \mathbb{Z}_3^4 \times \mathbb{Z}_4^4 \times \mathbb{Z}_5^3$. The kernel K_1 of the unique epimorphism with abelianized kernel $\mathbb{Z}^{11}$ is normal also in $C(5,5;2,2;3,3)$, $T_{\tau}(5,5;2,2;3,3)$, $T_{\mu}(5,5;2,2;3,3)$ and hence also in $C_{\tau\mu}(5,5;2,2;3,3)$. By Theorem 2.1 ii), $M_1 = \mathbb{H}^3/K_1$ is an orientable 3-manifold to type $48(g-1)$.

Since there is just one epimorphism of $C(5,5;2,2;3,3)$ to its vertex group $\mathbb{A}_5 \times \mathbb{Z}_2$, the kernel K_1 is the unique normal subgroup of index 120 in the Coxeter group $C(5,5;2,2;3,3)$.

This completes the proof of Theorem 2.2. □

In the follwing, we consider various other Coxeter groups.

2.3.2. The Coxeter group $C(4,4;2,2;3,3)$

Its subgroup $T(4,4;2,2;3,3)$ has a unique epimorphism to $\mathrm{PSL}(2,7)$, its kernel K_2 has abelianization $\mathbb{Z}^{13}$ and is normal also in $C(4,4;2,2;3,3)$. There are three epimorphisms of $C(4,4;2,2;3,3)$ to $\mathrm{PSL}(2,7) \times \mathbb{Z}_2$, exactly one with abelianization $\mathbb{Z}^{13}$, hence its kernel K_2 is normal also in the twisted groups $C_{\tau}(4,4;2,2;3,3)$, $C_{\mu}(4,4;2,2;3,3)$ and $C_{\tau\mu}(4,4;2,2;3,3)$ and Theorem 2.1 implies:

Proposition 2.3. *The quotient manifold $M_2 = \mathbb{H}^3/K_2$ is an orientable hyperbolic 3-manifold of type $48(g-1)$ and genus $g = 29$.*

2.3.3. The Coxeter group $C(5,5;2,2;2,3)$

As noted already in the proof of Theorem 2.2, $C(5,5;2,2;2,3)$ admits no epimorphism onto its vertex group $\mathbb{A}_5 \times \mathbb{Z}_2$. On the other hand, the tetrahedral group $T(5,5;2,2;2,3)$ has exactly two admissible surjections onto its vertex group $\mathbb{A}_5$, their kernels are conjugate in $C(5,5;2,2;2,3)$, normal in $T_\mu(5,5;2,2;2,3)$ with factor group $\mathbb{S}_5$, and uniformize the Seifert-Weber hyperbolic dodecahedral 3-manifold ([9]). By [6], $\mathbb{S}_5$ is in fact the full isometry group of the Seifert-Weber manifold which has no orientation-reversing isometries.

Proposition 2.4. *The Seifert-Weber hyperbolic dodecahedral 3-manifold is a closed orientable 3-manifold of interchanging type* $24(g-1)$ *and genus* $g=6$*, for the action of its isometry group* $\mathbb{S}_5$.

There are three epimorphisms of the tetrahedral group $T(5,5;2,2;2,3)$ to $\mathrm{PSL}(2,19)$, all admissible, and exactly one kernel K_3 has infinite abelianization $\mathbb{Z}^{56}$ and is normal also in the Coxeter group $C(5,5;2,2;2,3)$. There is exactly one epimorphism of $C(5,5;2,2;2,3)$ to $\mathrm{PSL}(2,19) \times \mathbb{Z}_2$, with kernel K_3, hence K_3 is normal also in the twisted Coxeter group $C_\mu(5,5;2,2;2,3)$ and Theorem 2.2 implies:

Proposition 2.5. *The quotient manifold* $M_3 = \mathbb{H}^3/K_3$ *is an orientable hyperbolic 3-manifold of type* $48(g-1)$ *and genus* $g=286$.

In the papers [4] and [8], infinite series of finite quotients of the hyperbolic Coxeter group $C(5,5;2,2;2,3)$ are considered, and these give infinite series of orientable hyperbolic 3-manifolds with actions of type $48(g-1)$.

2.3.4. The Coxeter group $T(5,2;2,2;3,3)$

There is exactly one admissible epimorphism of $T(5,2;2,2;3,3)$ to $\mathrm{PSL}(2,11)$, its kernel K_4 has abelianization $\mathbb{Z}^{10}$ and is also the kernel of the unique epimorphism of $C(5,2;2,2;3,3)$ to $\mathrm{PSL}(2,11) \times \mathbb{Z}_2$ and of the unique epimorphism of $T_\tau(5,2;2,2;3,3)$ to $\mathrm{PGL}(2,11) \times \mathbb{Z}_2$, hence Theorem 2.2 implies:

Proposition 2.6. *The quotient manifold* $M_4 = \mathbb{H}^3/K_4$ *is an orientable hyperbolic 3-manifold of non-interchanging type* $24(g-1)$ *and genus* $g=111$*, for an action of* $\mathrm{PGL}(2,11) \times \mathbb{Z}_2$.

It is shown in [5] that K_4 is the unique torsion-free subgroup of smallest possible index in the twisted Coxeter group $C_\tau(5,2;2,2;3,3)$.

2.3.5. The Euclidean Coxeter groups $C(3,3;2,2;3,3)$ and $C(4,4;2,2;2,3)$

There are five epimorphisms of the tetrahedral group $T(3,3;2,2;3,3)$ to its vertex group $\mathbb{A}_4$, the abelianizations of the kernels are $\mathbb{Z}^3$, two times $\mathbb{Z}_4^2$ and two times $\mathbb{Z}_2^5$. The kernel with abelianization $\mathbb{Z}^3$ uniformizes the 3-torus and is normal also in $C(3,3;2,2;3,3)$, $T_\tau(3,3;2,2;3,3)$ and hence $C_{\tau\mu}(3,3;2,2;3,3)$. The two epimorphisms with abelianized kernel $\mathbb{Z}_4^2$ are conjugate in $C(3,3;2,2;3,3)$ and uniformize the Hantzsche-Wendt manifold (the only of the six orientable Euclidean 3-manifolds with first homology $\mathbb{Z}_4^2$, see [10]), and the remaining two epimorphisms are not admissible.

Proposition 2.7. *i) The 3-torus is a 3-manifold of type* $48(g-1)$ *and genus* $g=3$ *which is also its ordinary Heegaard genus.*

ii) The Euclidean Hantzsche-Wendt manifold is of interchanging type $24(g-1)$ *and genus* $g=3$*, for the action of its orientation-preserving isometry group* $\mathbb{S}_4 \times \mathbb{Z}_2$ *(by section 1, it is also a 3-manifold of type* $12(g-1)$ *and genus* $g=2$*, for an action of the dihedral group* $\mathbb{D}_6$ *of order 12).*

By [12], the full isometry group of the Hantzsche-Wendt manifold has order 96 but the orientation-reversing elements do not preserve the Heegaard splitting of genus 3 of Proposition 2.7.

The 3-torus is a manifold of type $48(g-1)$ and genus $g=3$ in still a different way. The Euclidean tetrahedral group $T(4,4;2,2;2,3)$ has three epimorphisms to its vertex group $\mathbb{S}_4$, with abelianized kernels $\mathbb{Z}^3$ and two times $\mathbb{Z}_2^2 \times \mathbb{Z}_4$. The kernel with abelianization $\mathbb{Z}^3$ is normal also in the Coxeter group $C(4,4;2,2;2,3)$, the twisted tetrahedral group $T_\mu(4,4;2,2;2,3)$ and hence in the twisted Coxeter group $C_\mu(4,4;2,2;2,3)$, it uniformizes the 3-torus which is again a 3-manifold of type $48(g-1)$ (but for an action not equivalent to the action arising from $T(3,3;2,2;3,3)$).

2.3.6. The spherical Coxeter group $C(5,3;2,2;2,3)$

There is no epimorphism of $C(5,3;2,2;2,3)$ to its vertex group $\mathbb{A}_5 \times \mathbb{Z}_2$, and there are two epimorphisms of the tetrahedral group $T(5,3;2,2;2,3)$ to its vertex group $\mathbb{A}_5$; both kernels have trivial abelianization, are conjugate in $C(5,3;2,2;2,3)$ and uniformize the spherical Poincaré homology 3-sphere.

Proposition 2.8. *The spherical Poincaré homology 3-sphere is a 3-manifold of type* $12(g-1)$ *and genus* $g=6$*, for an action of* $\mathbb{A}_5$.

2.3.7. Non-orientable manifolds

The case of non-orientable manifolds is more elusive. It is shown in [1] that, for sufficiently large n, the alternating group $\mathbb{A}_n$ is a quotient of $C_\tau(5,2;2,2;3,3)$ by a torsion-free normal subgroup, and such a subgroup uniformizes a non-orientable 3-manifold: since $\mathbb{A}_n$ is simple, the orientation-preserving subgroup $T_\tau(5,2;2,2;3,3)$ of index 2 surjects onto $\mathbb{A}_n$, and hence the kernel contains an orientation-reversing element.

Theorem 2.9. *For all sufficiently large n, there is a non-orientable hyperbolic 3-manifold of non-interchanging type $24(g-1)$, for an action of the alternating group $\mathbb{A}_n$.*

Some other finite simple quotients of the nine hyperbolic Coxeter groups are listed in [3], and for the Coxeter groups of type $C(n,m;2,2;2,3)$ these define non-orientable hyperbolic 3-manifolds of non-interchanging type $24(g-1)$. At present, we don't know explicit examples of small genus of non-orientable manifolds of non-interchanging type $24(g-1)$, and no example of type $48(g-1)$. An explicit non-orientable example is as follows. Using a presentation of the Mathieu group $M(12)$ of order 95040, in [3, Section 8] an admissible epimorphism of $C(4,4;2,2;3,3)$ to $M(12)$ is exhibited, with torsion-free kernel K_5.

Proposition 2.10. *The quotient manifold $M_5 = \mathbb{H}^3/K_5$ is a non-orientable hyperbolic 3-manifold of non-interchanging type $12(g-1)$ and genus $g = 7921$, for an action of the Mathieu group $M(12)$.*

Note that in Proposition 2.10 we are not in the maximal case $24(g-1)$ for a non-interchanging action but in the next largest case $12(g-1)$ (for the maximal case $24(g-1)$ one should consider the twisted group $C_\tau(4,4;2,2;3,3)$ instead of $C(4,4;2,2;3,3)$).

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