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Finite group actions on the genus-2 surface, geometric generators and extendability

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Dedicated to Bruno Zimmermann on his 70th birthday

Abstract. *For each orientation-preserving finite group action on the closed orientable surface of genus 2, we determine its extendability over the handlebody of geuns 2 and over the 3-sphere; to do this we first find geometric generators.*

Keywords: symmetry of surface, symmetry of 3-sphere, extendable action.
MS Classification 2010: 57M60, 57S17, 57S25.

1. Introduction

Closed orientable surfaces are one of the most ordinary geometric and physical subjects to us, since they stay in our 3-dimensional space everywhere in various manners. The study of symmetries on closed orientable surfaces is also a classical topic in mathematics.

Let Σ_2 be the orientable closed surface of genus 2. If a finite group action G on Σ_2 can also act on the pair (Σ_2, S^3) for some embedding $e : \Sigma_2 \to S^3$, which is to say, for each $h \in G$ we have $h \circ e = e \circ h$, then we call such a group action on Σ_2 extendable over S^3. Similarly one can define the extendability over a genus 2 handlebody V_2.

There are many papers on such extending problems, for extending over a handlebody see [3, 7] and over S^3 see [2, 5, 6], and the references therein.

We determine extendability for all orientation-preserving finite group actions on Σ_2. To do this, for each finite group G acting on Σ_2, we first need to find geometric generators for G, that is to exhibit each generator as a primary and explicit symmetry (like rotations, reflections and antipodal maps, and their compositions), then check if those geometric generators are extendable. To find geometric generators itself is an interesting piece in the study of surfaces symmetries.

We start from the Table 1 below which is copied from [1]. Table 1 gives

G	$\lvert G\rvert$	Σ_2/G	presentation
$\mathbb{Z}_2$	2	$(S^2; 2, 2, 2, 2, 2, 2)$	$\langle x : x^2 = 1\rangle$
$\mathbb{Z}_2$	2	$(T^2; 2, 2)$	$\langle x : x^2 = 1\rangle$
$\mathbb{Z}_3$	2	$(S^2; 3, 3, 3, 3)$	$\langle x : x^3 = 1\rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	$(S^2; 2, 2, 2, 2, 2)$	$\langle x, y : x^2 = y^2 = 1 = [x, y] = 1\rangle$
$\mathbb{Z}_4$	4	$(S^2; 2, 2, 4, 4)$	$\langle x : x^4 = 1\rangle$
$\mathbb{Z}_5$	5	$(S^2; 5, 5, 5)$	$\langle x : x^5 = 1\rangle$
$\mathbb{Z}_6$	6	$(S^2; 3, 6, 6)$	$\langle x : x^6 = 1\rangle$
$\mathbb{Z}_6$	6	$(S^2; 2, 2, 3, 3)$	$\langle x : x^6 = 1\rangle$
D_3	6	$(S^2; 2, 2, 3, 3)$	$\langle x, y : x^2 = y^3 = 1, xyx^{-1} = y^{-1}\rangle$
$\mathbb{Z}_8$	8	$(S^2; 2, 8, 8)$	$\langle x : x^8 = 1\rangle$
$\widetilde{D_2}$	8	$(S^2; 4, 4, 4)$	$\langle x, y : x^4 = y^4 = 1, x^2 = y^2, xyx^{-1} = y^{-1}\rangle$
D_4	8	$(S^2; 2, 2, 2, 4)$	$\langle x, y : x^2 = y^4 = 1, xyx^{-1} = y^{-1}\rangle$
$\mathbb{Z}_{10}$	10	$(S^2; 2, 5, 10)$	$\langle x : x^{10} = 1\rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	12	$(S^2; 2, 6, 6)$	$\langle x, y : x^2 = y^6 = [x, y] = 1\rangle$
$D_{4,3,-1}$	12	$(S^2; 3, 4, 4)$	$\langle x, y : x^4 = y^3 = 1, xyx^{-1} = y^{-1}\rangle$
D_6	12	$(S^2; 2, 2, 2, 3)$	$\langle x, y : x^2 = y^6 = 1, xyx^{-1} = y^{-1}\rangle$
$D_{2,8,3}$	16	$(S^2; 2, 4, 8)$	$\langle x, y : x^2 = y^8 = 1, xyx^{-1} = y^3\rangle$
$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$	24	$(S^2; 2, 4, 6)$	$\langle x, y, z, w : x^2 = y^2 = z^2 = w^3 = [y, z] = [y, w] = [z, w] = 1, xyx^{-1} = y, xzx^{-1} = zy, xwx^{-1} = w^{-1}\rangle$
$SL_2(3)$	24	$(S^2; 3, 3, 4)$	$\langle x, y : x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\rangle$
$GL_2(3)$	48	$(S^2; 2, 3, 8)$	$\langle x, y : x = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}\rangle$

Table 1: Abstract finite group actions on Σ_2.

all the orientation-preserving finite group actions on Σ_2, including their orders, orbifold information, and group presentations. We remark that in this classification, two group actions are equivalent if and only if they differ by a conjugation via some topological homeomorphism of the Σ_2.

We need to explain more before we can state our results.

If an action of G is orientation-preserving on Σ_2 and is extendable over V_2, then the action of G on V_2 must also be orientation-preserving. However, if it is extendable over S^3, then there are two possibilities: (1) all elements $g \in G$ preserve the orientation of S^3, or (2) some element $g \in G$ reverses the orientation of S^3. We use the symbol "H" to denote an action which extends over V_2, use "+" to denote an action which extends over S^3 of type (1), and use "−" to denote an action which extends over S^3 of type (2).

G	Σ_2/G	generators	extendability
$\mathbb{Z}_2$	$(S^2; 2,2,2,2,2,2)$	$\rho_{2,1}$	$H, +$
$\mathbb{Z}_2$	$(T^2; 2,2)$	$\rho_{2,2}$	$H, +, -$
$\mathbb{Z}_3$	$(S^2; 3,3,3,3)$	ρ_3	$H, +$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(S^2; 2,2,2,2,2)$	$\rho_{2,1}, \rho_{2,2}$	$H, +, -$
$\mathbb{Z}_4$	$(S^2; 2,2,4,4)$	ρ_4	$H, -$
$\mathbb{Z}_5$	$(S^2; 5,5,5)$	ρ_5	$\varnothing$
$\mathbb{Z}_6$	$(S^2; 3,6,6)$	$\rho_{6,2}$	$-$
$\mathbb{Z}_6$	$(S^2; 2,2,3,3)$	$\rho_{6,1}$	$H, +$
D_3	$(S^2; 2,2,3,3)$	$\rho_{2,2}, \rho_3$	$H, +$
$\mathbb{Z}_8$	$(S^2; 2,8,8)$	ρ_8	$\varnothing$
$\widetilde{D_2}$	$(S^2; 4,4,4)$	ρ_4, ρ_4'	$\varnothing$
D_4	$(S^2; 2,2,2,4)$	$\rho_4, \rho_{2,2}$	$H, -$
$\mathbb{Z}_{10}$	$(S^2; 2,5,10)$	ρ_{10}	$\varnothing$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$(S^2; 2,6,6)$	$\rho_{2,1}, \rho_{6,2}$	$-$
$D_{4,3,-1}$	$(S^2; 3,4,4)$	ρ_3, ρ_4	$-$
D_6	$(S^2; 2,2,2,3)$	$\rho_{2,2}, \rho_{6,1}$	$H, +$
$D_{2,8,3}$	$(S^2; 2,4,8)$	$\rho_{2,2}, \rho_8$	$\varnothing$
$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$	$(S^2; 2,4,6)$	$\rho_{2,2}, \rho_{2,1}, \rho_{2,2}', \rho_3$	$-$
$SL_2(3)$	$(S^2; 3,3,4)$	ρ_3, ρ_4	$\varnothing$
$GL_2(3)$	$(S^2; 2,3,8)$	ρ_3, ρ_8	$\varnothing$

Table 2: Geometric generators and extendability of finite group actions on Σ_2.

Our main result is:

THEOREM 1.1. *For each orientation-preserving finite group action on the surface Σ_2,*

(1) its geometric generators, whose descriptions will be given in Section 2, are given in the third column of Table 2;

(2) its extendability is given in the last column of Table 2.

Here are some remarks about the symbols in Table 2:

1. We use ρ_n to denote a periodic map on Σ_2 of order n;
2. If there are more than one periodic maps, which are not conjugate to each other, then we denote them by $\rho_{n,1}, \rho_{n,2}, ...$;

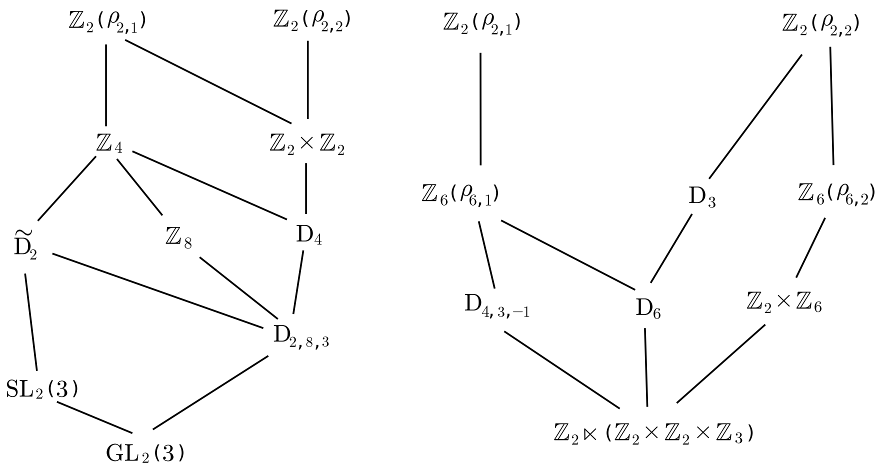

Figure 1

3. As the generators of $\widetilde{D_2}$, ρ_4 and ρ_4' are two different periodic maps on Σ_2, but they are conjugate to each other. The same for the symbols $\rho_{2,2}$ and $\rho_{2,2}'$ as the generators of $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$.

We finish the introduction by the following very useful fact:

If a group action is extendable of some type then its subgroups are also extendable, see Figure 1 for some subgroup inclusions. Two groups joined with an edge indicates that the upper one is a subgroup of the lower one (as group actions).

2. Examples

Example 2.1: This example is mostly from [2].

Figure 2(a) gives a realization of the actions of $\rho_{2,1}$ and $\rho_{2,2}$ on (Σ_2, S^3). This gives the examples of $\mathbb{Z}_2 \times \mathbb{Z}_2(H,+)$, $\mathbb{Z}_2(\rho_{2,1})(H,+)$ and $\mathbb{Z}_2(\rho_{2,2})(H,+)$.

Figure 2(b) gives a realization of the $\rho_{2,2}$-action on (Σ_2, S^3). The restriction of this action on each S^2 centered at O is an antipodal map, and if we view S^3 as the union of such S^2's together with O and ∞, then the action gives the orientation-reversing map of S^3 of order 2 with fixed points O and ∞. Here we view Σ_2 as the horizontal plane with two handles. This gives the example of $\mathbb{Z}_2(\rho_{2,2})(-)$.

Figure 3 gives a realization of the ρ_3 and $\rho_{2,1}$-actions on (Σ_2, S^3). The two actions commute and their composition is $\rho_3\rho_{2,1} = \rho_{6,1}$. This gives the example of $\mathbb{Z}_3(\rho_3)(H,+)$ and $\mathbb{Z}_6(\rho_{6,1})(H,+)$.

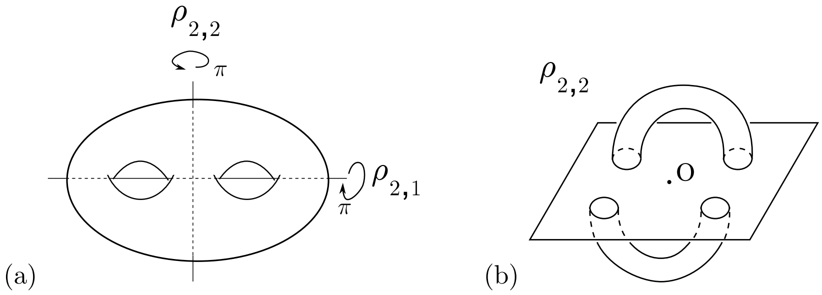

Figure 2

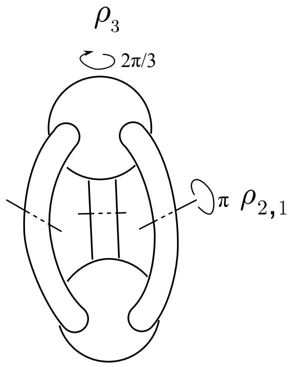

Figure 3

Figure 4(a) gives a realization of the ρ_4-action on (Σ_2, S^3). This action is the composition of a $\pi/2$-rotation together with a reflection about the horizontal plane. Here we view Σ_2 as the horizontal plane with two handles. This gives the example of $\mathbb{Z}_4(-)$.

Figure 4(b) gives a realization of the ρ_4-action on the handlebody V_2 as a solid 3-ball with two pairs of opposite disks identified. This gives the example of $\mathbb{Z}_4(H)$. This ρ_4 together with a $\rho_{2,2}$ give the example of $D_4(H)$.

Figure 5(a) gives a realization of the ρ_8-action on Σ_2 as an octagon with opposite sides identified.

Figure 5(b) gives a realization of the ρ_{10}-action on Σ_2. This also gives the $\rho_5 = \rho_{10}^2$ action.

Example 2.2: This example is mostly form [6]. Let S^3 be the unit sphere in $\mathbb{C}^2$:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

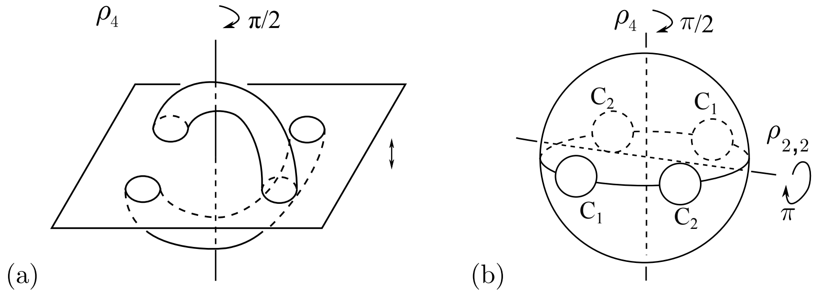

Figure 4

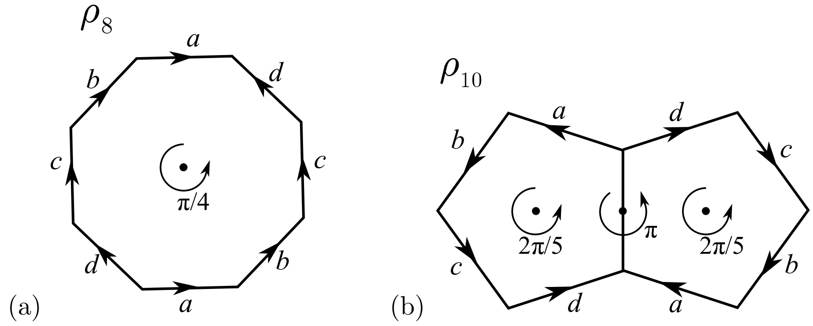

Figure 5

Let

$$a_m = (e^{\frac{m\pi}{2}i}, 0), \quad m = 0, 1, 2, 3,$$
$$b_n = (0, e^{\frac{n\pi}{3}i}), \quad n = 0, 1, \cdots, 5.$$

Connect each a_{2l} to each b_{2k} by the shortest geodesic in S^3, and connect each a_{2l+1} to each b_{2k+1} by the shortest geodesic in S^3, where $l = 0, 1$ and $k = 0, 1, 2$. Then we get two graphs $\Gamma, \Gamma' \in S^3$. Γ contains a_{2l}, b_{2k}, and Γ' contains a_{2l+1}, b_{2k+1}. Each graph has 5 vertices (3 of them are of degree 2) and 6 edges. They are in the dual positions as in Figure 6 where graphs have been projected into the 3-dimensional Euclidean space $E^3 = S^3 - \{(-1, 0)\}$; the vertex a_2 is at infinity.

Roughly speaking, there is a Σ_2 embedded at the "middle position" between the Γ and Γ', as a Heegaard surface bounding two handlebodies, so if a group action keeps $\Gamma \cup \Gamma'$ invariant, then this group acts on the pair (Σ_2, S^3). For details of such an embedding one can see [6].

Now we give some actions on S^3:

$$\begin{aligned} x &: (z_1, z_2) \mapsto (\bar{z_1}, \bar{z_2}) \\ y &: (z_1, z_2) \mapsto (-z_1, z_2) \\ z &: (z_1, z_2) \mapsto (-i\bar{z_1}, z_2) \\ w &: (z_1, z_2) \mapsto (z_1, e^{\frac{2}{3}\pi i} z_2) \end{aligned}$$

It is easy to check that all these maps keep $\Gamma \cup \Gamma'$ invariant, so they all act on the pair (Σ_2, S^3). Furthermore, x, y, w leave both Γ and Γ' invariant, and they preserve the orientation of S^3; z interchanges Γ and Γ', and it reverses the orientation of S^3, so we conclude that all these maps preserve the orientation of Σ_2. They generate the group

$$\begin{aligned} \langle x, y, z, w | x^2 = y^2 = z^2 = w^3 = [y, z] = [y, w] = [z, w] = 1, \\ xyx^{-1} = y, xzx^{-1} = zy, xwx^{-1} = w^{-1} \rangle \\ \cong \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3). \end{aligned}$$

So it gives the example for $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)(-)$.

Note that

(1) $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the subgroup generated by y and z, so it also gives the example for $\mathbb{Z}_2 \times \mathbb{Z}_2(-)$.

(2) $\mathbb{Z}_2 \times \mathbb{Z}_6$ is the subgroup generated by y, z, and w, so it also gives the example for $\mathbb{Z}_2 \times \mathbb{Z}_6(-)$.

(3) Z_6 is the subgroup generated by zw, so it also gives the example for $\mathbb{Z}_6(-)$.

(4) D_3 is the subgroup generated by x and w, so it also gives the example for $D_3(H, +)$.

(5) D_4 is the subgroup generated by x, y and z, so it also gives the example for $D_4(-)$.

(6) D_6 is the subgroup generated by x, y and w, so it also gives the example for $D_6(H, +)$.

(7) $D_{4,3,-1}$ is the subgroup generated by xz and w; we have

$$D_{4,3,-1} \cong \langle xz, w | (xz)^4 = w^3 = 1, (xz)w(xz)^{-1} = w^{-1} \rangle,$$

so it also gives the example for $D_{4,3,-1}(-)$.

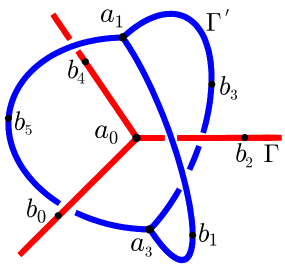

Figure 6

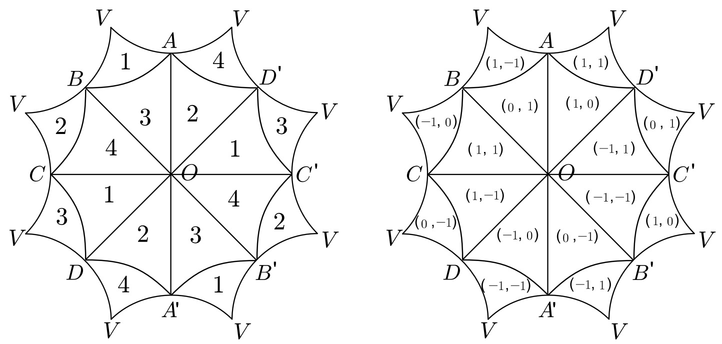

Figure 7

EXAMPLE 2.3: In this part, we provide some intuition of some actions on Σ_2. Although these actions are not extendable in any sense, we can see the group structures more directly.

First let us consider Σ_2 as the hyperbolic octagon with opposite sides identified. This octagon has all eight corners $\pi/4$, and can be divided into 16 hyperbolic equilateral triangles, each of which has inner angle $\pi/4$.

There is an obvious $\pi/4$ rotation around the center point O. Now we describe another order 3 rotation ρ_3. If we lift it to the universal cover of Σ_2, then it is a $2\pi/3$ rotation around the center of the triangle VBA. This action will permute these triangles as:

$$\begin{aligned} VBA &\mapsto BAV \\ OCD &\mapsto DOC \\ OC'D' &\mapsto D'OC' \\ A'VB' &\mapsto VB'A' \end{aligned}$$

$$
\begin{aligned}
&ABO \mapsto VA'D \mapsto B'VC' \mapsto ABO \\
&AOD' \mapsto VD'C' \mapsto B'C'O \mapsto AOD' \\
&BCO \mapsto A'OD \mapsto VDC \mapsto BCO \\
&VCB \mapsto B'OA' \mapsto AD'V \mapsto VCB
\end{aligned}
$$

If we map each triangle to a letter in $\{1, 2, 3, 4\}$ as in the left side of the picture, then this order 3 action induces a permutation

$$(2, 3, 4) \in S_4.$$

Note that the order 8 rotation ρ_8 induces a permutation

$$(1, 2, 3, 4).$$

Since $(2, 3, 4)$ and $(1, 2, 3, 4)$ generates the whole S_4, we have a surjective group homomorphism:

$$\langle \rho_3, \rho_8 \rangle \to S_4.$$

Furthermore, if we label the regions in Σ_2 with eight non-zero vectors in $\mathbb{Z}_3^2$ as in the right side, then we can check that

$$\rho_3((a, b)) = (a, b) \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

$$\rho_8((a, b)) = (a, b) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

So we have an isomorphism

$$\langle \rho_3, \rho_8 \rangle \cong \langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \rangle = GL_2(3),$$

together with a 2 to 1 surjective homomorphism

$$p : GL_2(3) \to S_4.$$

By take the subgroup $\langle \rho_3, \rho_4 \rangle = \langle \rho_3, \rho_8^2 \rangle$, we see another surjective group homomorphism:

$$
\begin{gathered}
\langle \rho_3, \rho_4 \rangle \to A_4 \\
\rho_3 \mapsto (2, 3, 4) \\
\rho_4 \mapsto (1, 2, 3, 4)^2 = (1, 3)(2, 4).
\end{gathered}
$$

And in fact we have

$$\langle \rho_3, \rho_4 \rangle \cong SL_2(3).$$

Consider the subgroup of S_4

$$\langle (1,2,3,4),(1,2)(3,4)\rangle \cong D_4,$$

its preimage under $p : GL_2(3) \to S_4$ is the $D_{2,8,3}$.

Finally consider the subgroup of S_4

$$\langle (1,2)(3,4),(1,3)(2,4)\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

its preimage under p is the $\widetilde{D_2}$, and $\widetilde{D_2}$ is isomorphic to the unit quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$.

So this example shows the group actions of $GL_2(3)$, $SL_2(3)$, $D_{2,8,3}$ and $\widetilde{D_2}$ on Σ_2.

3. Extendabilities

3.1. Extension to a handlebody

The following lemma follows from results of [7] or [3]; alternatively, it is a consequence of the equivariant Dehn's Lemma ([4]).

LEMMA 3.1. *If a group G acts on some handlebody V orientation-preservingly, and the orbifold $\partial V/G$ is a sphere with no more than* 3 *branch points, then V is in fact the* 3 *ball. So $\partial V/G$ must be $(S^2; n, n)$, $(S^2; 2, 2, n)$, $(S^2; 2, 3, 3)$, $(S^2; 2, 3, 4)$ or $(S^2; 2, 3, 5)$.*

Now we proof the handlebody part of the main theorem:

$\mathbb{Z}_2(\rho_{2,1})(H)$, $\mathbb{Z}_2(\rho_{2,2})(H)$, $\mathbb{Z}_3(H)$, $\mathbb{Z}_2 \times \mathbb{Z}_2(H)$, $\mathbb{Z}_4(H)$, $\mathbb{Z}_6(\rho_{6,1})(H)$ and $D_4(H)$ are from Example 2.1.

$D_3(H)$ and $D_6(H)$ are from Example 2.2.

All the other actions cannot extend to a handlebody by Lemma 3.1.

3.2. Extension to S^3

The results for cyclic group actions are proved in [2], and the existence of the extensions of the cyclic group actions is actually proved also in Section 2. In the following we consider only the non-cyclic group actions.

By Lemma 2.4 of [5], if some G-action on Σ_2 with orbifold Σ_2/G a sphere with no more than 4 singular points, then this action extends to S^3 orientation-preservingly implies that it also extends to some handlebody. So in this case if we do not have $G(H)$, we can not have $G(+)$.

$\mathbb{Z}_2 \times \mathbb{Z}_2(+)$ is from Example 2.1. $\mathbb{Z}_2 \times \mathbb{Z}_2(-)$ is from Example 2.2.

$D_3(+)$ is from Example 2.2.

$D_3(-)$ does not extend: otherwise, suppose $D_3 = \langle \rho_2, \rho_3 \rangle$ acts on S^3, where ρ_2 reverse the orientation of S^3. Denote by Θ_1 and Θ_2 the two 3-orbifold bounded by $X = \Sigma_2/\mathbb{Z}_3 = S^2(3,3,3,3)$, and by $\{A, B, C, D\}$ the four branched points on X. By applying Smith theory, we may suppose that two branched arcs in Θ_1 are AB and CD, and two branched arcs in Θ_2 are BC and DA , see Figure 8. Note the induced involution $\bar{\rho_2}$ on X is a π-rotation about two ordinary points and interchanges Θ_1 and Θ_2. Since $\bar{\rho_2}(A) \neq A$, if $\bar{\rho_2}$ interchanges A and B, $\bar{\rho_2}$ will keep the singular arc AB invariant; if $\bar{\rho_2}$ interchanges the pairs (A, B) and (C, D), then $\bar{\rho_2}$ interchanges the singular arcs AB and CD. In either case we would have $\bar{\rho_2}(\Theta_1) = \Theta_1$ which is a contradiction.

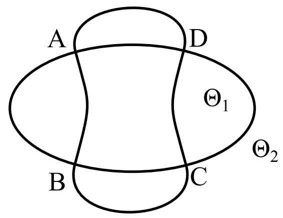

Figure 8

$\widetilde{D_2}(+)$ does not extend because it does not even extend to a handlebody.

$\widetilde{D_2}(-)$ does not extend: otherwise, suppose $\widetilde{D_2}$ acts on S^3, consider the orientation-preserving subgoup, which is an index 2 subgroup and must be isomorphic to $\mathbb{Z}_4$, but there is no $\mathbb{Z}_4(+)$.

So we have $\widetilde{D_2}\{\varnothing\}$. Since $\widetilde{D_2} \subset SL_2(3) \subset GL_2(3)$ and $\widetilde{D_2} \subset D_{2,8,3}$, we also conclude that $SL_2(3)\{\varnothing\}$, $GL_2(3)\{\varnothing\}$ and $D_{2,8,3}\{\varnothing\}$.

$D_4(+)$ does not extend because there is no $\mathbb{Z}_4(+)$ as a subgroup.

$\mathbb{Z}_2 \times \mathbb{Z}_6(+)$ and $D_{4,3,-1}(+)$ do not extend because they cannot even extend to a handlebody.

$D_6(-)$ does not extend because an element of order 2 which reverses the orientation of S^3 together with an element of order 3 will form a subgroup of either $\mathbb{Z}_6(\rho_{6,1})$ or D_3, but there is no $\mathbb{Z}_6(\rho_{6,1})(-)$ or $D_3(-)$ as a subgroup.

$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)(+)$ does not extend because it cannot even extend to a handlebody.

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