

Canonical Cohen-Macaulay property and Lyubeznik numbers under Gröbner deformations

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ABSTRACT. *In this note we draw some interesting consequences of the recent results on squarefree Gröbner degenerations obtained by Conca and the second author.*

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1. Introduction

Let $R = K[x_1, \dots, x_n]$ be a positively graded polynomial ring over a field K , where x_i is homogeneous of degree $g_i \in \mathbb{N}_{>0}$, and $\mathfrak{m} = (x_1, \dots, x_n)$ denotes its homogeneous maximal ideal. Also denote the canonical module of R by $\omega_R = R(-g)$, where $g = g_1 + \dots + g_n$.

DEFINITION 1.1. *A graded finitely generated R -module M is called canonical Cohen-Macaulay (CCM for short) if $\text{Ext}_R^{n-\dim M}(M, \omega_R)$ is Cohen-Macaulay.*

This notion was introduced by Schenzel in [9], who proved in the same paper the following result that contributes to make it interesting: given a homogeneous prime ideal $I \subset R$, the ring R/I is CCM if and only if it admits a birational Macaulayfication (that is a birational extension $R/I \subset A \subset Q(R/I)$ such that A is a finitely generated Cohen-Macaulay R/I -module, where $Q(R/I)$ is the fraction field of R/I). In this case, furthermore, A is the endomorphism ring of $\text{Ext}_R^{n-\dim R/I}(R/I, \omega_R)$.

In this note, we will derive from the recent result obtained by Conca and the second author in [4] the following: if a homogeneous ideal $I \subset R$ has a radical initial ideal $\text{in}_{\prec}(I)$ for some monomial order \prec , then R/I is CCM whenever $R/\text{in}_{\prec}(I)$ is CCM. In fact we prove something more general, from which we can also infer that, in positive characteristic, under the same assumptions the Lyubeznik numbers of R/I are bounded above from those of $R/\text{in}_{\prec}(I)$. As a consequence of the latter result, we can infer that, also in characteristic 0 by

reduction to positive characteristic, if $\text{in}_{\prec}(I)$ is a radical monomial ideal the following are equivalent:

1. The dual graph (a.k.a. Hochster-Huneke graph) of R/I is connected.
2. The dual graph of $R/\text{in}_{\prec}(I)$ is connected.

Motivated by these results, in the last section we study the CCM property for Stanley-Reisner rings $K[\Delta]$. We show that $K[\Delta]$ is CCM whenever Δ is a simply connected 2-dimensional simplicial complex.

2. CCM property and Gröbner deformations

Throughout, let us fix a monomial order \prec on R . We start with the following crucial lemma:

LEMMA 2.1. *Let I be a homogeneous ideal of R such that $\text{in}_{\prec}(I)$ is radical. Then, for all $i, j, k \in \mathbb{Z}$, we have:*

$$\dim_K \text{Ext}_R^i(\text{Ext}_R^j(R/I, \omega_R), \omega_R)_k \leq \dim_K \text{Ext}_R^i(\text{Ext}_R^j(R/\text{in}_{\prec}(I), \omega_R), \omega_R)_k$$

Proof. Let $w = (w_1, \dots, w_n) \in \mathbb{N}^n$ be a weight such that $\text{in}_w(I) = \text{in}_{\prec}(I)$. Let t be a new indeterminate over R . Set $P = R[t]$ and $S = P/\text{hom}_w(I)$, where $\text{hom}_w(I)$ is the w -homogenization of I . By providing P with the graded structure given by $\deg(x_i) = g_i$ and $\deg(t) = 0$, $\text{hom}_w(I)$ is homogeneous. If $x \in \{t, t-1\}$, apply the functor $\text{Ext}_P^i(\text{Ext}_P^j(S, P), -)$ to the short exact sequence

$$0 \rightarrow P \xrightarrow{\cdot x} P \rightarrow P/xP \rightarrow 0$$

getting the short exact sequences

$$0 \rightarrow \text{Coker} \mu_x^{i,j} \rightarrow \text{Ext}_P^i(\text{Ext}_P^j(S, P), P/xP) \rightarrow \text{Ker} \mu_x^{i+1,j} \rightarrow 0.$$

where $\mu_x^{i,j}$ is the multiplication by x on $\text{Ext}_P^i(\text{Ext}_P^j(S, P), P)$. So, for all $k \in \mathbb{Z}$ we have exact sequences of K -vector spaces:

$$0 \rightarrow (\text{Coker} \mu_x^{i,j})_k \rightarrow \text{Ext}_P^i(\text{Ext}_P^j(S, P), P/xP)_k \rightarrow (\text{Ker} \mu_x^{i+1,j})_k \rightarrow 0.$$

Since $E_k^{i,j} = \text{Ext}_P^i(\text{Ext}_P^j(S, P), P)_k$ is a finitely generated graded module over $K[t]$ (w.r.t. the standard grading on $K[t]$), we can write $E_k^{i,j} = F_k^{i,j} + T_k^{i,j}$ where $F_k^{i,j} = K[t]^{f_k^{i,j}}$ and $T_k^{i,j} = \bigoplus_{r=1}^{g_k^{i,j}} K[t]/(t^{d_r})$ with $d_r \geq 1$. Therefore we have:

$$\dim_K(\text{Coker} \mu_{t-1}^{i,j})_k = f_k^{i,j} \leq f_k^{i,j} + g_k^{i,j} = \dim_K(\text{Coker} \mu_t^{i,j})_k$$

and

$$\dim_K(\text{Ker} \mu_{t-1}^{i+1,j})_k = 0 \leq g_k^{i+1,j} = \dim_K(\text{Ker} \mu_t^{i+1,j})_k.$$

So $\dim_K \text{Ext}_P^i(\text{Ext}_P^j(S, P), P/(t-1)P)_k \leq \dim_K \text{Ext}_P^i(\text{Ext}_P^j(S, P), P/tP)_k$.

Note that, by using [3, Proposition 1.1.5] one can infer the following: if A is a ring, M and N are A -modules, and $a \in \text{Ann}(N)$ is A -regular and M -regular, then

$$\text{Ext}_A^i(M, N) \cong \text{Ext}_{A/aA}^i(M/aM, N) \quad \forall i \in \mathbb{N}.$$

Because $\text{in}_<(I)$ is radical, by [4, Proposition 2.4] $\text{Ext}_P^j(S, P)$ is a flat $K[t]$ -module, the multiplication by x on it is injective: that is, x is $\text{Ext}_P^j(S, P)$ -regular. Therefore we have:

$$\text{Ext}_P^i(\text{Ext}_P^j(S, P), P/xP) \cong \text{Ext}_{P/xP}^i(\text{Ext}_P^j(S, P)/x\text{Ext}_P^j(S, P), P/xP).$$

Again because the multiplication by x is injective on $\text{Ext}_P^j(S, P)$ and by the property mentioned above, we have

$$\text{Ext}_P^j(S, P)/x\text{Ext}_P^j(S, P) \cong \text{Ext}_P^j(S, P/xP) \cong \text{Ext}_{P/xP}^j(S/xS, P/xP).$$

Putting all together we get:

$$\begin{aligned} \dim_K \text{Ext}_{P/(t-1)P}^i(\text{Ext}_{P/(t-1)P}^j(S/(t-1)S, P/(t-1)P), P/(t-1)P)_k \\ \leq \dim_K \text{Ext}_{P/tP}^i(\text{Ext}_{P/tP}^j(S/tS, P/tP), P/tP)_k, \end{aligned}$$

that, because $\omega_R \cong R(-g)$, is what we wanted:

$$\dim_K \text{Ext}_R^i(\text{Ext}_R^j(R/I, R), R)_k \leq \dim_K \text{Ext}_R^i(\text{Ext}_R^j(R/\text{in}_<(I), R), R)_k.$$

□

COROLLARY 2.2. *Let I be a homogeneous ideal of R such that $\text{in}_<(I)$ is radical. Then, R/I is canonical Cohen-Macaulay whenever $R/\text{in}_<(I)$ is so.*

Proof. For a homogeneous ideal $J \subset R$, R/J is CCM if and only if

$$\text{Ext}_R^{n-i}(\text{Ext}_R^{n-\dim R/J}(R/J, \omega_R), \omega_R) = 0 \quad \forall i < \dim R/J,$$

so the result follows from Lemma 2.1. □

REMARK 2.3: Lemma 2.1 and Corollary 2.2 fail without assuming that $\text{in}_<(I)$ is radical. In fact, in characteristic 0, if I is in generic coordinates, by [7, Theorem 2.2] $R/\text{in}_<(I)$ is sequentially Cohen-Macaulay, thus CCM (for example see [7, Theorem 1.4]). However, it is plenty of homogeneous ideals I such that R/I is not CCM.

We do not know whether the implication of Corollary 2.2 can be reversed. Without assuming that $\text{in}_<(I)$ is radical, we already noticed that Corollary 2.2 fails in Remark 2.3. The following example shows that in general R/I CCM but $R/\text{in}_<(I)$ not CCM can also happen:

EXAMPLE 2.4: Let $R = K[x_1, \dots, x_9]$ and

$$I = (x_1^3 + x_2^3, x_3^2x_9 + x_4^2x_8, x_5^3x_7 + x_6^3x_9, x_7^2x_1 + x_6^2x_5, x_3x_9 - x_4x_8).$$

Since I is a complete intersection, R/I is CCM. However one can check that, if \prec is the lexicographic order extending $x_1 > \dots > x_9$, $R/\text{in}_\prec(I)$ is not CCM.

3. Lyubeznik numbers and connectedness

Let $I \subset R = K[x_1, \dots, x_n]$. In [8] Lyubeznik introduced the following invariants of $A = R/I$:

$$\lambda_{i,j}(A) = \dim_K \text{Ext}_R^i(K, H_I^{n-j}(R)) \quad \forall i, j \in \mathbb{N}.$$

It turns out that these numbers, later named *Lyubeznik numbers*, depend only on A , i and j , in the sense that if $A \cong S/J$ where $J \subset S = K[y_1, \dots, y_m]$,

$$\lambda_{i,j}(A) = \dim_K \text{Ext}_S^i(K, H_J^{m-j}(S)) \quad \forall i, j \in \mathbb{N}.$$

Also, $\lambda_{i,j}(A) = 0$ whenever $i > j$ or $j > \dim A$, and $\lambda_{d,d}(A)$ is the number of connected components of the *dual graph*, also known as the *Hochster-Huneke graph*, of $A \otimes_K \overline{K}$, where \overline{K} denotes the algebraic closure of K , [18]. For the convenience of the reader, we recall here that the dual graph of a Noetherian ring A of dimension d is the graph with the minimal primes of A as vertices and such that $\{\mathfrak{p}, \mathfrak{q}\}$ is an edge if and only if $\dim A/(\mathfrak{p} + \mathfrak{q}) = d - 1$.

We will refer to the upper triangular matrix $\Lambda(A) = (\lambda_{i,j}(A))$ of size $(\dim A + 1) \times (\dim A + 1)$ as the *Lyubeznik table* of A . By *trivial Lyubeznik table* we mean that $\lambda_{\dim A, \dim A}(A) = 1$ and $\lambda_{i,j}(A) = 0$ otherwise.

COROLLARY 3.1. *Let I be a homogeneous ideal of R such that $\text{in}_\prec(I)$ is radical. If K has positive characteristic,*

$$\lambda_{i,j}(R/I) \leq \lambda_{i,j}(R/\text{in}_\prec(I)) \quad \forall i, j \in \mathbb{N}.$$

Proof. By [19, Theorem 1.2], if $J \subset R$ is a homogeneous ideal,

$$\lambda_{i,j}(R/J) = \dim_K [\text{Ext}_R^{n-i}(\text{Ext}_R^{n-j}(R/J, \omega_R), \omega_R)_0]_s,$$

where the subscript $[-]_s$ stands for the stable part under the natural Frobenius action. In particular

$$\lambda_{i,j}(R/J) \leq \dim_K \text{Ext}_R^{n-i}(\text{Ext}_R^{n-j}(R/J, \omega_R), \omega_R)_0.$$

On the other hand, if $J \subset R$ is a radical monomial ideal, Yanagawa proved in [17, Corollary 3.10] that:

$$\lambda_{i,j}(R/J) = \dim_K \text{Ext}_R^{n-i}(\text{Ext}_R^{n-j}(R/J, \omega_R), \omega_R)_0.$$

So the result follows from Lemma 2.1. □

The following two examples show that Corollary 3.1 is false without assuming both that $\text{in}_\prec(I)$ is radical and that K has positive characteristic:

EXAMPLE 3.2: [12, Example 4.11] Let $R = K[x_1, \dots, x_6]$ and $\text{char}(K) = 5$. Let

$$I = (x_4^3 + x_5^3 + x_6^3, x_4^2x_1 + x_5^2x_2 + x_6^2x_3, x_1^2x_4 + x_2^2x_5 + x_3^2x_6, \\ x_1^3 + x_2^3 + x_3^3, x_5x_3 - x_6x_2, x_6x_1 - x_4x_3, x_4x_2 - x_5x_1).$$

Then

$$\Lambda(R/I) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 1 \end{bmatrix}.$$

If \prec is the degree reverse lexicographic term order extending $x_1 > \dots > x_6$ one has:

$$\text{in}_\prec(I) = (x_3x_5, x_3x_4, x_2x_4, x_4^3, x_1x_4^2, x_1^2x_4, x_1^3).$$

One can check that $R/\text{in}_\prec(I)$ has a trivial Lyubeznik table.

EXAMPLE 3.3: Let K be a field of characteristic 0 and $R = K[X]$ where X is a 2×3 matrix of variables. Let $I \subset R$ be the ideal generated by the 2-minors of X . Since $\text{Proj } R/I \cong \mathbb{P}^1 \times \mathbb{P}^2$, by [14, Theorem 1.2], we have:

$$\Lambda(R/I) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ & & & & 1 \end{bmatrix}.$$

If \prec is a diagonal term order it is easy to check that $\text{in}_\prec(I)$ is a radical monomial ideal such that $R/\text{in}_\prec(I)$ is Cohen-Macaulay, so that $\Lambda(R/\text{in}_\prec(I))$ is trivial.

In Corollary 3.1 we have an equality when R/I is generalized Cohen-Macaulay, which means that $H_m^i(R/I)$ has finite length for all $i < \dim R/I$:

PROPOSITION 3.4. *Let I be a homogeneous ideal of R such that $\text{in}_\prec(I)$ is radical. If K has positive characteristic and R/I is generalized Cohen-Macaulay,*

$$\lambda_{i,j}(R/I) = \lambda_{i,j}(R/\text{in}_\prec(I)) \quad \forall i, j \in \mathbb{N}.$$

Proof. Since R/I is generalized Cohen-Macaulay so is $R/\text{in}_\prec(I)$ by [4, Corollary 2.11]. Therefore it is enough to show that $\lambda_{0,j}(R/I) = \lambda_{0,j}(R/\text{in}_\prec(I))$ for all j (see [1, Subsection 4.3]). By [4, Proposition 3.3], both $R/\text{in}_\prec(I)$ and R/I are cohomologically full. So from [5, Proposition 4.11]:

$$\lambda_{0,j}(R/I) = \dim_K H_m^j(R/I)_0, \\ \lambda_{0,j}(R/\text{in}_\prec(I)) = \dim_K H_m^j(R/\text{in}_\prec(I))_0.$$

Now by [4, Theorem 1.3] we get the result. □

REMARK 3.5: Let I be an ideal of $R = K[x_1, \dots, x_n]$ such that $\text{in}_{\prec}(I)$ is generated by monomials u_1, \dots, u_r . Suppose that K has characteristic 0. Since I is finitely generated, there exists a finitely generated \mathbb{Z} -algebra $A \subset K$ such that I is defined over A , i.e. $I'R = I$ if $I' = I \cap A[x_1, \dots, x_n]$. Given a prime number p and a prime ideal $\mathfrak{p} \in \text{Spec}A$ minimal over (p) , let $Q(\mathfrak{p})$ denote the field of fractions of A/\mathfrak{p} (note that $Q(\mathfrak{p})$ has characteristic p), $R(\mathfrak{p}) = Q(\mathfrak{p})[x_1, \dots, x_n]$ and $I(\mathfrak{p}) = I'R(\mathfrak{p})$. We call the objects $R(\mathfrak{p}), I(\mathfrak{p}), R(\mathfrak{p})/I(\mathfrak{p})$ reductions mod p of $R, I, R/I$, and by abusing notation we denote them by $R_p, I_p, R_p/I_p$.

Seccia proved in [10, Lemma 2.3] that

$$\text{in}_{\prec}(I_p) = \text{in}_{\prec}(I)_p$$

for any reduction mod p if p is a large enough prime number, i.e. $\text{in}_{\prec}(I_p)$ is generated by u_1, \dots, u_r .

REMARK 3.6: Let A be a Noetherian ring of dimension d . The ring A is said to be connected in codimension 1 if $\text{Spec} A \setminus V(\mathfrak{a})$ is connected whenever $\dim A/\mathfrak{a} < d-1$ (here $V(\mathfrak{a})$ denotes the set of primes containing \mathfrak{a}). A result of Hartshorne [6, Proposition 1.1] implies that the dual graph of A is connected if and only if A is connected in codimension 1.

PROPOSITION 3.7. *Let I be a homogeneous ideal of R such that $\text{in}_{\prec}(I)$ is radical. Then:*

1. *$\text{Proj}R/I$ is connected if and only if $\text{Proj}R/\text{in}_{\prec}(I)$ is connected.*
2. *The dual graph of R/I is connected if and only if the dual graph of $R/\text{in}_{\prec}(I)$ is connected.*

Proof. The “only if” parts hold without the assumption that $\text{in}_{\prec}(I)$ is radical and they have been proved in [15]. So we will concentrate on the “if” parts.

Since computing initial ideal, as well as the connectedness properties concerning $R/\text{in}_{\prec}(I)$, are not affected extending the field, while the connectedness properties concerning R/I follow from the corresponding connectedness properties of $R/I \otimes_K \overline{K}$, it is harmless to assume that K is algebraically closed. Under this assumption, if $J \subset R$ is a homogeneous radical ideal, we have that:

- (a) $\text{Proj}R/J$ is connected if and only if $H_{\mathfrak{m}}^1(R/J)_0 = 0$.
- (b) The dual graph of R/J is connected if and only if $\lambda_{\dim R/J, \dim R/J}(R/J) = 1$ by the main theorem of [18].

Under our hypothesis I is radical, so (1) follows at once from (a) and the fact that the Hilbert function of the local cohomology modules of R/I is bounded above by that of the ones of $R/\text{in}_{\prec}(I)$ (in this case we even have equality

by [4]). Concerning the “if-part” of (2), since $\lambda_{\dim R/I, \dim R/I}(R/I)$ counts the number of connected components of some graph, it is nonzero in any case; if K has positive characteristic, hence the conclusion follows from (b) and Corollary 3.1. So, assume that K has characteristic 0. If, by contradiction, R/I were not connected in codimension 1, there would be two ideals $H \supsetneq I$ and $J \supsetneq I$ such that $H \cap J = I$ and $\dim R/(H + J) < \dim R/I - 1$ (see [2, Lemma 19.1.15]). By Remark 3.5, it is not difficult to check that we can choose a prime number $p \gg 0$ such that $H_p \supsetneq I_p$ and $J_p \supsetneq I_p$, $H_p \cap J_p = I_p$, $\dim R_p/(H_p + J_p) < \dim R_p/I_p - 1$ and $\text{in}_{\prec}(I_p) = \text{in}_{\prec}(I)_p$ (for instance, to compute the intersection of two ideals amounts to perform a Gröbner basis calculation). Clearly the dual graph of a Stanley-Reisner ring does not depend on the characteristic of the base field. So the dual graph of $R_p/\text{in}_{\prec}(I_p)$ would be connected but that of R_p/I_p would be not, and this contradicts the fact that we already proved the result in positive characteristic. \square

4. CCM Simplicial Complexes

Let Δ be a simplicial complex on the vertex set $[n] = \{1, \dots, n\}$. We denote the Stanley-Reisner ring R/I_{Δ} by $K[\Delta]$. See [11] for generalities on these objects. The aim of this section is to examine the CCM property for the Stanley-Reisner rings $K[\Delta]$, especially when Δ has dimension 2.

Recall that a \mathbb{N}^n -graded R -module M is *squarefree* if, for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, the multiplication by x_j from M_{α} to $M_{\alpha+e_j}$ is bijective whenever $\alpha_j \neq 0$. It turns out that $K[\Delta]$, I_{Δ} and $\text{Ext}_R^i(K[\Delta], \omega_R)$ are squarefree modules by [16].

LEMMA 4.1. *Let M be a nonzero squarefree module. If $M_0 = 0$, then $\text{depth}M > 0$.*

Proof. Assume, by way of contradiction, that $\text{depth}M = 0$. Then $\mathfrak{m} \in \text{Ass}M$. So there exist $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $0 \neq u \in M_{\alpha}$ such that $\mathfrak{m} = \text{Ann}(u)$. So for $j = 1, \dots, n$, $x_j \cdot u = 0$. It follows that the multiplication map on M_{α} by x_j is not injective for all j . So, because M is a squarefree module, $\alpha = 0$ and $u \in M_0 = 0$, a contradiction. Hence $\text{depth}M > 0$. \square

LEMMA 4.2. *For any homogeneous ideal $I \subset R$, for all $i < 3$ the R -module $\text{Ext}_R^{n-i}(\text{Ext}_R^{n-\dim R/I}(R/I, R), R)$ has finite length.*

Proof. If $(\cap_{i=1}^r \mathfrak{q}_i) \cap (\cap_{j=1}^s \mathfrak{q}'_j)$ is an irredundant primary decomposition of I with $\dim R/\mathfrak{q}_i = \dim R/I$ and $\dim R/\mathfrak{q}'_j > \dim R/I$, one has

$$\text{Ext}_R^{n-\dim R/I}(R/I, R) \cong \text{Ext}_R^{n-\dim R/I}(R/\cap_{i=1}^r \mathfrak{q}_i, R).$$

So we can assume that $\dim R/\mathfrak{p} = \dim R/I$ for all $\mathfrak{p} \in \text{Ass}R/I$.

Let $\mathfrak{p} \neq \mathfrak{m}$ be a homogeneous prime ideal of R containing I , and set $M_i = \text{Ext}_R^{n-i}(\text{Ext}_R^{n-\dim R/I}(R/I, R), R)$. We have:

$$(M_i)_{\mathfrak{p}} = \text{Ext}_{R_{\mathfrak{p}}}^{\text{ht}(\mathfrak{p})-(i-n+\text{ht}(\mathfrak{p}))}(\text{Ext}_{R_{\mathfrak{p}}}^{\text{ht}(\mathfrak{p})-(\dim R_{\mathfrak{p}}/IR_{\mathfrak{p}})}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, R_{\mathfrak{p}}), R_{\mathfrak{p}}).$$

We have $(M_i)_{\mathfrak{p}} = 0$, since $i - n + \text{ht}(\mathfrak{p}) \leq 1$ by the assumptions and $\text{Ext}_{R_{\mathfrak{p}}}^{\text{ht}(\mathfrak{p})-(\dim R_{\mathfrak{p}}/IR_{\mathfrak{p}})}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, R_{\mathfrak{p}})$ has depth at least 2 by [9, Proposition 2.3]. □

COROLLARY 4.3. *Let Δ be a 2-dimensional simplicial complex. Then $K[\Delta]$ is CCM if and only if $\lambda_{2,3}(K[\Delta]) = 0$.*

Proof. Since $\text{Ext}_R^{n-3}(K[\Delta], \omega_R)$ satisfy Serre’s condition (S_2) by [9, Proposition 2.3], it is enough to show that $\text{Ext}_R^{n-2}(\text{Ext}_R^{n-3}(K[\Delta], \omega_R), \omega_R) = 0$. By Lemma 4.2 $\text{Ext}_R^{n-2}(\text{Ext}_R^{n-3}(K[\Delta], \omega_R), \omega_R)$ has finite length; so, since it is a squarefree module,

$$\begin{aligned} \text{Ext}_R^{n-2}(\text{Ext}_R^{n-3}(K[\Delta], \omega_R), \omega_R) &= 0 \\ \iff \text{Ext}_R^{n-2}(\text{Ext}_R^{n-3}(K[\Delta], \omega_R), \omega_R)_0 &= 0. \end{aligned}$$

We conclude because $\lambda_{2,3}(K[\Delta]) = \text{Ext}_R^{n-2}(\text{Ext}_R^{n-3}(K[\Delta], \omega_R), \omega_R)_0$ by [17, Corollary 3.10]. □

REMARK 4.4: If Δ is a $(d - 1)$ -dimensional simplicial complex, it is still true that if $K[\Delta]$ is CCM, then $\lambda_{j,d}(K[\Delta]) = 0$ for all $j < d$. The converse, however, is not true as soon as $\dim(\Delta) > 2$:

Let $R = K[x_1, \dots, x_6]$ and I be the monomial ideal of R generated by

$$x_1x_2x_3x_4, x_1x_3x_4x_5, x_1x_2x_3x_6, x_1x_2x_5x_6, x_1x_4x_5x_6 \text{ and } x_3x_4x_5x_6.$$

The ring R/I has a trivial Lyubeznik table but it is not CCM. Here I is the Stanley-Reisner ring of a 3-dimensional simplicial complex.

PROPOSITION 4.5. *Assume that Δ is a 2-dimensional simplicial complex such that $H_1(\Delta; K)$ vanishes. Then $K[\Delta]$ is CCM.*

Proof. Since $H_1(\Delta; K) = 0$, by Hochster formula we get $\text{Ext}_R^{n-2}(K[\Delta], \omega_R)_0 = 0$. If $\text{Ext}_R^{n-2}(K[\Delta], \omega_R) \neq 0$, since it is a squarefree module it has positive depth by Lemma 4.1.

So, in any case, $\text{Ext}_R^n(\text{Ext}_R^{n-2}(K[\Delta], \omega_R), \omega_R) = 0$, and hence

$$\lambda_{0,2}(K[\Delta]) = \text{Ext}_R^n(\text{Ext}_R^{n-2}(K[\Delta], \omega_R), \omega_R)_0 = 0.$$

By [1, Remark 2.3], $\lambda_{2,3}(K[\Delta]) = \lambda_{0,2}(K[\Delta]) = 0$. Now by Corollary 4.3 $K[\Delta]$ is CCM. □

The converse of this corollary does not hold in general:

EXAMPLE 4.6: Let Δ be the simplicial complex on 6 vertices with facets $\{1, 2, 3\}$, $\{1, 4, 5\}$ and $\{3, 4, 6\}$. Then $K[\Delta]$ is CCM but $H_1(\Delta; K) \neq 0$

PROPOSITION 4.7. *Let Δ be a $(d - 1)$ -dimensional Buchsbaum simplicial complex. The ring $K[\Delta]$ is CCM if and only if $H_i(\Delta; K) = 0$ for all $1 \leq i < d - 1$.*

Proof. Let $K[\Delta]$ be CCM and fix $i \in \{1, \dots, d - 2\}$. Since Δ is Buchsbaum, $K[\Delta]$ behaves cohomologically like an isolated singularity, hence:

$$\lambda_{0,i+1}(K[\Delta]) = \lambda_{d-i,d}(K[\Delta])$$

(see [1, Subsection 4.3]). On the other hand, since the canonical module of $K[\Delta]$ is a d -dimensional Cohen-Macaulay module, $\lambda_{d-i,d}(K[\Delta]) = 0$ by [17, Corollary 3.10]. So

$$\lambda_{0,i+1}(K[\Delta]) = \dim_K \text{Ext}_R^n(\text{Ext}_R^{n-i-1}(K[\Delta], \omega_R), \omega_R)_0 = 0.$$

By local duality $H_m^0(\text{Ext}_R^{n-i-1}(K[\Delta], \omega_R))_0 = 0$. Since $\text{Ext}_R^{n-i-1}(K[\Delta], \omega_R)$ is of finite length

$$H_m^0(\text{Ext}_R^{n-i-1}(K[\Delta], \omega_R))_0 = \text{Ext}_R^{n-i-1}(K[\Delta], \omega_R)_0 = 0.$$

Therefore Hochster formula tells us that $H_i(\Delta; K) = 0$.

Conversely, assume that $H_i(\Delta; K) = 0$ for all $1 \leq i < d - 1$. Then we have that $\text{Ext}_R^{n-i-1}(K[\Delta], \omega_R)_0 = 0$ by Hochster formula. As Δ is Buchsbaum, $\text{Ext}_R^{n-i-1}(K[\Delta], \omega_R)$ is of finite length, so

$$\text{Ext}_R^{n-i-1}(K[\Delta], \omega_R) = \text{Ext}_R^{n-i-1}(K[\Delta], \omega_R)_0 = 0 \quad \forall 1 \leq i < d - 1.$$

Now [13, Theorem 4.9] and local duality imply that for $1 \leq i < d - 1$,

$$H_m^{i+1}(\text{Ext}_R^{n-d}(K[\Delta], \omega_R)) \cong \text{Ext}_R^{n-d+i}(K[\Delta], \omega_R) = 0.$$

Thus $K[\Delta]$ is CCM. □

EXAMPLE 4.8: Propositions 4.5 and 4.7 provide the following situation concerning CCM 2-dimensional simplicial complexes:

- (i) $H_1(\Delta; K) = 0 \implies K[\Delta]$ is CCM.
- (ii) If Δ is Buchsbaum, $H_1(\Delta; K) = 0 \iff K[\Delta]$ is CCM.

Item (ii) above yields many examples of Buchsbaum 2-dimensional nonCCM simplicial complexes. We conclude this note with an example of a 2-dimensional simplicial complex which is neither Buchsbaum nor CCM:

Let $R = K[x_1, \dots, x_8]$ and Δ be the simplicial complex with facets $\{x_1, x_2, x_6\}$, $\{x_2, x_6, x_4\}$, $\{x_2, x_4, x_5\}$, $\{x_2, x_3, x_5\}$, $\{x_3, x_5, x_6\}$, $\{x_1, x_3, x_6\}$, $\{x_1, x_7, x_8\}$. One can check that Δ is not Buchsbaum and $K[\Delta]$ is not CCM. Accordingly with Proposition 4.5, $H_1(\Delta; K) \neq 0$.

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