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Volume estimates for right-angled hyperbolic polyhedra

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"To Bruno Zimmermann on his 70-th birthday"

Abstract. *By Andreev theorem acute-angled polyhedra of finite volume in a hyperbolic space $\mathbb{H}^3$ are uniquely determined by combinatorics of their 1-skeletons and dihedral angles. For a class of compact right-angled polyhedra and a class of ideal right-angled polyhedra estimates of volumes in terms of the number of vertices were obtained by Atkinson in 2009. In the present paper upper estimates for both classes are improved.*

Keywords: right-angled polyhedron, ideal polyhedron, hyperbolic volume.
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1. Introduction

In the present paper we consider polyhedra of finite volume in a hyperbolic 3-space $\mathbb{H}^3$. A hyperbolic polyhedron is said to be *right-angled* if all its dihedral angles are equal to $\pi/2$. Necessary and sufficient conditions for realization in $\mathbb{H}^3$ of a polyhedron with given combinatorial type and dihedral angles were described by Andreev [1]. Moreover, the realization is unique up to isometry. A hyperbolic polyhedron is said to be *ideal* if all its vertices belong to $\partial\mathbb{H}^3$. The smallest 825 compact right-angled hyperbolic polyhedra were determined by Inoue [6]. In [11] we listed 248 initial volumes of ideal right-angled hyperbolic polyhedra and formulated a conjecture about smallest volume polyhedra when number of vertices is fixed.

Right-angled polyhedra (both compact and ideal) are very useful building-blocks for construction of hyperbolic 3-manifolds with interesting properties [9, 10]. In particular, the right-angled decomposition gives an immersed totally geodesic surface in the 3-manifold arising from the faces of the polyhedra. Recently nice combinatorial characterization of right-angled hyperbolic 3-orbifolds was obtained in [7].

It is known that complements of the Whitehead link and of the Borromean rings admit decompositions into ideal right-angled octahedra. Following [3], a hyperbolic link in 3-sphere is said to be right-angled if its complement equipped with the complete hyperbolic structure admits a decomposition into ideal hyperbolic right-angled polyhedra. Among non-alternating links, the class of fully augmented links is right-angled (see recent results on augmented links in [5, 8]). The following conjecture was proposed in [3]: there does not exist a right-angled knot. In the same paper the conjecture was verified for knots up to 11 crossings, basing on the tabulation of volumes of ideal right-angled polyhedra given in [11].

2. Main results

Since right-angled hyperbolic polyhedra are determined by combinatorial structure, it is natural to expect that their geometrical invariants, e.g. volume, can be estimated by combinatorial invariants, e.g. number of vertices.

Two-sided volume estimates for ideal right-angled hyperbolic polyhedra in terms of the number of vertices were obtained by Atkinson [2].

THEOREM 2.1 ([2, Theorem 2.2.]). *If P is an ideal right-angled hyperbolic polyhedron with V vertices, then*

$$(V-2)\cdot\frac{v_8}{4}\leqslant \operatorname{vol}(P)\leqslant (V-4)\cdot\frac{v_8}{2},$$

where v_8 is the volume of a regular ideal hyperbolic octahedron. Both inequalities are equalities when P is the regular ideal hyperbolic octahedron. There is a sequence of ideal $\pi/2$-equiangular polyhedra P_i with V_i vertices such that $\operatorname{vol}(P_i)/V_i$ approaches $v_8/2$ as i goes to infinity.

Constant v_8 has an expression in terms of the Lobachevsky function,

$$\Lambda(x)=-\int_0^x \log|2\sin t|dt,$$

which dates back to N. I. Lobachevsky's 1832 paper. Namely, $v_8=8\Lambda(\pi/4)$. To fifteen decimal places, v_8 is 3.663862376708876.

Recall that if $P\subset\mathbb{H}^3$ is ideal right-angled, then $V\geqslant 6$. The case $V=6$ is realized for an octahedron that is an antiprism with a triangular top and bottom, the case $V=8$ is realized for an antiprism with quadrilateral top and bottom, and there no ideal right-angled hyperbolic polyhedra with $V=7$ vertices. The upper bound from Theorem 2.1 can be improved if we exclude from consideration the two smallest ideal right-angled hyperbolic polyhedra.

THEOREM 2.2. *If P is an ideal right-angled hyperbolic polyhedron with $V\geqslant 9$ vertices, then*

$$\operatorname{vol}(P)\leqslant (V-5)\cdot\frac{v_8}{2}.$$

The equality holds if and only if $V=9$.

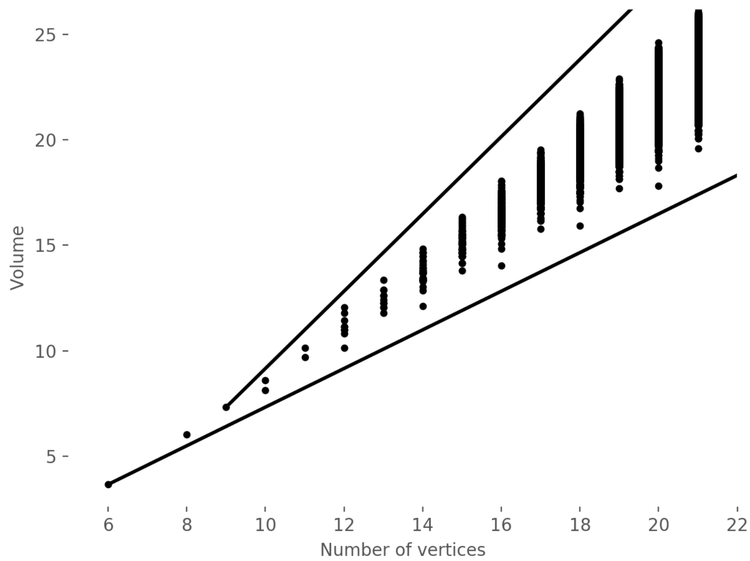

Figure 1: Volumes of ideal right-angled hyperbolic polyhedra and volume estimates from Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.2 is given in Section 3. In Figure 1 dots present volumes of ideal right-angled hyperbolic polyhedra with at most 21 vertices, calculated in [11], and lines present volume estimates from Theorem 2.1 (lower bound) and Theorem 2.2 (upper bound).

Two-sided volume estimates for compact right-angled polyhedra in terms of number of vertices were obtained by Atkinson in [2].

Theorem 2.3 ([2, Theorem 2.3]). *If P is a compact right-angled hyperbolic polyhedron with V vertices, then*

$$(V-8)\cdot\frac{v_8}{32}\leqslant \operatorname{vol}(P)<(V-10)\cdot\frac{5v_3}{8},$$

where v_3 is the volume of a regular ideal hyperbolic tetrahedron. There is a sequence of compact polyhedra P_i, with V_i vertices such that $\operatorname{vol}(P_i)/V_i$ *approaches* $5v_3/8$ *as i goes to infinity.*

Constant v_3 has an expression in terms of the Lobachevsky function, $v_3 = 2\Lambda(\pi/6)$. To fifteen decimal places, v_3 is 1.014941606409653.

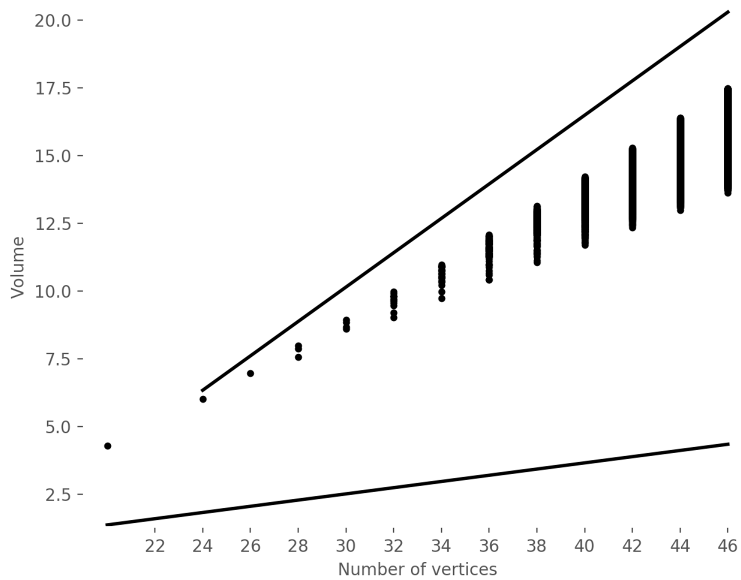

Figure 2: Volumes of compact right-angled hyperbolic polyhedra and volume estimates from Theorem 2.3 and Theorem 2.4.

Recall that if $P \subset \mathbb{H}^3$ is compact right-angled, then $V \geqslant 20$ and even. The case $V = 20$ is realized for a dodecahedron, and there no compact right-angled hyperbolic polyhedra with $V = 22$ vertices. The upper bound from Theorem 2.3 can be improved if we exclude a dodecahedron from our considerations.

Theorem 2.4. *If P is a compact right-angled hyperbolic polyhedron with $V \geqslant 24$ vertices, then*

$$\mathrm{vol}(P) \leqslant (V - 14) \cdot \frac{5v_3}{8}.$$

Proof of Theorem 2.4 is given in Section 4. In Figure 2 dots presents volumes of compact right-angled hyperbolic polyhedra with at most 46 vertices, and lines present volume estimates from Theorem 2.3 (lower bound) and Theorem 2.4 (upper bound). Previously, volumes of the first 825 compact right-angled hyperbolic polyhedra were calculated in [6] and volumes of compact right-angled polyhedra with at most 64 vertices having only pentagonal and hexagonal faces were calculated in [4].

3. Proof of Theorem 2.2

We recall that 1-skeleton of any ideal right-angled hyperbolic polyhedra is 4-regular planar graph. Therefore, if P is such a polyhedron with V vertices and F faces, then by the Euler formula $V = F - 2$.

The upper estimate from Theorem 2.1 was generalized in [11] as in Proposition 3.1 and Proposition 3.3.

Proposition 3.1 ([11, Theorem 3.2]). *Let P be an ideal right-angled hyperbolic polyhedron with $V > 6$ vertices, If P has two faces F_1 and F_2 such that F_1 is n_1-gonal, and F_2 is n_2-gonal, where $n_1, n_2 \geqslant 4$, then*

$$\operatorname{vol}(P) \leqslant \left(V - \frac{n_1}{2} - \frac{n_2}{2}\right) \cdot \frac{v_8}{2}.$$

Considering $n_1 \geqslant 5$ and $n_2 \geqslant 5$ in Proposition 3.1, we immediately get the following result.

Corollary 3.2. *If P is an ideal right-angled hyperbolic polyhedron with V vertices that has two faces such that each of them is at least 5-gonal, then*

$$\operatorname{vol}(P) \leqslant (V - 5) \cdot \frac{v_8}{2}.$$

Proposition 3.3 ([11, Theorem 3.3]). *If P is an ideal right-angled hyperbolic polyhedron with $V \geqslant 15$ vertices having only triangular or quadrilateral faces, then*

$$\operatorname{vol}(P) \leqslant (V - 5) \cdot \frac{v_8}{2}.$$

Proposition 3.4. *Let P be an ideal right-angled hyperbolic polyhedron with V vertices. Let F_1, F_2 and F_3 be three faces of P such that F_1 is n_1-gonal, F_2 is n_2-gonal and F_3 is n_3-gonal. Assume that F_2 is adjacent to both F_1 and F_3. Then*

$$\operatorname{vol}(P) \leqslant \left(V + 1 - \frac{n_1}{2} - \frac{n_2}{2} - \frac{n_3}{2}\right) \cdot \frac{v_8}{2}.$$

Proof. Let us consider union $P' = P \cup_{F_2} P$ of two copies of P along F_2. Polyhedron P' is ideal right-angled and has $V' = 2V - n_2$ vertices. Its volume is twice volume of P. Remark that P' has a face F_{11} which is a union of two copies of F_1 along a common edge. Hence F_{11} is $(2n_1 - 2)$-gonal. Similar, P' has a face F_{13} which is a union of two copies of F_3 along a common edge. Hence F_{13} is $(2n_3 - 2)$-gonal. Applying Proposition 3.1 to polyhedron P' and its faces F_{11} and F_{13}, we obtain

$$2\operatorname{vol}(P) = \operatorname{vol}(P') \leqslant \left((2V - n_2) - \frac{2n_1 - 2}{2} - \frac{2n_3 - 2}{2}\right) \cdot \frac{v_8}{2}.$$

Dividing both sides of the inequality by 2, we get the result. □

Corollary 3.5. *If P is an ideal right-angled hyperbolic polyhedron with V vertices that has at least one k-gonal face, $k \geqslant 6$, then*

$$\operatorname{vol}(P) \leqslant (V-5) \cdot \frac{v_8}{2}.$$

Proof. Consider three faces F_1, F_2 and F_3 of P, where one of them, say F_2, is k-gonal, $k \geqslant 6$, and F_1, F_3 are adjacent to F_2. Then we can apply Proposition 3.4 for the triple (n_1, n_2, n_3), where $n_1 \geqslant 3$, $n_2 \geqslant 6$ and $n_3 \geqslant 3$. Then

$$\begin{aligned} \operatorname{vol}(P) &\leqslant \left(V + 1 - \frac{n_1}{2} - \frac{n_2}{2} - \frac{n_3}{2}\right) \cdot \frac{v_8}{2} \\ &\leqslant \left(V + 1 - \frac{3}{2} - \frac{3}{2} - \frac{6}{2}\right) \cdot \frac{v_8}{2} = (V-5) \cdot \frac{v_8}{2}, \end{aligned}$$

and the statement is proved. □

Proposition 3.6. *Let P be an ideal right-angled polyhedron with $V \geqslant 16$ vertices. Assume that P has one pentagonal face and all other faces are triangles or quadrilaterals. Then*

$$\operatorname{vol}(P) \leqslant (V-5) \cdot \frac{v_8}{2}.$$

Proof. Denote by F number of faces and by p_k the number of k-gonal faces ($k \geqslant 3$) of polyhedron P. Then from $F = V + 2$ we obtain

$$p_3 = 8 + \sum_{k \geq 5} (k-4) p_k.$$

By the assumption, P has one pentagonal face, and therefore, nine triangular faces. Hence $F = 10 + p_4 \geqslant 18$ and $p_4 \geqslant 8$.

Observe that P contains three quadrilateral or pentagonal faces F_1, F_2, F_3 such that F_2 is adjacent to F_1 and F_3 both. By a contradiction, assume that any quadrilateral or pentagonal face of P has at most one adjacent quadrilateral or pentagonal face. Hence any quadrilateral face has at least 3 adjacent triangular faces and the pentagonal face has at least 4 adjacent triangular faces. Therefore, totally there are at least $4 + 3p_4$ sides of triangles which are adjacent to sides of a quadrilateral and pentagonal faces. Since the total number of triangular faces is 9, we get $4 + 3p_4 \leqslant 27$. Since $p_4 \geqslant 8$, the inequality is not satisfied.

This contradiction implies that there is a triple of sequentially adjacent quadrilateral or pentagonal faces F_1, F_2, F_3, where F_2 is adjacent to F_1 and F_3. Applying Proposition 3.4 we get

$$\begin{aligned} \operatorname{vol}(P) &\leqslant \left(V + 1 - \frac{n_1}{2} - \frac{n_2}{2} - \frac{n_3}{2}\right) \cdot \frac{v_8}{2} \\ &\leqslant \left(V + 1 - \frac{4}{2} - \frac{4}{2} - \frac{4}{2}\right) \cdot \frac{v_8}{2} = (V-5) \cdot \frac{v_8}{2}, \end{aligned}$$

and the statement is proved. □

# of vertices	# of polyhedra	# of volumes	min volume	max volume
6	1	1	3.663863	3.663863
7	0	0	-	-
8	1	1	6.023046	6.023046
9	1	1	7.327725	7.327725
10	2	2	8.137885	8.612415
11	2	2	9.686908	10.149416
12	9	7	10.149416	12.046092
13	11	7	11.801747	13.350771
14	37	17	12.106298	14.832681
15	79	31	13.813278	16.331571

Table 1: Ideal right-angled hyperbolic polyhedra.

Summarizing Proposition 3.3 (the case if there is no k-gonal faces for $k \geqslant 5$), Corollary 3.5 (the case if there is at least one k-gonal face, $k \geqslant 6$), Proposition 3.6 (the case if there is one pentagonal face and other faces are k-gonal, $k \leqslant 4$), and Corollary 3.2 (the cases if there are two faces such that each of them has at least 5 edges) we have the statement for polyhedra with $V \geqslant 16$ vertices. For polyhedra P with $9 \leqslant V \leqslant 15$ vertices the inequality $\mathrm{vol}(P) \leqslant (V-5) \cdot \frac{v_8}{2}$ holds by direct computation [11], see Table 1. Proof of Theorem 2.2 is completed.

4. Proof of Theorem 2.4

The proof of Theorem 2.4 uses the same method as the proof of Theorem 2.2 presented in previous section. But with the difference that instead of ideal right-angled polyhedra with 4-regular 1-skeletons we consider compact right-angled hyperbolic polyhedra with 3-regular 1-skeletons. Therefore, if P is a compact right-angled hyperbolic polyhedron with V vertices and F faces, then by the Euler formula $2F = V + 4$.

The upper estimate from Theorem 2.2 was generalized in [4] as in Proposition 4.1, Proposition 4.3 and Proposition 4.4 below. For the reader's convenience we present these statements with proofs.

Proposition 4.1 ([4, Theorem 3.4]). *Let P be a compact right-angled hyperbolic polyhedron with V vertices. Let F_1 and F_2 be two faces of P such that F_1 is n_1-gonal, and F_2 is n_2-gonal, where $n_1, n_2 \geqslant 6$. Then*

$$\mathrm{vol}(P) \leqslant (V - n_1 - n_2) \cdot \frac{5v_3}{8}.$$

Proof. Case (1). Let us assume that faces F_1 and F_2 are not adjacent. We construct a family $\{P_i\}$ of right-angled polyhedra by induction, attaching on each step a copy of the polyhedron P. Put $P_1 = P$. Define $P_2 = P_1 \cup_{F_1} P_1$ by identifying two copies of the polyhedron P_1 along the face F_1. It has $V_2 = 2V - 2n_1$ vertices. Indeed, faces of P_1 adjacent to F_1 form dihedral angles $\pi/2$ with F_1, so vertices of F_1 aren't vertices in the glued polyhedron $P_2 = P_1 \cup_{F_1} P_1$. For the volumes we have $\mathrm{vol}(P_2) = 2\,\mathrm{vol}(P)$. The polyhedron P_2 has at the least one face isometric to F_2. Attaching the polyhedron P to the polyhedron P_2 along this face we get $P_3 = P_2 \cup_{F_2} P = P \cup_{F_1} P \cup_{F_2} P$. Evidently, P_3 is a compact right-angled polyhedron with $V_3 = 3V - 2n_1 - 2n_2$ vertices and volume $\mathrm{vol}(P_3) = 3\,\mathrm{vol}(P)$. Continuing the process of attaching P alternately through the faces isometric to F_1 and F_2, we get the polyhedron $P_{2k+1} = P_{2k-1} \cup_{F_1} P \cup_{F_2} P$, which is a compact right-angled polyhedron with $V_{2k+1} = (2k+1)V - 2kn_1 - 2kn_2$ vertices and volume $\mathrm{vol}(P_{2k+1}) = (2k+1)\,\mathrm{vol}(P)$. Now let us apply the upper bound from Theorem 2.3 to polyhedron P_{2k+1}:

$$(2k+1)\,\mathrm{vol}(P) < ((2k+1)V - 2kn_1 - 2kn_2 - 10)\,\frac{5v_3}{8}.$$

Dividing both sides of the inequality by $(2k+1)$ and passing to the limit as $k \to \infty$, we obtain the required inequality.

Case (2). Let us assume that faces F_1 and F_2 are adjacent. Put $P' = P \cup_{F_1} P$. Then polyhedron P' has $V' = 2V - 2n_1$ vertices and $\mathrm{vol}(P') = 2\,\mathrm{vol}(P)$. By construction, the polyhedron P' has a face F_2', which is a union of two copies of F_2 along a common edge. Hence, F_2' is a $(2n_2 - 4)$-gon. Since the face F_1 has at least 5 sides, there is a face F_3 in P adjacent to F_1, but not adjacent to F_2. As a result of attaching P along F_1, the face F_3 will turn into a face F_3' in P' that has at least 6 sides. Thus, in P' there is a pair of non-adjacent faces F_2' and F_3', each of which has at least 6 sides. Applying case (1) for the polyhedron P' and its non-adjacent faces F_2' and F_3' we get:

$$2\,\mathrm{vol}(P) \leqslant (V' - (2n_2 - 4) - 6)\,\frac{5v_3}{8} = (2V - 2n_1 - 2n_2 - 2)\,\frac{5v_3}{8}$$

and hence, $\mathrm{vol}(P) < (V - n_1 - n_2)\,\dfrac{5v_3}{8}$. □

Considering $n_1 \geqslant 7$ and $n_2 \geqslant 7$ in Proposition 4.1 we immediately get the following result.

Corollary 4.2. *If P is a compact right-angled hyperbolic polyhedron with V vertices having two faces such that each of them is at least 7-gonal. Then*

$$\mathrm{vol}(P) \leqslant (V - 14) \cdot \frac{5v_3}{8}.$$

PROPOSITION 4.3 ([4, Corollary 3.2]). *Let P be a compact right-angled hyperbolic polyhedron with V vertices. Let F_1, F_2 and F_3 be three faces of P such that F_1 is n_1-gonal, F_2 is n_2-gonal and F_3 is n_3-gonal. Assume that F_2 is adjacent to both F_1 and F_3. Then*

$$\operatorname{vol}(P) \leqslant (V - n_1 - n_2 - n_3 + 4) \cdot \frac{5v_3}{8}.$$

Proof. Let us consider polyhedron $P' = P \cup_{F_2} P$. It has $V' = 2V - 2n_2$ vertices and $\operatorname{vol}(P') = 2\operatorname{vol}(P)$. Remark that P' has a face F_1' which is a union of two copies of F_1 and has $2n_1 - 4$ vertices. Similar, P' has a face F_3' which is a union of two copies of F_3 and has $2n_3 - 4$ vertices. Applying Proposition 4.1 to polyhedron P' and its faces F_1' and F_3'. We get

$$2\operatorname{vol}(P) = \operatorname{vol}(P') \leqslant (V' - (2n_1 - 4) - (2n_3 - 4)) \cdot \frac{5v_3}{8}.$$

Using formula for V' and dividing both sides of the inequality by 2 we get the result. □

PROPOSITION 4.4 ([4, Theorem 3.5]). *If P is a compact right-angled hyperbolic polyhedron with $V \geqslant 46$ vertices and with only pentagonal or hexagonal faces, then*

$$\operatorname{vol}(P) \leqslant (V - 14) \cdot \frac{5v_3}{8}.$$

Proof. Denote by F number of faces and by p_k the number of k–gonal faces ($k \geqslant 5$) of polyhedron P. Then from $2F = V + 4$ we obtain

$$p_5 = 12 + \sum_{k \geqslant 7} (k-6) p_k.$$

By the assumption, P has only pentagonal and hexagonal faces. Hence $F = 12 + p_6 \geqslant 25$ and $p_6 \geqslant 13$. We observe that in the polyhedron P there are three hexagonal faces F_1, F_2, F_3 such that F_2 is adjacent to both F_1 and F_3. Assume by contradiction that there is no such triple of faces. Then each hexagonal face is adjacent to at most one hexagonal face. If a hexagonal face has no adjacent hexagons (we will say that it is isolated), then it is adjacent to 6 pentagonal faces. If two hexagons are adjacent to each other and none of them is adjacent to another hexagon (we will say that the faces form a pair), then their union is adjacent to 10 pentagonal faces. If we have k_1 isolated hexagonal faces and k_2 pairs of hexagonal faces, then they are adjacent to pentagonal faces through $6k_1 + 10k_2$ sides. Since $p_5 = 12$ we have $6k_1 + 10k_2 \leqslant 60$. But $k_1 + 2k_2 = p_6 \geqslant 13$ implies $2k_2 \geqslant 13 - k_1$, hence $6k_1 + 10k_2 \geqslant 65 + k_1 > 60$. This contradiction implies that there is a triple of hexagonal faces F_1, F_2, F_3, where F_2 is adjacent to F_1 and F_3. By applying Proposition 4.3 we get the result. □

Corollary 4.5. *If P is a compact right-angled hyperbolic polyhedron with V vertices and at least one k-gonal face, $k \geqslant 8$. Then*

$$\mathrm{vol}(P) \leqslant (V-14) \cdot \frac{5v_3}{8}.$$

Proof. Consider three faces F_1, F_2 and F_3 of P, where one of them, say F_2, is k-gonal, $k \geqslant 8$, and F_1, F_3 are adjacent to F_2. Then we can apply Proposition 4.3 for the triple (n_1, n_2, n_3), where $n_1 \geqslant 5$, $n_2 \geqslant 8$ and $n_3 \geqslant 5$. Then

$$\begin{aligned}\mathrm{vol}(P) &\leqslant (V - n_1 - n_2 - n_3 + 4) \cdot \frac{5v_3}{8} \\ &\leqslant (V - 5 - 8 - 5 + 4) \cdot \frac{5v_3}{8} = (V-14) \cdot \frac{5v_3}{8},\end{aligned}$$

that gives the result. □

Proposition 4.6. *Let P be a compact right-angled polyhedron with $V \geqslant 48$ vertices. Assume that P has one heptagonal face and all other faces are pentagons or hexagons. Then*

$$\mathrm{vol}(P) \leqslant (V-14) \cdot \frac{5v_3}{8}.$$

Proof. Denote by F number of faces and by p_k the number of k–gonal faces ($k \geqslant 5$) of polyhedron P. Then from $2F = V + 4$ we obtain

$$p_5 = 12 + \sum_{k \geqslant 7} (k-6) p_k.$$

By the assumption, P has one heptagonal face, and therefore, 13 pentagonal faces. Hence $F = 14 + p_6 \geqslant 26$ and $p_6 \geqslant 12$.

Observe that P contains three hexagonal or heptagonal faces F_1, F_2, F_3 such that F_2 is adjacent to F_1 and F_3 both. By a contradiction, assume that any hexagonal or heptagonal face of P has at most one adjacent hexagonal or heptagonal face. Hence any hexagonal face has at least 5 adjacent pentagonal faces and the heptagonal face has at least 6 adjacent pentagonal faces. Therefore, totally there are at least $6 + 5p_6$ sides of pentagons which are adjacent to sides of hexagonal and heptagonal faces. Since the total number of pentagonal faces is 13, we get $6 + 5p_6 \leqslant 65$. Since $p_6 \geqslant 12$, the inequality is not satisfied.

This contradiction implies that there is a triple of sequentially adjacent hexagonal or heptagonal faces F_1, F_2, F_3, where F_2 is adjacent to F_1 and F_3. Applying Proposition 4.4 we get

$$\begin{aligned}\mathrm{vol}(P) &\leqslant (V - n_1 - n_2 - n_3 + 4) \cdot \frac{5v_3}{8} \\ &\leqslant (V - 6 - 6 - 6 + 4) \cdot \frac{5v_3}{8} = (V-14) \cdot \frac{5v_3}{8},\end{aligned}$$

that completes the proof. □

# of vertices	# of polyhedra	# of volumes	min volume	max volume
20	1	1	4.306208	4.306208
22	0	0	-	-
24	1	1	6.023046	6.023046
26	1	1	6.967011	6.967011
28	3	3	7.563249	8.000234
30	4	4	8.612415	8.946606
32	12	12	9.019053	9.977170
34	23	23	9.730847	10.986057
36	71	71	10.416044	12.084191
38	187	187	11.058763	13.138893
40	627	627	11.708462	14.222648
42	1970	1952	12.352835	15.300168
44	6833	6771	12.996118	16.397833
46	23384	23082	13.637792	17.486616

Table 2: Compact right-angled hyperbolic polyhedra.

Summarizing Proposition 4.4 (the case if there is no k-gonal faces for $k \geqslant 7$), Corollary 4.5 (the case if there is at least one k-gonal face, $k \geqslant 8$), Proposition 4.6 (the case if there is one heptagonal face and other faces are k-gonal, $k \leqslant 6$), and Corollary 4.2 (the cases if there are two faces such that each of them has at least 7 edges) we have the statement for polyhedra with $V \geqslant 48$ faces. For polyhedra with $24 \leqslant V \leqslant 46$ vertices the inequality holds by direct computation, see Table 2. This completes the proof of Theorem 2.4.

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