

Classification of parabolic generating pairs of Kleinian groups with two parabolic generators

SHUNSUKE AIMI, DONGHI LEE, SHUNSUKE SAKAI
AND MAKOTO SAKUMA

Dedicated to Professor Bruno Zimmermann on the occasion of his 70-th birthday

ABSTRACT. *We give an alternative proof to Agol's classification of parabolic generating pairs of non-free Kleinian groups generated by two parabolic transformations. As an application, we give a complete characterisation of epimorphisms between 2-bridge knot groups and a complete characterisation of degree one maps between the exteriors of hyperbolic 2-bridge links.*

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1. Introduction

In [1, Theorem 4.3], Adams proved that a torsion free Kleinian group of cofinite volume is generated by two parabolic transformations if and only if the quotient hyperbolic manifold is homeomorphic to the complement of a 2-bridge link which is not a torus link. This refines the result of Boileau and Zimmermann [14, Corollary 3.3] that a link in S^3 is a 2-bridge link if and only if its link group is generated by two meridians.

In 2002, Agol [3] announced the following classification theorem of non-free Kleinian groups generated by two parabolic transformations, which generalises Adams' result.

THEOREM 1.1. *A non-free Kleinian group Γ is generated by two non-commuting parabolic elements if and only if one of the following holds.*

- (1) Γ is conjugate to the hyperbolic 2-bridge link group, $G(r) \cong \pi_1(S^3 - K(r))$, for some rational number $r = q/p$, where p and q are relatively prime integers such that $q \not\equiv \pm 1 \pmod{p}$.

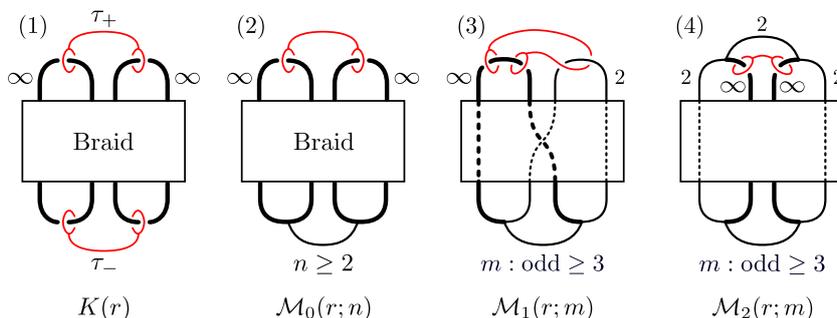


Figure 1: The black graphs illustrate weighted graphs representing 2-bridge links and Heckoid orbifolds, where the thick edges with weight ∞ correspond to parabolic loci and thin edges with integral weights represent the singular set. The red thin graphs represent the parabolic generating pairs of the hyperbolic 2-bridge link groups and the Heckoid groups.

(2) Γ is conjugate to the Heckoid group, $G(r; n)$, for some $r \in \mathbb{Q}$ and some $n \in \frac{1}{2}\mathbb{N}_{\geq 3}$.

Adams also proved that each hyperbolic 2-bridge link groups has only finitely many distinct parabolic generating pairs up to equivalence [1, Corollary 4.1] and moreover that the figure-eight knot group has precisely two such pairs up to equivalence [1, Corollary 4.6]. Here, a *parabolic generating pair* of a non-elementary Kleinian group Γ is an unordered pair of two parabolic transformations that generate Γ . Two parabolic generating pairs $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ of Γ are said to be *equivalent* if $\{\alpha', \beta'\}$ is equal to $\{\alpha^{\epsilon_1}, \beta^{\epsilon_2}\}$ for some $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ up to simultaneous conjugation.

Agol [3] also announced the following theorem, which generalises the above results of Adams. (See Figure 1 for intuitive description and Section 2 for precise description.)

THEOREM 1.2. (1) *Every hyperbolic 2-bridge link group $G(r)$ has precisely two parabolic generating pairs up to equivalence.*

(2) *Every Heckoid group $G(r; n)$ has a unique parabolic generating pair up to equivalence.*

In the last author’s joint paper [7] with Hirotaka Akiyoshi, Ken’ichi Ohshika, John Parker, and Han Yoshida, a full proof to Theorem 1.1 was given. The main purpose of this companion paper of [7] is to give an alternative ‘homological’ proof to Theorem 1.2. As an application, we give a complete characterisation of epimorphisms between 2-bridge knot groups (Theorem 8.1 and Remark 8.2) and a complete characterisation of degree one maps between the exteriors of

hyperbolic 2-bridge links (Theorem 8.3). Since it is proved by Boileau, Boyer, Reid, and Wang [10, Corollary 1.3] that a knot group which is a “homomorph” of a 2-bridge knot group is itself a 2-bridge knot group, Theorem 8.1 gives an alternative “precise” proof for 2-bridge knot groups to Simon’s conjecture that a knot group admits epimorphisms onto only finitely many knot groups, which was established by Agol and Liu [4] in full generality. Theorem 8.1 has been already used in [31, 45, 46]. We apologise the delay in writing up the result.

In the preceding paper [35] by the second and last authors, a proof to Theorem 1.2(1) for the genus-one 2-bridge knot groups was given by using the small cancellation theory. In the first author’s master thesis [5] supervised by the last author, a proof to the result for all hyperbolic 2-bridge knots was given by using the Alexander invariants. In this paper, we give a simple proof for all hyperbolic 2-bridge link groups and for all Heckoid groups, by using the homology of the double branched coverings.

It should be noted that the parabolicity of the generating pairs is essential in Theorem 1.2. In fact, Heusener and Porti [30] proved that every hyperbolic knot admits infinitely many generating pairs up to Nielsen equivalence. Moreover, the same conclusion for torus knots, which include the non-hyperbolic 2-bridge knots, had been proved by Zieschang [47].

Our proof of Theorem 1.2 is based on Boileau’s suggestion [9] to use the well-known fact that, for a parabolic generating pair $\{\alpha, \beta\}$ of Γ , there is an order 2 elliptic element h in the normaliser $N(\Gamma)$ of Γ in $\text{Isom}^+(\mathbb{H}^3)$ such that $(h\alpha h^{-1}, h\beta h^{-1}) = (\alpha^{-1}, \beta^{-1})$ (see Propositions 3.1 and 3.2). Since we can determine the isometry groups of the hyperbolic 2-bridge link complements and the Heckoid orbifolds (Propositions 4.1 and 7.2), Boileau’s suggestion leads us to a finite list of possible parabolic generating pairs (see Propositions 4.4, 4.5, and Section 7). For the hyperbolic 2-bridge link groups, we can exclude all fake parabolic generating pairs through simple calculations on the homology of the double branched coverings (Sections 5 and 6). For the Heckoid groups, we can also do so through a simple argument by using the orbifold theorem and by using natural epimorphisms from Heckoid groups onto the π -orbifold groups of 2-bridge links (Section 7). This is the strategy of our proof of the main Theorem 1.2.

At the end of the introduction, we note that Agol [3] obtained Theorem 1.2(1) as a corollary of the following much stronger theorem.

THEOREM 1.3. *For any hyperbolic 2-bridge link group $G(r)$ and for any meridian pair in $G(r)$ which is not equivalent to the upper nor lower meridian pair, the subgroup of $G(r)$ generated by the meridian pair is a free group.*

He proved this theorem through a beautiful ping-pong argument applied to the ideal boundary of the universal covering space of the natural local CAT(0)

cubical complex of the 2-bridge link complement, which is constructed from a reduced alternating diagram of the 2-bridge link.

This paper is organised as follows. In Section 2, we give a detailed description of Theorem 1.2. In Section 3, we recall Boileau's key suggestion (Proposition 3.1) which relates parabolic generating pairs of hyperbolic 2-bridge links to strong inversions of the links, and present its extension to Heckoid orbifolds (Proposition 3.2). In Section 4, we classify the strong inversions of hyperbolic 2-bridge links and list all possible parabolic generating pairs of the link groups (Propositions 4.4 and 4.5). In Sections 5 and 6, we show that all possible parabolic generating pairs, except for the upper and lower meridian pairs, are not generating pairs, by using the homology of the double branched coverings (Propositions 5.2 and 6.1). Thus the proof of the first assertion of Theorem 1.2 is completed in this section. In Section 7, we prove the second assertion of Theorem 1.2, after determining the orientation-preserving isometry groups of Heckoid orbifolds (Proposition 7.2) by using [7, Proposition 12.6]. In Section 8, we give a characterisation of epimorphisms between 2-bridge knot groups (Theorem 8.1) and a characterisation of degree one maps between the exteriors of hyperbolic 2-bridge links (Theorem 8.3), by using the main Theorem 1.2.

2. Precise description of Theorem 1.2

We first give a precise description of Theorem 1.2(1). For a rational number $r = q/p$, let $K(r)$ be the 2-bridge link of slope r . Recall that $K(r)$ is hyperbolic if and only if $q \not\equiv \pm 1 \pmod{p}$. When r satisfies this condition, the *hyperbolic 2-bridge link group* $G(r)$ is defined to be the Kleinian group, which is unique up to conjugation, such that $\mathbb{H}^3/G(r)$ is homeomorphic to $S^3 - K(r)$ as an oriented manifold. Thus $G(r)$ is isomorphic to $\pi_1(E(K(r)))$, where $E(K(r)) = S^3 - \text{int } N(K(r))$ is the *exterior* (the complement of an open regular neighbourhood) of $K(r)$. The Kleinian group $G(r)$ is generated by two parabolic transformations, and it is proved by Adams [1, Theorem 4.3] that any parabolic generating pair of $G(r)$ consists of meridians. Here an element of $G(r) \cong \pi_1(E(K(r)))$ is called a *meridian* if it is freely homotopic to a meridional loop in $\partial E(K(r))$, i.e. a simple loop in $\partial E(K(r))$ that bounds an essential disc in $N(K(r))$. This implies that any parabolic generating pair $\{\alpha, \beta\}$ is represented by an arc properly embedded in $E(K(r))$, together with a pair of meridional loops on $\partial E(K(r))$ passing through the endpoints of the arc. (See [35, Section 2] for more detailed explanation.) It is obvious that the meridian pair represented by the upper tunnel τ_+ (resp. the lower tunnel τ_-) forms a parabolic generating pair of $G(r) \cong \pi_1(E(K(r)))$, and it is called the *upper meridian pair* (resp. *lower meridian pair*) of $\pi_1(E(K(r)))$. See Figure 1(1), where the black bold graph represent a 2-bridge link, and the two red thin graphs represent the upper and lower meridian pairs, respectively.

The ‘weight’ ∞ of the 2-bridge link indicates that we consider the link exterior and that its boundary tori form the parabolic locus of the hyperbolic structure. For precise definitions of 2-bridge links and upper/lower tunnels, please see the companion paper [7, Section 2]. *Theorem 1.2(1) says that, for each hyperbolic 2-bridge link group $G(r)$, (i) the upper meridian pair and the lower meridian pair are the only parabolic generating pairs of $G(r)$, and (ii) they are not equivalent.*

REMARK 2.1: The upper/lower tunnels are actually *unknotting tunnels*, namely the complement of an open regular neighbourhood of the tunnel is a genus 2 handlebody. Moreover, the upper (resp. lower) meridian pair is the image of a free generating pair of the fundamental group of the complementary handlebody of the lower (resp. upper) tunnel. Note that we do not know a priori that the tunnel associated to a parabolic generating pair is an unknotting tunnel.

Next, we give a precise description of Theorem 1.2(2). The Heckoid groups were first introduced by Riley [40] as an analogy of the classical Hecke group, and it was reformulated by Lee and Sakuma [34] following Agol [3] as the orbifold fundamental groups of the Heckoid orbifolds illustrated in Figure 1(2)-(4). (See [12, 13, 18] for basic terminologies and facts concerning orbifolds.) These figures illustrate weighted graphs (S^3, Σ, w) whose explicit descriptions are given by Definition 7.1. For each weighted graph (S^3, Σ, w) in the figure, let (M_0, P) be the pair of a compact 3-orbifold M_0 and a compact 2-suborbifold P of ∂M_0 determined by the rules described below. Let Σ_∞ be the subgraph of Σ consisting of the edges with weight ∞ , and let Σ_s be the subgraph of Σ consisting of the edges with integral weight.

1. The underlying space $|M_0|$ of M_0 is the complement of an open regular neighbourhood of the subgraph Σ_∞ .
2. The singular set of M_0 is $\Sigma_0 := \Sigma_s \cap |M_0|$, where the index of each edge of the singular set is given by the weight $w(e)$ of the corresponding edge e of Σ_s .
3. For an edge e of Σ_∞ , let P be the 2-suborbifold of ∂M_0 defined as follows.
 - (a) In Figure 1(2), P consists of two annuli in ∂M_0 whose cores, respectively, are meridians of the two edges of Σ_∞ .
 - (b) In Figure 1(3), P consists of an annulus in ∂M_0 whose core is a meridian of the single edge of Σ_∞ .
 - (c) In Figure 1(4), P consists of two copies of the annular orbifold $D^2(2, 2)$ (the 2-orbifold with underlying space the disc and with two cone points of index 2) in ∂M_0 each of which is bounded by a meridian of an edge of Σ_∞ .

By [34, Lemmas 6.3 and 6.6], the orbifold pair (M_0, P) is a Haken pared orbifold (see [13, Definition 8.3.7]) and admits a unique complete hyperbolic structure, which is geometrically finite (see [34, Proposition 6.7], [7, Section 3]). Namely there is a geometrically finite Kleinian group Γ , unique up to conjugation, such that $M = \mathbb{H}^3/\Gamma$ is homeomorphic to the interior of the compact orbifold M_0 , such that P represents to the parabolic locus. To be more precise, there are positive constants δ and μ , such that

$$(M_0, P) \cong (\text{thick}_\mu(C_\delta(M)), \partial(\text{thick}_\mu(C_\delta(M))) \cap (\text{thin}_\mu(C_\delta(M))), \quad (1)$$

where $C_\delta(M)$ is the closed δ -neighbourhood of the convex core $C(M)$ of M , and $\text{thick}_\mu(C_\delta(M))$ and $\text{thin}_\mu(C_\delta(M))$ are the μ -thick part and μ -thin part (see [13] for terminology). The μ -thin part $\text{thin}_\mu(C_\delta(M))$ consists of only cuspidal components and it is isomorphic to $P \times [0, \infty)$. We denote by P_α and P_β the components of P corresponding to the parabolic transformations α and β , respectively, i.e., P_α (resp. P_β) is the component of P which is the image of a subsurface of an α -invariant (resp. β -invariant) horosphere. The pair (M_0, P) is also regarded as a relative compactification of the pair consisting of a non-cuspidal part of M and its boundary (see [7, Section 5]).

We denote the pared orbifold $\mathcal{M} := (M_0, P)$ by $\mathcal{M}_0(r; n)$, $\mathcal{M}_1(r; m)$, or $\mathcal{M}_2(r; m)$ according as it is described by the weighted graph in Figure 1(2), (3), or (4). We also denote the Kleinian group Γ that uniformises the pared orbifold \mathcal{M} by $\pi_1(\mathcal{M})$. (More generally, we use the symbol $\pi_1(\cdot)$ to denote the orbifold fundamental group.) For $r = q/p \in \mathbb{Q}$ and $n \in \frac{1}{2}\mathbb{N}_{\geq 3}$, the *Heckoid group* $G(r; n)$ is the Kleinian group that is defined as follows.

$$G(r; n) \cong \begin{cases} \pi_1(\mathcal{M}_0(r; n)) & \text{if } n \in \mathbb{N}_{\geq 2}, \\ \pi_1(\mathcal{M}_1(\hat{r}; m)) & \text{if } n = m/2 \text{ for some odd } m \geq 2 \text{ and if } p \text{ is odd,} \\ \pi_1(\mathcal{M}_2(\hat{r}; m)) & \text{if } n = m/2 \text{ for some odd } m \geq 2 \text{ and if } p \text{ is even,} \end{cases}$$

where \hat{r} is defined from $r = q/p$ by the following rule.

$$\hat{r} = \begin{cases} \frac{q/2}{p} & \text{if } p \text{ is odd and } q \text{ is even,} \\ \frac{(p+q)/2}{p} & \text{if } p \text{ is odd and } q \text{ is odd,} \\ \frac{q}{p/2} & \text{if } p \text{ is even.} \end{cases}$$

For the background of this rather complicated definition, please see [7, 35].

By [1, p.197] (cf. [7, Lemma 7.2]), any parabolic generating pair $\{\alpha, \beta\}$ of the Heckoid group $G(r; n)$ consists of primitive elements. Since the parabolic locus $P_\alpha \cong P_\beta$ is an annular orbifold $S^1 \times I$ or $D^2(2, 2)$, this implies that α and β are freely homotopic to the unique (up to isotopy) essential simple loop in P_α and P_β , respectively. Thus they are freely homotopic to a meridional loop of the corresponding edge of Σ_∞ . Since $G(r; n)$ is identified with a quotient

of the usual fundamental group $\pi_1(|M_0| - \Sigma_0)$, it follows that any parabolic generating pair is represented by a graph embedded in $|M_0| - \Sigma_0$ consisting of two meridional loops of Σ_∞ and an arc joining them. It follows from the definition of Heckoid groups that the pairs of parabolic elements represented by the red graphs in Figure 1(2)-(4) are generating pairs (see [34, Section 3], [7, Proposition 3.3]). *Theorem 1.2(2) says that each Heckoid group $G(r; n)$ admits a unique parabolic generating pair, and it is illustrated by Figure 1 (2), (3) or (4) according to the type of $G(r; n)$.*

In the special case where $r \in \mathbb{Z}$, the Heckoid group $G(r; n)$ with $n = \frac{m}{2} \in \frac{1}{2}\mathbb{N}_{\geq 3}$ is conjugate to $G(0; n)$, and it is a Fuchsian group, which is essentially equal to the classical *Hecke group* $H(m) \cong \pi_1(S^2(2, m, \infty))$, generated by the following matrices ([29]):

$$A_m := \begin{pmatrix} 1 & 2 \cos \frac{\pi}{m} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Q := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

To be precise, $G(r; n) \cong G(0; n)$ is conjugate to the subgroup of $H(m)$, with $n = \frac{m}{2}$, generated by $\{A_m, QA_mQ^{-1}\}$. Thus according to whether (a) n is an integer ≥ 2 or (b) a half-integer $n = m/2$ (m : odd ≥ 3), the Heckoid group $G(0; n)$ is conjugate to (a) the index 2 subgroup of the Hecke group $H(m)$ isomorphic to $\pi_1(S^2(n, \infty, \infty))$ or (b) the Hecke group $H(m)$, with $m = 2n$. These follow from the following topological observation.

1. For every $r \in \mathbb{Z}$ and $n \in \mathbb{N}_{\geq 2}$, we have

$$\mathcal{M}_0(r; n) \cong \mathcal{M}_0(0; n) \cong S^2(n, \infty, \infty) \times I,$$

and $S^2(n, \infty, \infty)$ is a double orbifold covering of $S^2(2, m, \infty)$ with $m = 2n$.

2. For every $r \in \mathbb{Z}$ and an odd integer $m \in \mathbb{N}_{\geq 3}$, we have $\mathcal{M}_1(\hat{r}; m) \cong \mathcal{M}_1(0; m) \cong S^2(2, m, \infty) \times I$.

In [32, Proposition 4.2], Knapp completely determined when a given pair of non-commuting parabolic transformations of the hyperbolic plane generate a discrete group, by using Poincaré’s theorem on fundamental polyhedra. In our terminology, it is described as follows, which particularly implies Theorem 1.2(2) for the special case where $r \in \mathbb{Z}$.

PROPOSITION 2.2 ([32, Proposition 4.2]). *A non-elementary non-free Fuchsian group is generated by two parabolic transformations, if and only if it is conjugate to the Fuchsian group $G(0; n)$ for some $n \in \frac{1}{2}\mathbb{N}_{\geq 3}$. Moreover, $\{A_m, QA_mQ^{-1}\}$, where $m = 2n$, is the unique parabolic generating pair of $G(0; n)$, up to equivalence.*

3. Inverting elliptic elements for the parabolic generating pairs

In this section, we recall Boileau's key suggestion [9], and present its slight extension. His suggestion is to use the following well-known fact. Let $\Gamma = \langle \alpha, \beta \rangle$ be a non-elementary Kleinian group generated by two parabolic transformations α and β , and $M = \mathbb{H}^3/\Gamma$ the quotient hyperbolic 3-manifold or orbifold. Let η be the geodesic joining the parabolic fixed points of α and β , and let h be the π -rotation around η . Then we have

$$(h\alpha h^{-1}, h\beta h^{-1}) = (\alpha^{-1}, \beta^{-1}). \quad (2)$$

We call h the *inverting elliptic element* for the parabolic generating pair $\{\alpha, \beta\}$ of the Kleinian group Γ . If $h \notin \Gamma$, then h induces an isometric involution on M , whereas if $h \in \Gamma$, then M is an orbifold with non-empty singular set and the axis η projects to a 'geodesic' contained in the subset of the singular set consisting of even degree edges. (By a geodesic (arc) in a complete hyperbolic orbifold, we mean the image of a geodesic (arc) in the universal cover \mathbb{H}^3 in the orbifold.)

We first treat the case where Γ is a hyperbolic 2-bridge link group. We prepare some terminologies. For a link K in S^3 , a *tunnel* for K is an arc, τ , in S^3 such that $\tau \cap K = \partial\tau$. We assume τ intersects a fixed regular neighbourhood $N(K)$ of K in two arcs, each of which forms a radius of a meridian disc of $N(K)$. Then the intersection of τ with the exterior $E(K) = S^3 - \text{int } N(K)$ is an arc properly embedded in $E(K)$. Thus, as described in the first paragraph of Section 2 (cf. [35, Section 2]), it determines a pair of meridians in the link group $\pi_1(E(K))$, up to equivalence. We call it the *meridian pair determined by the tunnel* τ . Here two meridian pairs $\{m_1, m_2\}$ and $\{m'_1, m'_2\}$ of $\pi_1(E(K))$ is said to be *equivalent*, if $\{m'_1, m'_2\}$ is equal to $\{m_1^{\epsilon_1}, m_2^{\epsilon_2}\}$ for some $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ up to simultaneous conjugation.

A *strong inversion* of K is an orientation-preserving involution, which we often denote by the symbol h , of S^3 preserving K setwise such that the fixed point set $\text{Fix}(h)$ is a circle intersecting each component of K in two points. Note that $\text{Fix}(h)$ consists of 2μ tunnels, where μ is the number of components of K . Then the following proposition, which holds a key to the proof of Theorem 1.2(1), was suggested by Boileau [9] (cf. [33, Proposition 2.1]). (See [2, 8, 20] for interesting related results.)

PROPOSITION 3.1. *Let $K(r)$ be a hyperbolic 2-bridge link, and let $\{\alpha, \beta\}$ be a parabolic generating pair of the link group $G(r)$. Then there is a strong inversion, h , of $K(r)$ such that $\{\alpha, \beta\}$ is a meridian pair represented by a tunnel contained in $\text{Fix}(h)$.*

Proof. Though the proof is given in [33], we recall the proof as a warm-up for the treatment of Heckoid groups. By definition, $G(r)$ is a non-elementary

Kleinian group generated by the parabolic transformations α and β . Let h be the inverting elliptic element for the parabolic generating pair $\{\alpha, \beta\}$ of $G(r)$. Since h is an element of the normaliser of $G(r)$ in $\text{Isom}^+(\mathbb{H}^3)$ which does not belong to $G(r)$ (recall that $G(r)$ is torsion free), h descends to an orientation-preserving involution, \bar{h} , of $\mathbb{H}^3/G(r) \cong S^3 - K(r)$. Since each component of $\partial E(K(r))$ corresponds to one of the parabolic loci P_α and P_β (which in turn follows from the fact that α and β generate $H_1(E(K(r)))$), it follows that the restriction of \bar{h} to each of the components of $\partial E(K(r))$ is a hyper-elliptic involution. Hence \bar{h} extends to an involution of the pair $(S^3, K(r))$, which we continue to denote by \bar{h} . Then \bar{h} is a strong inversion of $K(r)$, and $\text{Fix}(\bar{h})$ contains the image $\bar{\eta}$ of the axis η of the inverting elliptic element h . Then, by [1, Theorem 4.3], $\{\alpha, \beta\}$ is the meridian pair represented by the tunnel for $K(r)$ that is obtained as the closure of $\bar{\eta}$ in $\text{Fix}(\bar{h})$. By denoting the strong inversion \bar{h} by h , we obtain the desired result. \square

We next treat the case where Γ is a Heckoid group $G(r; n)$. Let $\mathcal{M} = (M_0, P)$ be the corresponding Heckoid orbifold, and identify M_0 and P with a subspace of the quotient hyperbolic orbifold $M = \mathbb{H}^3/\Gamma$ through the isomorphism (1) in the introduction. As is noted in the introduction (see [1, p.197], [7, Lemma 7.2]), any parabolic generating pair $\{\alpha, \beta\}$ of $G(r; n)$ consists of primitive elements, and α and β are freely homotopic to the unique (up to isotopy) essential simple loops in the annular orbifolds P_α and P_β , respectively. Thus the equivalence class of $\{\alpha, \beta\}$ is uniquely determined by the proper geodesic arc $\bar{\eta} \cap M_0$, where $\bar{\eta}$ is the ‘geodesic’ in the orbifold M obtained as the image of the axis η of the inverting elliptic element h . (Note that if η intersects orthogonally the axis of an even order elliptic transformation in Γ , then the underlying space of $\bar{\eta}$ has an endpoint in an even degree singular locus of the orbifold M .) If $h \notin \Gamma$, then it induces an isometric involution, \bar{h} , on M , and $\bar{\eta}$ is contained in $\text{Fix}(\bar{h})$. If $h \in \Gamma$ then $\bar{\eta}$ is a ‘geodesic edge path’ contained in the subset of the singular set of M consisting of even degree edges. Thus we have the following analogy of Proposition 3.1, which holds a key to the proof of Theorem 1.2(2).

PROPOSITION 3.2. *Consider the Heckoid group $\Gamma = G(r; n)$, and let $\mathcal{M} = (M_0, P)$ be the corresponding Heckoid orbifold. Let $\{\alpha, \beta\}$ be a parabolic generating pair of $G(r; n)$, h the inverting elliptic element for the parabolic generating pair $\{\alpha, \beta\}$, and η the axis of h , and $\bar{\eta}$ be the image of η in $M = \mathbb{H}^3/\Gamma$. Then $\{\alpha, \beta\}$ is represented by $\bar{\eta} \cap M_0$. Moreover the following holds.*

1. *If $h \notin \Gamma$, then it descends to an involution, \bar{h} , of M , such that $\bar{\eta}$ is contained in $\text{Fix}(\bar{h})$.*
2. *If $h \in \Gamma$, then $\bar{\eta} \cap M_0$ is a geodesic edge path joining P_α and P_β , contained in the subset of the singular set of M_0 consisting of even degree edges.*

4. Symmetries of 2-bridge links and all possible generating meridian pairs for 2-bridge link groups

Let $K(r)$ with $r = q/p$ ($q \not\equiv \pm 1 \pmod{p}$) be a hyperbolic 2-bridge link, and let τ_+ and τ_- be the upper and lower tunnel for $K(r)$ (see [7, Section 2] for an explicit definition). In this section, we describe all strong inversions of $K(r)$, up to strong equivalence, and list all possible generating meridian pairs of the 2-bridge link group $G(r)$ by using Proposition 3.1. Here two strong inversions of a link K are said to be *strongly equivalent* if they are conjugate by a homeomorphism of (S^3, K) that is pairwise isotopic to the identity. To this end, we first describe the orientation-preserving isometry group $\text{Isom}^+(S^3 - K(r))$ of the hyperbolic manifold $S^3 - K(r)$.

PROPOSITION 4.1. *For a hyperbolic 2-bridge link $K(r)$ with $r = q/p$ ($q \not\equiv \pm 1 \pmod{p}$), the orientation-preserving isometry group $\text{Isom}^+(S^3 - K(r))$ is given by the following formula.*

$$\text{Isom}^+(S^3 - K(r)) \cong \begin{cases} (\mathbb{Z}_2)^2 & \text{if } q^2 \not\equiv 1 \pmod{p} \\ D_4 \cong (\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2 & \text{if } p \text{ is odd and } q^2 \equiv 1 \pmod{p}, \text{ or} \\ & \text{if } p \text{ is even and } q^2 \equiv p+1 \pmod{2p} \\ (\mathbb{Z}_2)^3 \cong (\mathbb{Z}_2)^2 \times \mathbb{Z}_2 & \text{if } p \text{ is even and } q^2 \equiv 1 \pmod{2p} \end{cases}$$

Here D_4 denotes the order 8 dihedral group, which is regarded as a semi-direct product of $(\mathbb{Z}_2)^2$ and \mathbb{Z}_2 . The subgroups $(\mathbb{Z}_2)^2$ in the formula consist of those isometries which preserve both the upper tunnel τ_+ and the lower tunnel τ_- setwise: the elements in $\text{Isom}^+(S^3 - K(r))$ which are not contained in the characteristic subgroup $(\mathbb{Z}_2)^2$ interchange τ_+ and τ_- .

Proof. By the work of Epstein and Penner [19], every cusped hyperbolic manifold M of finite volume admits a canonical decomposition into hyperbolic ideal polyhedra and the isometry group of M is isomorphic to the combinatorial automorphism group of the canonical decomposition. In [43], a topological candidate, \mathcal{D} , of the canonical decomposition of the cusped hyperbolic manifold $S^3 - K(r)$ was constructed [43, Theorem II.2.5], and the combinatorial automorphism group $\text{Aut}(\mathcal{D})$ of \mathcal{D} was calculated [43, Theorem II.3.2]. (As is noted in [36], there is an error in the proof of Theorem II.3.2, but this does not affect the conclusion of the theorem.) On the other hand, it was proved by Guéritaud [25] (cf. [6, 26]) that \mathcal{D} is combinatorially equivalent to the canonical decomposition. Hence $\text{Isom}^+(S^3 - K(r))$ is isomorphic to the orientation-preserving subgroup $\text{Aut}^+(\mathcal{D})$ of $\text{Aut}(\mathcal{D})$, which is described in [43, Theorem II.3.2]. □

REMARK 4.2: The action of $\text{Isom}^+(S^3 - K(r))$ on $S^3 - K(r)$ extends to an action on $(S^3, K(r))$, and this fact implies that $\text{Isom}^+(S^3 - K(r))$ is isomorphic

to the orientation-preserving symmetry group, $\text{Sym}^+(S^3, K(r))$, the group of diffeomorphisms of the pair $(S^3, K(r))$, up to pairwise isotopy, which preserves the orientation of S^3 . (In fact, it is observed by Riley [39, p.124] that this holds for all hyperbolic knots in S^3 .) We note that the symmetry groups of 2-bridge links had been already obtained, as described below. For 2-bridge knots, it is reported in [24] that Conway calculated the outer-automorphism groups of their knot groups, which is isomorphic to the full symmetry groups of the knots. Bonahon and Siebenmann [15] calculated the full symmetry group of every 2-bridge link, by using the uniqueness of 2-bridge decompositions up to isotopy established by Schubert [44]. Another calculation is given by [42, Theorems 4.1 and 6.1] by using the orbifold theorem. (As is noted in [7, the paragraph preceding Proposition 12.5], though there are misprints in the statement [42, Theorem 4.1], the correct formula can be found in the tables in [42, p.184].)

We now visualise the action of $\text{Isom}^+(S^3 - K(r)) \cong \text{Sym}^+(S^3, K(r))$ on $(S^3, K(r))$ and classify the strong inversions of $K(r)$, up to strong equivalence, generalising and refining the result for the knot case given by [41, Proposition 3.6]. To this end, note that, by virtue of Schubert's classification of 2-bridge links (cf. [16, Chapter 12]), we may assume that the slope $r = q/p$ of the hyperbolic 2-bridge link $K(r)$ satisfies the inequality $0 < q \leq p/2$ and the condition that one of p and q is even. The following lemma is well-known and can be proved by an argument similar to that in [27, Lemma 2].

LEMMA 4.3. *Let p and q be relatively prime integers such that $0 < q \leq p/2$ and that one of p and q is even. Then $r = q/p$ has a unique continued fraction expansion*

$$r = [2b_1, 2b_2, \dots, 2b_n] = \frac{1}{2b_1 + \frac{1}{2b_2 + \dots + \frac{1}{2b_n}}},$$

where b_i is a non-zero integer ($1 \leq i \leq n$). The length n is even or odd according to whether p is odd or even, i.e., $K(r)$ is a knot or a two-component link. Moreover the following hold.

1. Suppose p is odd, i.e., n is even. Then $q^2 \equiv 1 \pmod{p}$ if and only if $b_i = -b_{n+1-i}$ ($1 \leq i \leq n$).
2. Suppose p is even, i.e., n is odd. Then $q^2 \equiv 1 \pmod{2p}$ if and only if $b_i = b_{n+1-i}$ ($1 \leq i \leq n$).

By using Lemma 4.3, we see that every 2-bridge link $K(r)$ admits a $(\mathbb{Z}_2)^2$ -action generated by the two involutions f and h illustrated in Figure 2(1) or (2)

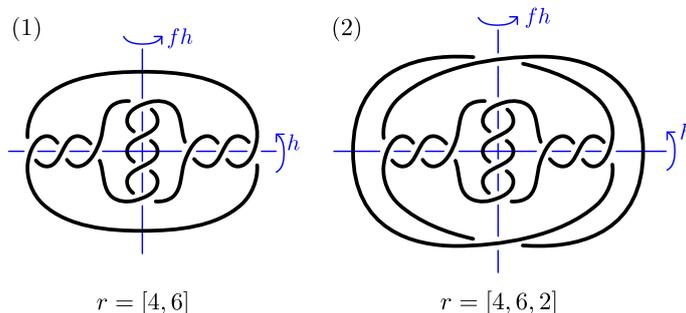


Figure 2: The action of the characteristic subgroup $(\mathbb{Z}_2)^2 < \text{Sym}^+(S^3, K(r))$ for the knot case (1) and for the two component link case (2). In both cases, f is the π -rotation about the axis which intersects the projection plane perpendicularly at the central point (the intersection of the axes of h and fh).

according to whether $K(r)$ is a knot or a 2-component link. If $K(r)$ is hyperbolic, then the $(\mathbb{Z}_2)^2$ -action projects faithfully onto the characteristic subgroup $(\mathbb{Z}_2)^2$ of $\text{Sym}^+(S^3, K(r)) \cong \text{Isom}^+(S^3 - K(r))$. (This can be seen either by appealing to the result of Borel (see [17]) that a finite group action on an aspherical manifold M with centerless fundamental group projects injectively into $\text{Out}(\pi_1(M))$, or by looking at the action on the canonical decomposition \mathcal{D} of $S^3 - K(r)$.) Moreover, any two such $(\mathbb{Z}_2)^2$ -actions on $(S^3, K(r))$ are conjugate to each other, because (a) the restriction of any such $(\mathbb{Z}_2)^2$ -action to $E(K(r))$ is conjugate to the restriction of the action of the characteristic subgroup $(\mathbb{Z}_2)^2 < \text{Isom}^+(S^3 - K(r))$ by virtue of the orbifold theorem, and because (b) the restriction of any such $(\mathbb{Z}_2)^2$ -action to $N(K(r))$ is determined by its restriction to $\partial N(K(r)) = \partial E(K(r))$. Thus we obtain the following classification of the strong inversions of $K(r)$, up to strong equivalence, whose image in $\text{Sym}^+(S^3, K(r))$ is contained in the characteristic subgroup $(\mathbb{Z}_2)^2$.

1. Suppose $K(r)$ is a hyperbolic 2-bridge knot. Then the involutions $h_+ := h$ and $h_- := fh$ in Figure 2(1) are the only strong inversions of $K(r)$ which projects to an element of the characteristic subgroup $(\mathbb{Z}_2)^2 < \text{Sym}^+(S^3, K(r))$. $\text{Fix}(h_+)$ consists of two tunnels, one of which is the upper tunnel τ_+ . We denote the other tunnel by τ_+^c , and call the meridian pair of $\pi_1(E(K(r))) \cong G(r)$ represented by τ_+^c the *long upper meridian pair* of $G(r)$. The tunnel τ_-^c and the *long lower meridian pair* of $G(r)$ are defined similarly.
2. Suppose $K(r)$ is a two-component hyperbolic 2-bridge link. Then the involution h illustrated in Figure 2(2) is the unique strong inversion of $K(r)$ which projects to an element of the characteristic subgroup

$(\mathbb{Z}_2)^2 < \text{Sym}^+(S^3, K(r))$. $\text{Fix}(h)$ consists of four tunnels, two of which are τ_+ and τ_- . We denote the remaining tunnels by τ_L and τ_R , and call the meridian pair of $G(r)$ represented by one of the two tunnels an *intermediate meridian pair* of $G(r)$. It should be noted that the two intermediate meridian pairs are equivalent modulo the automorphism f_* of $G(r)$ induced by the involution f in Figure 2(2), because the two additional tunnels are mapped to each other by f .

If $q^2 \not\equiv 1 \pmod{p}$, then $\text{Isom}^+(S^3 - K(r)) \cong (\mathbb{Z}_2)^2$ by Proposition 4.1, and so the above $(\mathbb{Z}_2)^2$ -action projects faithfully onto the full group $\text{Isom}^+(S^3 - K(r))$. Thus the list of strong inversions of $K(r)$ in the above give all strong inversions of $K(r)$. Hence, by Proposition 3.1, the lists of meridian pairs in the above give all possible parabolic generating pairs of $G(r)$.

If $q^2 \equiv 1 \pmod{p}$, then $\text{Isom}^+(S^3 - K(r))$ is a \mathbb{Z}_2 -extension of the characteristic subgroup $(\mathbb{Z}_2)^2$, and $(S^3, K(r))$ admits extra symmetries, which interchange τ_+ and τ_- .

1. Suppose $K(r)$ is a hyperbolic 2-bridge knot (i.e., p is odd) and $q^2 \equiv 1 \pmod{p}$. Then n is even and $b_i = -b_{n+1-i}$ ($1 \leq i \leq n$) by Lemma 4.3(1). Thus $(S^3, K(r))$ admits an extra symmetry g illustrated in Figure 3(1). Note that g and h defined in the caption generate the order 8 dihedral group $D_4 = \langle g, h \mid g^4, h^2, hgh^{-1} = g^{-1} \rangle$ which acts faithfully on $(S^3, K(r))$. As in the preceding arguments, we see that the D_4 -action projects faithfully onto the whole group $\text{Sym}^+(S^3, K(r))$ and that any such D_4 -action on $\text{Sym}^+(S^3, K(r))$ is conjugate to the D_4 -action in Figure 3(1). Hence the involutions $g^i h$ ($0 \leq i \leq 3$) are the only strong inversions of $K(r)$ (cf. [41, Section 3]). Thus, in addition to $h_+ = h$ and $h_- = g^2 h$, we have two extra strong inversions gh and $g^3 h$, and each of them determines two tunnels for $K(r)$. We call the meridian pairs of $G(r)$ determined by such an extra tunnel an *extra meridian pair* of $G(r)$. The four extra meridian pairs are divided into two classes, where each class consists of two extra meridian pairs which are equivalent modulo the automorphism g_* of $G(r)$ induced by g , because gh and $g^3 h = g(gh)g^{-1}$ are conjugate in D_4 .
2. Suppose $K(r)$ is a two-component hyperbolic 2-bridge link (i.e., p is even) and $q^2 \equiv 1 \pmod{2p}$. Then n is odd and $b_i = b_{n+1-i}$ ($1 \leq i \leq n$) by Lemma 4.3(2). Thus $(S^3, K(r))$ admits a $(\mathbb{Z}_2)^3$ -action generated by three involutions f , h and g , as shown in Figure 3(2a) and (2b) according to whether $b_{n/2}$ is odd or even. Then $h' := gh$ or $h' := g$ is a strong inversion of $K(r)$ accordingly, and this is the only extra strong inversion of $K(r)$. We call the meridian pair represented by one of the four arc components of $\text{Fix}(h') \cap E(K(r))$ an *extra meridian pair*. The four extra meridian

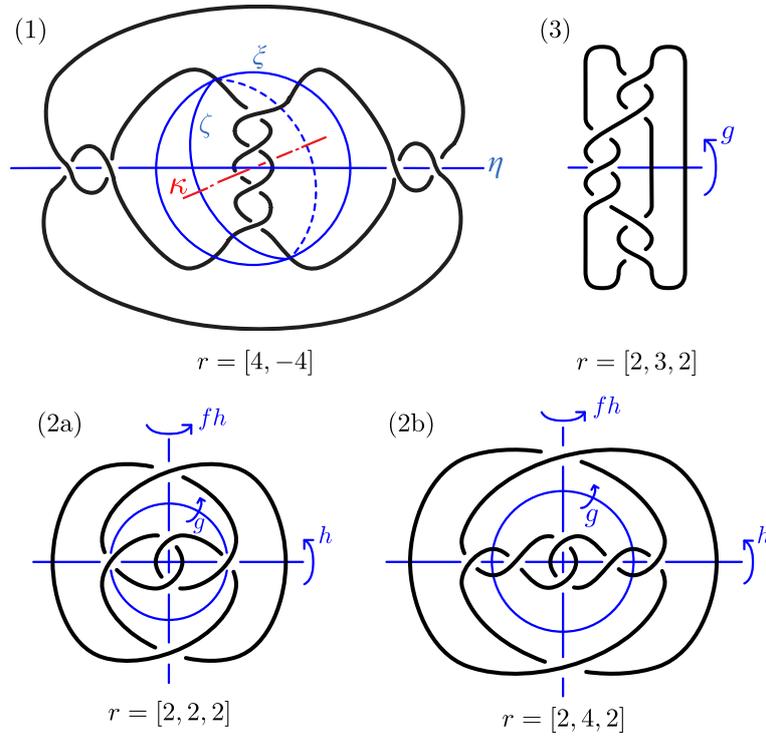


Figure 3: Additional symmetry of $K(r)$. (1) $p : \text{odd}, q^2 \equiv 1 \pmod{p}$. Then $\text{Sym}^+(S^3, K(r)) \cong \langle g, h \mid g^4, h^2, (gh)^2 \rangle$, where $g = (\pi/2\text{-rotation about } \kappa) \circ (\pi\text{-rotation about } \xi)$, $h = (\pi\text{-rotation around } \eta)$, and $gh = (\pi\text{-rotation around } \zeta)$. In this case, $g^i h$ ($i = 1, 3$) are the extra strong inversions. (2a) $p : \text{even}, q^2 \equiv 1 \pmod{2p}, b_{n/2} : \text{odd}$. In this case, gh is the extra strong inversion. (2b) $p : \text{even}, q^2 \equiv 1 \pmod{2p}, b_{n/2} : \text{even}$. In this case, g is the extra strong inversion. (3) $p : \text{even}, q^2 \equiv p + 1 \pmod{2p}$. In this case, there are no extra strong inversions.

pairs are divided into two classes, where each class consists of two extra meridian pairs which are equivalent modulo the automorphism f_* of $G(r)$ induced by f .

3. Suppose $K(r)$ has two components and $q^2 \equiv p + 1 \pmod{2p}$. In this case, it is not easy to draw a link diagram in which one can see the whole symmetry. (A simple conceptual way to understand the whole symmetry is to use the decomposition of $(S^3, K(r))$ into two rational tangles, as described in Section 6). However, it is easy to visualise a single additional symmetry, as described below. By [43, Lemma II.3.3] (cf. Lemma 6.2(1) in Section 6), the continued fraction expansion of $r = q/p$ consisting of positive integers is symmetric, and $(S^3, K(r))$ admits an extra symmetry g which interchanges τ_+ and τ_- , as shown in Figure 3(3). The extra involution g preserves each of the two components of $K(r)$, acts on one component preserving orientation and on the other component reversing orientation. Since $\text{Sym}^+(S^3, K(r)) \cong D_4$ is generated by the characteristic subgroup $(\mathbb{Z}_2)^2$ and the extra element g , we see that $K(r)$ does not have an extra strong inversion.

By the above arguments, we obtain the following propositions (cf. [35, Corollary 2.2]).

PROPOSITION 4.4. *For a hyperbolic 2-bridge knot $K(r)$ with $r = q/p$ (p :odd, $q \not\equiv \pm 1 \pmod{p}$), the following hold.*

- (1) *Suppose $q^2 \not\equiv 1 \pmod{p}$. Then any parabolic generating pair of $G(r)$ is equivalent to the upper, lower, long upper or long lower meridian pair.*
- (2) *Suppose $q^2 \equiv 1 \pmod{p}$. Then any parabolic generating pair of $G(r)$ is equivalent to the upper, lower, long upper, long lower meridian pair or one of the four extra meridian pairs. Moreover, the four extra meridian pairs are divided into two classes up to automorphisms of $G(r)$.*

PROPOSITION 4.5. *For a hyperbolic 2-component 2-bridge link $K(r)$ with $r = q/p$ (p :even, $q \not\equiv \pm 1 \pmod{p}$), the following hold.*

- (1) *Suppose $q^2 \not\equiv 1 \pmod{2p}$. Then any parabolic generating pair of $G(r)$ is equivalent to the upper or lower meridian pair, or one of the two intermediate meridian pairs. Moreover, the two intermediate meridian pairs are equivalent up to automorphisms of $G(r)$.*
- (2) *Suppose $q^2 \equiv 1 \pmod{2p}$. Then any parabolic generating pair of $G(r)$ is equivalent to the upper or lower meridian pair, one of the two intermediate meridian pairs, or one of the four extra meridian pairs. Moreover, the two intermediate meridian pairs are equivalent up to automorphisms of*

$G(r)$, and the four extra meridian pairs are divided into two classes up to automorphisms of $G(r)$.

By Propositions 4.4 and 4.5, the proof of the assertion in Theorem 1.2(1) that each hyperbolic 2-bridge link group $G(r)$ admits at most two parabolic generating pairs (i.e., the upper/lower meridian pairs) is reduced to the proof of the fact that none of the long upper/lower meridian pairs, intermediate meridian pairs, and the extra meridian pairs can generate the hyperbolic 2-bridge link group $G(r)$. The next two sections are devoted to the proof of this fact.

At the end of this section, we prove the following proposition, which, together with the above, completes the proof of Theorem 1.2(1).

PROPOSITION 4.6. *For each hyperbolic 2-bridge link group $G(r)$, the upper meridian pair and the lower meridian pair are not equivalent.*

Proof. Suppose the upper and lower meridian pairs of a hyperbolic 2-bridge link group $G(r)$ are equivalent. Then the upper tunnel τ_+ and the lower tunnel τ_- are properly homotopic in $E(K(r))$. This contradicts [37, Example (3.4)], which implies that τ_+ and τ_- are not properly homotopic. (Though it is only claimed that they are not isotopic, the proof actually shows that they are not properly homotopic.) \square

5. Long upper/lower meridian pairs and intermediate meridian pairs

For a link K in S^3 , let $M(K)$ be the double branched covering of S^3 branched over K . Then its fundamental group is intimately related with the π -orbifold group $O(K)$ of K , which is defined as the quotient of the link group $\pi_1(E(K))$ by the normal closure of the squares of meridians (see [14]). In fact, $O(K)$ is the semidirect product $\pi_1(M(K)) \rtimes \mathbb{Z}_2$, where the action of \mathbb{Z}_2 on $\pi_1(M(K))$ is given by the action of the covering transformation group. If K is a 2-bridge link $K(r)$ with $r = q/p$, then $O(r) := O(K(r))$ is isomorphic to the semidirect product

$$\pi_1(M(K(r))) \rtimes \mathbb{Z}_2 \cong H_1(M(K(r))) \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_p \rtimes \mathbb{Z}_2 \cong D_p,$$

where D_p is the dihedral group of order $2p$.

For a meridian pair $\{m_1, m_2\}$ in $\pi_1(E(K))$, let $\omega(m_1, m_2)$ be the element of $O(K)$ represented by the product $m_1 m_2 \in \pi_1(E(K))$. The following simple observation is a key tool for the proof of Theorem 1.2(1).

LEMMA 5.1. (1) *The element $\omega(m_1, m_2) \in O(K)$, up to inversion and conjugation, is uniquely determined by the equivalence class of $\{m_1, m_2\}$. Namely,*

if two meridian pairs $\{m_1, m_2\}$ and $\{m'_1, m'_2\}$ of $\pi_1(E(K))$ are equivalent, then $\omega(m'_1, m'_2)$ is conjugate to $\omega(m_1, m_2)$ or its inverse in $O(K)$.

- (2) If $\{m_1, m_2\}$ is a generating meridian pair of a 2-bridge link group $G(r)$ with $r = q/p$, then $\omega(m_1, m_2)$ is a generator of the the index 2 subgroup $H_1(M(K(r))) \cong \mathbb{Z}_p$ of $O(r) \cong H_1(M(K(r))) \rtimes \mathbb{Z}_2 \cong D_p$.

Proof. (1) follows from the fact that $m_1^2 = m_2^2 = 1$ in $O(K)$ and the definition of the equivalence of meridian pairs.

(2) Suppose that $\{m_1, m_2\}$ is a generating meridian pair of a 2-bridge link group $G(r)$ with $r = q/p$. Then there is an epimorphism from the infinite dihedral group $D_\infty = \langle m_1, m_2 \mid m_1^2, m_2^2 \rangle$ generated by the symbols m_1 and m_2 onto $O(r) \cong D_p$ which maps the elements m_1 and m_2 of D_∞ to the (images of the) meridians m_1 and m_2 in $O(r)$, respectively. Since D_∞ is isomorphic to the semi-direct product $\langle m_1 m_2 \rangle \rtimes \mathbb{Z}_2$, it follows that $O(r) \cong D_p$ is the quotient of D_∞ by the infinite cyclic normal subgroup $\langle (m_1 m_2)^p \rangle$. Hence $\omega(m_1, m_2) = m_1 m_2$ generates $H_1(M(K(r))) \cong \langle m_1 m_2 \mid p(m_1 m_2) = 0 \rangle$. \square

Note that the long upper/lower meridian pairs are defined for all 2-bridge knots and that the intermediate meridian pairs are defined for all 2-component 2-bridge links. The following proposition together with the above lemma shows that the long upper/lower meridian pairs and the intermediate meridian pairs are not generating pairs of hyperbolic 2-bridge link groups.

PROPOSITION 5.2. (1) Let $K(r)$ with $r = q/p$ (p : odd) be a 2-bridge knot, and let $\{m_1, m_2\}$ be the long upper (or lower) meridian pair of the knot group $G(r)$. Then $\omega(m_1, m_2) = 0$ in $H_1(M(K(r))) \cong \mathbb{Z}_p$. In particular, if $K(r)$ is a nontrivial knot (i.e. $p \geq 3$), then $\omega(m_1, m_2)$ is not a generator of $H_1(M(K(r)))$.

(2) Let $K(r)$ with $r = q/p$ (p even) be a nontrivial 2-component 2-bridge link, and let $\{m_1, m_2\}$ be an intermediate meridian pair of the link group $G(r)$. Then $2\omega(m_1, m_2) = 0$ in $H_1(M(K(r))) \cong \mathbb{Z}_p$. In particular, if $K(r)$ is not a Hopf link (i.e. $p \geq 4$), then $\omega(m_1, m_2)$ is not a generator of $H_1(M(K(r)))$.

Proof. (1) We prove the assertion for the long lower meridian pair. (The assertion for the long upper meridian pair is a consequence of this, because there is an isomorphism $G(q/p) \cong G(q'/p)$, where $qq' \equiv 1 \pmod{p}$, which maps the long upper meridian pair of $G(q/p)$ to the long lower meridian pair of $G(q'/p)$ (cf. [7, Proposition 2.1]). Recall that the long lower meridian pair $\{m_1, m_2\}$ is represented by the tunnel $\tau^c \subset \text{Fix}(h_-)$. Then τ^c is properly embedded in the checker-board surface, F , illustrated in Figure 4(1), associated with the link diagram in Figure 2(1), and moreover, τ^c is separating in F . Let M_F be the

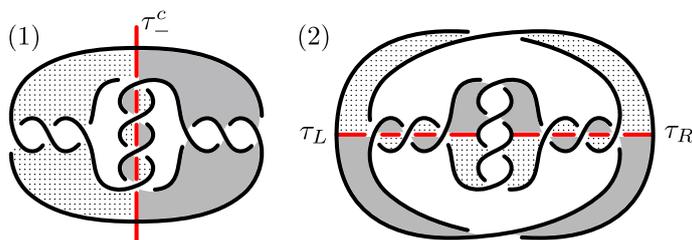


Figure 4: Checkerboard surfaces associated with the diagrams in Figure 2 containing τ_-^c in (1) and $\tau_L \cup \tau_R$ in (2), respectively, as separating arc systems

3-manifold obtained by cutting S^3 along F , and let σ be the involution on ∂M_F such that $\text{Fix}(\sigma)$ is the copy of L in ∂M_F and that $(M_F, \partial M_F)/\sigma \cong (S^3, F)$. (Here the symbol $/\sigma$ means to identify x with $\sigma(x)$ for all $x \in \partial M_F$.) The double branched covering $M(K(r))$ of S^3 branched over $K(r)$ is obtained from two copies $M_F^{(0)}$ and $M_F^{(1)}$ of M_F by gluing their boundaries through the homeomorphism $\partial M_F^{(0)} \rightarrow \partial M_F^{(1)}$ induced by σ , i.e. the homeomorphism that maps the copy in $M_F^{(0)}$ of a point $x \in \partial M_F$ to its copy in $M_F^{(1)}$ of the point $\sigma(x) \in \partial M_F$. Now, let $\tilde{\tau}_-^c$ be the simple loop in ∂M_F obtained as the inverse image of the proper arc $\tau_-^c \subset F$ in ∂M_F under the projection $\partial M_F \rightarrow \partial M_F/\sigma = F$. By using the fact that the long lower meridian pair $\{m_1, m_2\}$ is represented by τ_-^c , we can see that the element $\omega(m_1, m_2) \in H_1(M(K(r))) < O(r)$ is represented by the simple loop $\tilde{\tau}_-^c$. To be precise, $\omega(m_1, m_2)$ is represented by the copy of the loop $\tilde{\tau}_-^c$ in $\partial M_F^{(0)} \subset M(K(r))$ with a suitable orientation. However, the loop $\tilde{\tau}_-^c$ is separating in ∂M_F , because it is the inverse image of the separating arc τ_-^c of F . Thus $\tilde{\tau}_-^c$ is null-homologous in M_F , and hence $\omega(m_1, m_2) = [\tilde{\tau}_-^c] = 0 \in H_1(M(K(r)))$, as desired.

(2) We prove the assertion for the intermediate meridian pair $\{m_1, m_2\}$ represented by the tunnel $\tau_L \subset \text{Fix}(h)$. (The assertion for that represented by τ_R is a consequence of this by Proposition 4.5(1).) Observe that the checkerboard surface, F , for $K(r)$ in Figure 4(2) contains $\tau_L \cup \tau_R$ as a separating arc system. Observe also that the involution f of $(S^3, K(r))$ introduced in Figure 2(2) preserves the surface F and interchanges τ_L and τ_R . Let M_F and σ be as in the proof of (1), and recall that the double branched covering $M_2(K(r))$ is obtained by gluing two copies $M_F^{(0)}$ and $M_F^{(1)}$ of M_F through the homeomorphism $\partial M_F^{(0)} \rightarrow \partial M_F^{(1)}$ induced by the involution σ on ∂M_F . Now, let $\tilde{\tau}_L$ and $\tilde{\tau}_R$ be the simple loops in ∂M_F obtained as the inverse image of the arcs τ_L and τ_R in F , respectively, under the projection $\partial M_F \rightarrow \partial M_F/\sigma = F$ ($i = 1, 2$). As in (1), the element $\omega(m_1, m_2) \in H_1(M(K(r))) < O(r)$ is represented by the simple loop $\tilde{\tau}_L \subset \partial M_F \cong \partial M_F^{(0)} \subset M_2(K(r))$ with a suitable orientation. We

endow $\tilde{\tau}_L$ with this orientation.

Now let \hat{f} be the orientation-preserving involution on M_F induced by the involution f . Then $\tilde{\tau}_R = \hat{f}(\tilde{\tau}_L)$, and we orient $\tilde{\tau}_R$ as the image by \hat{f} of the oriented loop $\tilde{\tau}_L$. On the other hand, since $\tau_L \cup \tau_R$ is a separating arc system in F , $\tilde{\tau}_L \cup \tilde{\tau}_R$ is a separating loop system in ∂M_F (where we forget the orientation), and \hat{f} interchanges the two components of $\partial M_F - (\tilde{\tau}_L \cup \tilde{\tau}_R)$. Since (the restriction to ∂M_F of) \hat{f} is orientation-preserving, this implies that the cycle $\tilde{\tau}_L - \hat{f}(\tilde{\tau}_L) = \tilde{\tau}_L - \tilde{\tau}_R$ is null homologous in ∂M_F . Hence we have $[\tilde{\tau}_L] = [\hat{f}(\tilde{\tau}_L)]$ in $H_1(M(K(r)))$.

Let \tilde{f} be the lift of f to $M_2(K(r)) = M_F^{(0)} \cup M_F^{(1)}$ that is obtained by gluing the copies of the involution \hat{f} on $M_F^{(i)}$ ($i = 1, 2$). Note that the double branched covering $M(K(r))$ is the union of two solid tori, whose cores are the circles, $\tilde{\tau}_+$ and $\tilde{\tau}_-$, obtained as the inverse images of the upper tunnel τ_+ and the lower tunnel τ_- , respectively (cf. the second paragraph of Section 6). Since $\text{Fix}(f)$ intersects τ_{\pm} transversely in a single point and since $\tau_{\pm} \subset M_F$, we see that $\text{Fix}(\tilde{f})$ intersects each of the core circle $\tilde{\tau}_{\pm}$ in two points, and so \tilde{f} acts on the circles $\tilde{\tau}_{\pm}$ as an orientation-reversing involution. Since each of $[\tilde{\tau}_{\pm}]$ is a generator of $H_1(M(K(r)))$, this implies that \tilde{f}_* acts on $H_1(M(K(r)))$ as multiplication by -1 . Hence, we have

$$[\tilde{\tau}_L] = [\hat{f}(\tilde{\tau}_L)] = \tilde{f}_*([\tilde{\tau}_L]) = -[\tilde{\tau}_L] \in H_1(M(K(r)))$$

Thus we have $2\omega(m_1, m_2) = 2[\tilde{\tau}_L] = 0$ in $H_1(M(K(r)))$, as desired. □

6. Extra meridian pairs

In this section, we prove the follog proposition, which, together with Lemma 5.1, implies that the extra meridian pairs are not generating pairs of hyperbolic 2-bridge link groups.

PROPOSITION 6.1. *Let $\{m_1, m_2\}$ be an extra meridian pair of a hyperbolic 2-bridge link group $G(r)$. Then $\omega(m_1, m_2) \in H_1(M(K(r)))$ is not a generator of $H_1(M(K(r)))$.*

To this end, we regard $(S^3, K(r))$ as the union of two rational tangles $(\mathbf{B}^3, t(\infty))$ and $(\mathbf{B}^3, t(r))$ of slopes ∞ and r , as in [7, Section 2]. Here the common boundary $\partial(\mathbf{B}^3, t(\infty)) = \partial(\mathbf{B}^3, t(r))$ is identified with the Conway sphere $(S^2, P^0) := (\mathbb{R}^2, \mathbb{Z}^2)/\mathcal{J}$, where \mathcal{J} is the group of isometries of the Euclidean plane \mathbb{R}^2 generated by the π -rotations around the points in the lattice \mathbb{Z}^2 . For each rational number $s \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, a line of slope s in $\mathbb{R}^2 - \mathbb{Z}^2$ projects to an essential simple loop, denoted by α_s , in the 4-times punctured sphere $\check{S}^2 := S^2 - P^0$. Similarly, a line of slope s in \mathbb{R}^2 passing through a point \mathbb{Z}^2 determines an essential simple proper arc in \check{S}^2 . For each s , there are

exactly two essential simple proper arcs in \check{S}^2 obtained in this way, and the union of the two arcs is denoted by δ_s . The rational number s is called the *slope* of α_s and δ_s . By the definition of the rational tangles, the loops α_∞ and α_r bound discs in $\mathbf{B}^3 - t(\infty)$ and $\mathbf{B}^3 - t(r)$, respectively.

The double branched covering $M(K(r))$ of $(S^3, K(r))$ is the union of the solid tori V_∞ and V_r that are obtained as the double branched coverings of $(\mathbf{B}^3, t(\infty))$ and $(\mathbf{B}^3, t(r))$, respectively. Let $\tilde{\alpha}_0$ and $\tilde{\alpha}_\infty$ be lifts in ∂V_∞ of the simple loops α_0 and α_∞ , respectively. (Here a lift means a connected component of the inverse image.) Then $\tilde{\alpha}_0$ and $\tilde{\alpha}_\infty$ form the meridian and the longitude of V_∞ . Similarly a lift $\tilde{\alpha}_r$ of α_r in ∂V_r is a meridian of V_r . Thus the loops $\tilde{\alpha}_\infty$ and $\tilde{\alpha}_r$ represent the trivial elements of $H_1(V_\infty)$ and $H_1(V_r)$, respectively. Since $[\tilde{\alpha}_r] = p[\tilde{\alpha}_0] + q[\tilde{\alpha}_\infty]$ in $H_1(\partial V_\infty)$, where $r = q/p$, we have

$$H_1(M(K(r))) \cong \langle [\tilde{\alpha}_0], [\tilde{\alpha}_\infty] \mid [\tilde{\alpha}_\infty], [\tilde{\alpha}_r] \rangle \cong \langle \tilde{\alpha}_0 \mid p[\tilde{\alpha}_0] \rangle \cong \mathbb{Z}_p.$$

Now recall the following well-known facts (cf. [43, Lemma II.3.3]).

LEMMA 6.2. *For a rational number $r = q/p$ with $0 < q \leq p/2$, consider the continued fraction expansion $r = [a_1, a_2, \dots, a_n]$ into positive integers a_i such that $a_1 \geq 2$ and $a_n \geq 2$. Then the following hold.*

- (1) *The following conditions are equivalent.*
 - (a) *p is even and $q^2 \equiv 1 \pmod{2p}$.*
 - (b) *n is odd, $a_{(n+1)/2}$ is even, and (a_1, \dots, a_n) is symmetric.*
- (2) *The following conditions are equivalent.*
 - (a) *Either (i) p is odd and $q^2 \equiv 1 \pmod{p}$, or (ii) p is even and $q^2 \equiv p+1 \pmod{2p}$.*
 - (b) *n is odd, $a_{(n+1)/2}$ is odd, and (a_1, \dots, a_n) is symmetric.*

The symmetry of the continued fraction expansion in the above lemma is realised by the symmetry of the Farey tessellation, \mathcal{F} , as illustrated by Figure 5. Recall that the *Farey tessellation* is the tessellation of the upper half space \mathbb{H}^2 by ideal triangles that are obtained from the ideal triangle with the ideal vertices $0, 1, \infty \in \hat{\mathbb{Q}}$ by repeated reflection in the edges. Then $\hat{\mathbb{Q}}$ is identified with the set of the vertices of \mathcal{F} . The automorphism group of the Farey tessellation is identified with $\text{PGL}(2, \mathbb{Z})$ as follows. For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, consider its action on \mathbb{R}^2 by the left multiplication $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}$. Then A maps a line of a slope $s \in \hat{\mathbb{Q}}$ to a line of slope $s' = (c + ds)/(a + bs)$. The correspondence $s \mapsto s'$ gives a bijection of the Farey vertices, which extends to

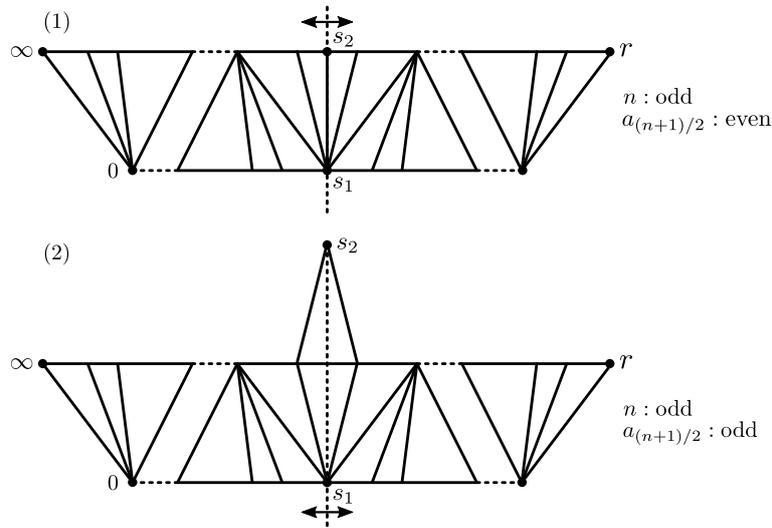


Figure 5: If $r = q/p$ satisfies one of the conditions in Lemma 6.2, then there is an orientation-reversing involution of the Farey tessellation \mathcal{F} which interchanges ∞ and r .

an automorphism of the Farey tessellation. (When $A \in \text{SL}(2, \mathbb{Z})$, the transformation $z \mapsto (c + dz)/(a + bz)$ of $\mathbb{H}^2 \cup \{\infty\}$ gives the desired automorphism.) Conversely, every automorphism of the Farey tessellation \mathcal{F} is obtained in this way.

Now suppose $q^2 \equiv 1 \pmod{p}$. Then, by Lemma 6.2, there is an orientation-reversing involution of the Farey tessellation which interchanges ∞ with r (see Figure 5). Let R be one of the two matrices in $\text{GL}(2, \mathbb{Z})$ that realises the involution. Then the linear action of R on \mathbb{R}^2 is orientation-reversing and interchanges the 1-dimensional vector subspace of slope ∞ with that of slope r . Since R normalizes the subgroup \mathcal{J} in the affine transformation group of \mathbb{R}^2 , it descends to an orientation-reversing involution on (S^2, \mathbf{P}^0) which swaps δ_∞ with δ_r and α_∞ with α_r . Hence R induces an orientation-preserving involution, g_R , of $(S^3, K(r))$ which interchanges $(\mathbf{B}^3, t(\infty))$ with $(\mathbf{B}^3, t(r))$. This gives one of the extra symmetries of $(S^3, K(r))$ described in Section 4. We note that the generators f and h of the $(\mathbb{Z}_2)^2$ -action on $(S^3, K(r))$ of the characteristic subgroup come from the π -rotations of \mathbb{R}^2 about the point $(1/2, 0)$ and $(0, 1/2)$, respectively. The extra symmetry g_R and the $(\mathbb{Z}_2)^2$ -action generate a finite group action on $(S^3, K(r))$ which gives a realisation of $\text{Sym}^+(S^3, K(r))$.

Case 1. Suppose that the mutually equivalent conditions in Lemma 6.2(1) hold, namely, $n = 2n_0 + 1$ is odd, $a_{n_0+1} = 2a'_{n_0+1}$ is even, and (a_1, \dots, a_n) is

symmetric. Then the reflection on \mathcal{F} in the Farey edge, spanned by the vertices $s_1 = q_1/p_1$ and $s_2 = q_2/p_2$ in Figure 5(1), interchanges ∞ and r . Thus the matrix $\pm R$ is the conjugate of the matrix $R_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which induces the reflection on \mathcal{F} in the Farey edge spanned by $0/1$ and $1/0$, by the the matrix $A = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}$, which maps the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to the vectors $\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ and $\begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$, respectively. Hence we have

$$\pm R = AR_0A^{-1} = \begin{pmatrix} p_1q_2 + p_2q_1 & -2p_1p_2 \\ 2q_1q_2 & -p_1q_2 - p_2q_1 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} p \\ q \end{pmatrix} = \pm R \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \pm \begin{pmatrix} -2p_1p_2 \\ -p_1q_2 - p_2q_1 \end{pmatrix}.$$

Hence we have $p = 2p_1p_2$.

Observe that the involution on $(\mathcal{S}^2, \mathbf{P}^0)$ induced by R_0 is the reflection in the circle that is the union of the four arcs in $\delta_{0/1}$ and $\delta_{1/0}$. Hence the restriction of the extra involution g_R to $(\mathcal{S}^2, \mathbf{P}^0)$ is the the reflection in the circle that is the union the four arcs in δ_{q_1/p_1} and δ_{q_2/p_2} . This confirms that g_R is a strong inversion of the 2-component link $K(r)$, and the four extra meridian pairs that arise from the strong inversion g_R are represented by the four arcs in δ_{q_1/p_1} and δ_{q_2/p_2} , respectively (cf. Figure 3(2a), (2b)). Let $\{m_1, m_2\}$ be one of the four extra meridian pairs. Then it is represented by an arc on $\hat{\mathcal{S}}^2$ of slope q_i/p_i for $i = 1$ or 2 . This implies that the element $\omega(m_1, m_2) \in H_1(M(K(r))) < O(r)$ is represented the simple loop α_{q_i/p_i} for $i = 1$ or 2 . Hence we have

$$\omega(m_1, m_2) = [\alpha_{q_i/p_i}] = p_i[\tilde{\alpha}_0] + q_i[\tilde{\alpha}_\infty] = p_i[\tilde{\alpha}_0] \in H_1(M(K(r))) \cong \langle \tilde{\alpha}_0 \mid p[\tilde{\alpha}_0] \rangle.$$

If $K(r)$ is hyperbolic, then $q \not\equiv \pm 1 \pmod{p}$ and so we can see that $1 < p_i < p = 2p_1p_2$ (see Figure 5(1)). So, $\omega(m_1, m_2)$ is not a generator of $H_1(M(K(r)))$. Hence, none of the extra meridian pairs is a generating pair of $G(r)$. This completes the proof of Proposition 6.1 for the case where the condition (1) in Lemma 6.2 is satisfied.

Case 2. Suppose that the mutually equivalent conditions in Lemma 6.2(2) hold, namely, $n = 2n_0 + 1$ is odd, $a_{n_0+1} = 2a'_{n_0+1} + 1$ is odd, and (a_1, \dots, a_n) is symmetric. Then the reflection on \mathcal{F} in the geodesic, joining the vertices $s_1 = q_1/p_1$ and $s_2 = q_2/p_2$ in Figure 5(2), interchanges ∞ and r . Thus the matrix $\pm R$ realising the symmetry is conjugate to the matrix $R_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which induces the reflection on \mathcal{F} in the geodesic joining the vertices $1/1$ and

$-1/1$, by the matrix A that maps the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ to the vectors $\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ and $\begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$, respectively. The matrix A is given by

$$A = \frac{1}{2} \begin{pmatrix} p_1 + p_2 & p_1 - p_2 \\ q_1 + q_2 & q_1 - q_2 \end{pmatrix},$$

and hence we have

$$\pm R = AR_0A^{-1} = \begin{pmatrix} \frac{-1}{2}(p_1q_2 + p_2q_1) & p_1p_2 \\ -q_1q_2 & \frac{1}{2}(p_1q_2 + p_2q_1) \end{pmatrix}.$$

Thus

$$\begin{pmatrix} p \\ q \end{pmatrix} = \pm R \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \pm \begin{pmatrix} p_1p_2 \\ \frac{1}{2}(p_1q_2 + p_2q_1) \end{pmatrix}.$$

Hence we have $p = p_1p_2$.

Observe that the involution on $(\mathbf{S}^2, \mathbf{P}^0)$ induced by R_0 is the reflection in the circle that is the union of an arc of slope $1/1$ and an arc of slope $-1/1$. Hence the restriction of the extra involution g_R to $(\mathbf{S}^2, \mathbf{P}^0)$ is the reflection in the circle that is the union of an arc of slope q_1/p_1 and an arc of slope q_2/p_2 . Hence $\text{Fix}(g_R)$ is a circle which intersects $K(r)$ in two points. This confirms that g_R is a strong inversion if and only if $K(r)$ is a knot (cf. Figures 3(1) and (3)). So, we assume $K(r)$ is a knot. Then the two extra meridian pairs that arise from the strong inversion g_R are represented by an arc of slope q_1/p_1 and an arc of slope q_2/p_2 , respectively.

If an extra meridian pair $\{m_1, m_2\}$ corresponds to an arc of slope q_i/p_i , then the element $\omega(m_1, m_2) \in H_1(M(K(r))) < O(r)$ is represented the simple loop α_{q_i/p_i} . Hence we have

$$\omega(m_1, m_2) = [\alpha_{q_i/p_i}] = p_i[\tilde{\alpha}_0] + q_i[\tilde{\alpha}_\infty] = p_i[\tilde{\alpha}_0] \in H_1(M(K(r))) \cong \langle \tilde{\alpha}_0 \mid p[\tilde{\alpha}_0] \rangle.$$

We can observe that if $K(r)$ is hyperbolic then $1 < p_i < p_1p_2$ (see Figure 5(2)). So $\omega(m_1, m_2)$ is not a generator of $H_1(M(K(r)))$. This completes the proof of Proposition 6.1 for the case where the condition (2) in Lemma 6.2 is satisfied.

Thus we have proved that all possible parabolic generating pairs listed in Propositions 4.4 and 4.5, except for the upper/lower meridian pairs, are not generating pairs. This completes the proof of Theorem 1.2(1).

7. Parabolic generating pairs of Heckoid groups

In this section, we prove Theorem 1.2(2) which asserts that every Heckoid group has a unique parabolic generating pair up to equivalence.

We first give an explicit description of the weighted graph in Figure 1 representing the Heckoid orbifolds (cf. [7, Definition 3.4], [34, Section 5]).

DEFINITION 7.1. (1) For $r \in \mathbb{Q}$ and for an integer $n \geq 2$, $\mathcal{M}_0(r; n)$ denotes the orbifold pair determined by the weighted graph $(S^3, K(r) \cup \tau_-, w_0)$, where w_0 is given

$$w_0(K(r)) = \infty, \quad w_0(\tau_-) = n.$$

(2) For $r = q/p \in \mathbb{Q}$ with p odd and an odd integer $m \geq 3$, $\mathcal{M}_1(r; m)$ denotes the orbifold pair determined by the weighted graph $(S^3, K(r) \cup \tau_-, w_1)$, where w_1 is given by the following rule. Let J_1 and J_2 be the edges of the graph $K(r) \cup \tau_-$ distinct from τ_- . Then

$$w_1(J_1) = \infty, \quad w_1(J_2) = 2, \quad w_1(\tau_-) = m.$$

(3) For $r \in \mathbb{Q}$ and an odd integer $m \geq 3$, $\mathcal{M}_2(r; m)$ denotes the orbifold pair determined by the weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, w_2)$, where w_2 is given by the following rule. Let J_1 and J_2 be unions of two mutually disjoint edges of the graph $K(r) \cup \tau_+ \cup \tau_-$ distinct from τ_{\pm} , such that both J_1 and J_2 are preserved by the ‘vertical involution’ f of $K(r)$. Then

$$w_2(J_1) = \infty, \quad w_2(J_2) = 2, \quad w_2(\tau_+) = 2, \quad w_2(\tau_-) = m.$$

In Definition 7.1(3), the *vertical involution* of $(S^3, K(r))$ is an involution of $(S^3, K(r))$ which preserves both $(\mathbf{B}^3, t(\infty))$ and $(\mathbf{B}^3, t(r))$ and whose restriction to the common boundary (S^2, \mathbf{P}^0) is given by the π -rotation of \mathbb{R}^2 about the point $(1/2, 0)$ (see [7, Figure 4]).

We first calculate the orientation-preserving isometry group of the Heckoid orbifolds. To this end, we recall the spherical dihedral orbifold $\mathcal{O}(r; d_+, d_-)$ introduced in [7, Theorem 4.1], that is represented by a weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, w)$, where d_+ and d_- are mutually prime positive integers d_+ and d_- , and w is given by the rule.

$$w(K(r)) = 2, \quad w(\tau_+) = d_+, \quad w(\tau_-) = d_-.$$

PROPOSITION 7.2. *The orientation-preserving isometry group of the Heckoid orbifolds are given by the following formula.*

$$\text{Isom}^+(\mathcal{M}_0(r; n)) \cong (\mathbb{Z}_2)^2, \quad \text{Isom}^+(\mathcal{M}_1(r; m)) \cong \mathbb{Z}_2, \quad \text{Isom}^+(\mathcal{M}_2(r; m)) \cong \mathbb{Z}_2.$$

Proof. Let \mathcal{M} be a Heckoid orbifold. Then the isometry group $\text{Isom}^+(\mathcal{M})$ is finite, because the hyperbolic structure of \mathcal{M} is geometrically finite. Let \mathcal{O} be the orbifold obtained from \mathcal{M} by the order 2 orbifold surgery, namely the orbifold represented by the weighted graph obtained from that representing the Heckoid orbifold \mathcal{M} , by replacing the label ∞ with 2 (cf. [7, Definition 6.3]). Then \mathcal{O} is the spherical dihedral orbifold $\mathcal{O}(r; 1, n)$, $\mathcal{O}(r; 1, m)$ or $\mathcal{O}(r; 2, m)$ according as $\mathcal{M} = \mathcal{M}_0(r; n)$, $\mathcal{M}_1(r; m)$, or $\mathcal{M}_2(r; m)$. The action of $\text{Isom}^+(\mathcal{M})$

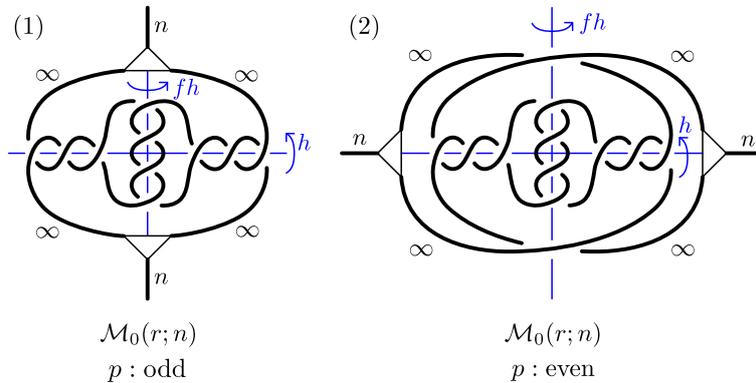


Figure 6: The inverting elliptic element for any generating pair induces the involution h of $\mathcal{M}_0(r; n)$ in the figure. According to whether p is odd (1) or even (2), $\text{Fix}(h)$ contains two or one geodesic paths joining the parabolic locus P to itself.

descends faithfully to an orientation-preserving finite group action on the spherical orbifold \mathcal{O} . By the orbifold theorem (see [11]), we may assume the action is an isometric action on the spherical orbifold. Thus $\text{Isom}^+(\mathcal{M})$ is identified with the subgroup of the spherical isometry group $\text{Isom}^+(\mathcal{O})$ consisting of those isometries which map the union of the edges of the singular set that arise from the parabolic locus of \mathcal{M} . The orientation-preserving isometry groups of the spherical dihedral orbifolds are given by [7, Proposition 12.6], which together with the above fact enables us to obtain the desired isomorphisms. \square

Let $\mathcal{M} = (M_0, P)$ be a Heckoid orbifold, and identify (M_0, P) with a subspace of the hyperbolic orbifold $M = \mathbb{H}^3/\Gamma$ ($\Gamma = \pi_1(\mathcal{M})$) by the isomorphism (1) in Section 1. Let $\{\alpha, \beta\}$ be a parabolic generating pair of the Heckoid group $\pi_1(\mathcal{M})$, and let h be the inverting elliptic element for $\{\alpha, \beta\}$, i.e., the π -rotation about the geodesic η joining the parabolic fixed points of α and β . Let $\bar{\eta}$ be the image of η in M . Then, by Proposition 3.2, $\{\alpha, \beta\}$ is represented by $\bar{\eta} \cap M_0$.

Case 1. $\mathcal{M} = \mathcal{M}_0(r; n)$ with $r = q/p$ and $n \geq 2$. Then $\text{Isom}^+(\mathcal{M}) \cong (\mathbb{Z}_2)^2$ is illustrated in Figure 6(1) or (2) according to whether p is odd or even. Since the singular set of \mathcal{M} consists of a single arc of index n which joins the two components of $\text{cl}(\partial M_0 - P)$, the inverting elliptic element h does not belong to Γ , and so it descends to an isometric involution of \mathcal{M} , which we continue to denote by h . Since its fixed point set $\text{Fix}(h)$ contains the geodesic path $\bar{\eta}$ joining the parabolic loci P_α and P_β , the involution h on \mathcal{M} must be equivalent to the involution h in Figure 6(1) or (2) according to whether p is odd or even.

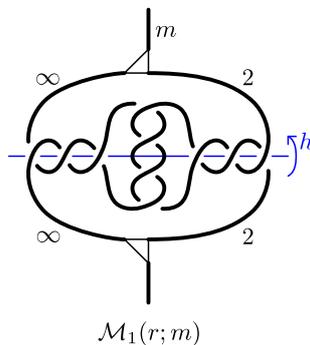


Figure 7: The inverting elliptic element for any generating pair induces the involution h of the orbifold $\mathcal{M}_1(r; m)$ in the figure. $\text{Fix}(h)$ contains two geodesic paths joining the parabolic locus P to itself.

Subcase 1.1. p is odd. By Proposition 2.2, we have only to treat the case $p \geq 3$. Note that $\text{Fix}(h)$ of the involution h in Figure 6(1) contains two geodesic paths which join the parabolic locus P to itself, namely τ_+ and τ_+^c , the images of the tunnels τ_+ and τ_+^c for $K(r)$ in $\mathcal{M} = \mathcal{M}_0(r; n)$ (see Figure 6(1)). If $\bar{\eta} = \tau_+$, then $\{\alpha, \beta\}$ is equivalent to the standard parabolic generating pair in Figure 1(2). We show that $\bar{\eta}$ cannot be τ_+^c . Suppose on the contrary that $\bar{\eta} = \tau_+^c$. Then the natural epimorphism from $\pi_1(\mathcal{M}_0(r; n))$ onto the 2-bridge knot group $G(r)$ maps the pair $\{\alpha, \beta\}$ to the long upper meridian pair of $G(r)$. Since $p \geq 3$, the long upper meridian pair of $G(r)$ is not a generating pair by Lemma 5.1 and Proposition 5.2(1). This contradicts the assumption that $\{\alpha, \beta\}$ is a generating pair of $\pi_1(\mathcal{M}_0(r; n))$. Hence, $\pi_1(\mathcal{M}_0(q/p; n))$ with p odd has a unique parabolic generating pair.

Subcase 1.2: p is even. Then we can see from Figure 6(2) that τ_+ is the unique geodesic path contained in $\text{Fix}(h)$ of the involution h in Figure 6(2) which joins the parabolic locus P to itself. Hence the pair $\{\alpha, \beta\}$ is represented by τ_+ , and so it is equivalent to the standard parabolic generating pair in Figure 1(2). Hence, $\pi_1(\mathcal{M}_0(q/p; n))$ with p even has a unique parabolic generating pair.

Case 2. $\mathcal{M} = \mathcal{M}_1(r; m)$ with $r = q/p$ (p : odd) and $m \geq 3$ odd. Then the singular set of \mathcal{M} contains a unique edge of even index (actually 2) and it joins the two components of $\text{cl}(\partial M_0 - P)$. Hence the inverting elliptic element h does not belong to Γ , and so it descends to an isometric involution of \mathcal{M} , which we continue to denote by h . The involution h generates $\text{Isom}^+(\mathcal{M}) \cong \mathbb{Z}_2$ and it is as illustrated Figure 7. Observe that $\text{Fix}(h)$ of the involution h in Figure 7 contains two geodesic paths which joins the parabolic locus P to itself, namely τ_+ and τ_+^c . If $\bar{\eta} = \tau_+$, then $\{\alpha, \beta\}$ is equivalent to the standard parabolic

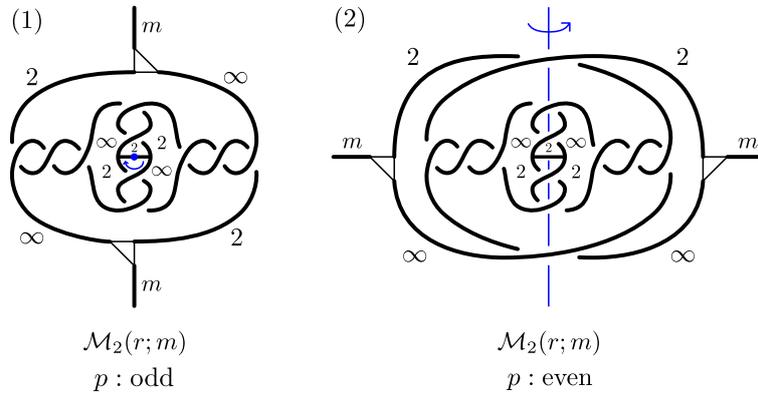


Figure 8: $\mathcal{M}_2(r; m)$ admits a unique orientation-preserving isometric involution. Its fixed point set is disjoint from the parabolic locus P . Hence the inverting elliptic element h must be contained in $\pi_1(\mathcal{M}_2(r; m))$.

generating pair in Figure 1(3). We show that $\bar{\eta}$ cannot be τ_+^c . Suppose on the contrary that $\bar{\eta} = \tau_+^c$. Note that the natural epimorphism from $\pi_1(\mathcal{M}_1(r; m))$ onto the π -orbifold group $O(r)$ maps the pair $\{\alpha, \beta\}$ to $\{m_1, m_2 m_1 m_2^{-1}\}$, where $\{m_1, m_2\}$ is the image in $O(r)$ of the long upper meridian pair of $G(r)$. By Lemma 5.1 and Proposition 5.2(1), $\{m_1, m_2\}$ cannot generate $O(r)$. (Here we use the fact that the conclusion of Lemma 5.1(2) holds under the weaker condition that the image of $\{m_1, m_2\}$ in $O(K)$ generates $O(K)$.) So $\{m_1, m_2 m_1 m_2^{-1}\}$ cannot generate $O(r)$, and hence $\{\alpha, \beta\}$ is not a generating pair, a contradiction. Hence, $\pi_1(\mathcal{M}_1(r; m))$ has a unique parabolic generating pair.

Case 3. $\mathcal{M} = \mathcal{M}_2(r; m)$ with $m \geq 3$ odd. Then $\text{Isom}^+(\mathcal{M}) \cong \mathbb{Z}_2$ is as illustrated Figure 8(1) or (2) according to whether p is odd or even. Thus the fixed point set of the unique orientation-preserving involution of $\mathcal{M}_2(r; m)$ does not contain a geodesic path connecting the parabolic locus P to itself, and so the inverting elliptic element h must belong to the Heckoid group $\pi_1(\mathcal{M}_2(r; m))$. Since m is odd, $\bar{\eta}$ must be the upper tunnel τ_+ . Hence the pair $\{\alpha, \beta\}$ is equivalent to the standard parabolic generating pair in Figure 1(4). Thus $\pi_1(\mathcal{M}_2(r; m))$ has a unique parabolic generating pair.

8. Applications to epimorphisms between 2-bridge knot groups and degree one maps between 2-bridge link exteriors

In [38], Ohtsuki, Riley, and Sakuma gave a systematic construction of epimorphisms between 2-bridge link groups. In this section, we show that all

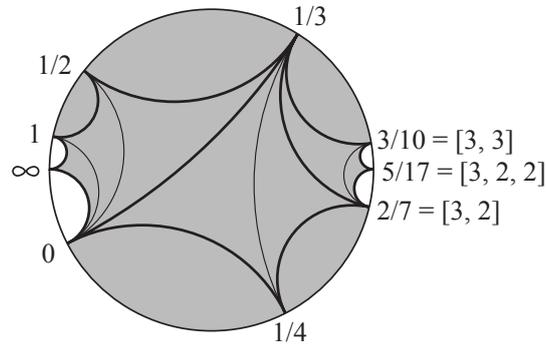


Figure 9: A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = [3, 2, 2]$.

epimorphisms between 2-bridge knot groups essentially arise from their construction.

We first recall the result of [38]. Let Γ_r be the group of automorphisms of the Farey tessellation \mathcal{F} generated by the reflections in the Farey edges with an endpoint r . It should be noted that Γ_r is isomorphic to the infinite dihedral group and the region bounded by two adjacent Farey edges with an endpoint r is a fundamental domain for the action of Γ_r on \mathbb{H}^2 . Let $\hat{\Gamma}_r$ be the group generated by Γ_∞ and Γ_r . When $r \in \mathbb{Q} - \mathbb{Z}$, $\hat{\Gamma}_r$ is equal to the free product $\Gamma_\infty * \Gamma_r$, having a fundamental domain shown in Figure 9. (Otherwise, $\hat{\Gamma}_r$ is equal to Γ_∞ or the group generated by the reflections in all Farey edges according to whether $r = \infty$ or $r \in \mathbb{Z}$.) It should be noted that Schubert's classification theorem of 2-bridge links says that two 2-bridge links $K(r)$ and $K(r')$ are equivalent if and only if there is an automorphism of \mathcal{F} which sends $\{\infty, r\}$ to $\{\infty, r'\}$. Thus the conjugacy class of the group $\hat{\Gamma}_r$ in the automorphism group of \mathcal{F} is uniquely determined by the link $K(r)$.

The following result is a consequence of [38, Theorem 1.1] and the well-known fact that there is an upper-meridian-preserving isomorphism $G(r) \cong G(r+1)$ for any $r \in \hat{\mathbb{Q}}$ (cf. [7, Proposition 2.1(1.a)]).

- (A) *There is an epimorphism from a 2-bridge link group $G(\tilde{r})$ to a 2-bridge link group $G(r)$, if \tilde{r} or $\tilde{r} + 1$ belongs to the $\hat{\Gamma}_r$ -orbit of r or ∞ . Moreover, we can choose the epimorphism so that it is upper-meridian-pair-preserving, namely, it sends the upper meridian pair of $G(\tilde{r})$ to that of $G(r)$.*

Lee and Sakuma proved the following converse to the above result ([33, Theorem 2.4]), by solving a certain word problem for the 2-bridge link groups, using the small cancellation theory.

- (B) *There is an upper-meridian-pair-preserving epimorphism from a 2-bridge link group $G(\tilde{r})$ to a 2-bridge link group $G(r)$ if and only if \tilde{r} or $\tilde{r} + 1$ belongs to the $\hat{\Gamma}_r$ -orbit of r or ∞ .*

On the other hand, it was proved by Boileau, Boyer, Reid, and Wang that any epimorphism from a hyperbolic 2-bridge knot group $G(\tilde{r})$ onto a non-trivial knot group $G(K)$ is induced by a non-zero degree map $S^3 - K(r) \rightarrow S^3 - K$ and that K is necessarily a two-bridge knot ([10, Corollary 1.3]). This in particular implies the following.

- (C) *Any epimorphism from a hyperbolic 2-bridge knot group $G(\tilde{r})$ to a non-trivial 2-bridge knot group $G(r)$ maps the upper meridian pair of $G(\tilde{r})$ to a pair consisting of peripheral elements. In particular, if $G(r)$ is also a hyperbolic 2-bridge knot group, then the image in $G(r)$ of the upper meridian pair of $G(\tilde{r})$ is a parabolic generating pair of $G(r)$.*

Furthermore, González-Acuña and Ramírez [21, Theorem 1.2] completely determined the 2-bridge knot groups that have epimorphisms onto a $(2, p)$ torus knot group $G(1/p)$. Their result can be reformulated as follows.

- (D) *There is an epimorphism from a 2-bridge knot group $G(\tilde{r})$ to a $(2, p)$ torus knot group $G(r)$ with $r = 1/p$ ($p > 1$ odd) if and only if \tilde{r} or $\tilde{r} + 1$ belongs to the $\hat{\Gamma}_r$ -orbit of $r = 1/p$ or ∞ .*

In fact, if there is an epimorphism $\varphi : G(\tilde{r}) \rightarrow G(1/p)$, then the composition of φ with the natural epimorphism $G(1/p) \rightarrow \mathbb{Z}_2 * \mathbb{Z}_p$ determines an epimorphism $G(\tilde{r}) \rightarrow \mathbb{Z}_2 * \mathbb{Z}_p$. Thus \tilde{r} satisfies the condition (iii) of [21, Theorem 1.2], which is equivalent to the condition (v) of the theorem. The latter condition in turn is equivalent to the condition that \tilde{r} or $\tilde{r} + 1$ belongs to the $\hat{\Gamma}_r$ -orbit of $r = 1/p$ or ∞ by [38, Proposition 5.1].

By using the above results (A)~(D) and Theorem 1.2(1), we obtain the following complete characterisation of epimorphisms between 2-bridge knot groups.

THEOREM 8.1. *There is an epimorphism from a 2-bridge knot group $G(\tilde{r})$ to a 2-bridge knot group $G(r)$ with $r = q/p$, if and only if one of the following conditions holds.*

1. \tilde{r} or $\tilde{r} + 1$ belongs to the $\hat{\Gamma}_r$ -orbit of r or ∞ .
2. \tilde{r} or $\tilde{r} + 1$ belongs to the $\hat{\Gamma}_{r'}$ -orbit of r' or ∞ , where $r' = q'/p$ with $qq' \equiv 1 \pmod{p}$.

Proof. Recall the well-known fact that if r and r' are as in (2) in the theorem then there is an isomorphism $G(r) \rightarrow G(r')$ which brings the upper meridian

pair of $G(r)$ to the lower meridian pair of $G(r')$ (cf. [7, Proposition 2.1(1.b)]). The if part of the theorem is a consequence of the result (A) and this fact.

So we prove the only if part. If $K(r)$ is a trivial knot, then every \tilde{r} satisfies the condition (1) or (2), and so the result holds trivially. If $K(r)$ is a nontrivial torus knot, then it is nothing other than the result (D). Thus we may assume $K(r)$ is a hyperbolic knot. Let $\varphi : G(\tilde{r}) \rightarrow G(r)$ be an epimorphism and $\{\tilde{\alpha}, \tilde{\beta}\}$ the upper meridian pair of $G(\tilde{r})$. Set $\{\alpha, \beta\} := \{\varphi(\tilde{\alpha}), \varphi(\tilde{\beta})\}$. Then, by the result (C), $\{\alpha, \beta\}$ is a parabolic generating pair of $G(r)$. Thus, by Theorem 1.2(1), it is equivalent to the upper or lower meridian pair of $G(r)$. If $\{\alpha, \beta\}$ is equivalent to the upper-meridian pair, then the result (B) implies that the condition (1) holds. If $\{\alpha, \beta\}$ is equivalent to the lower meridian pair of $G(r)$, its image in $G(r')$ by the isomorphism $G(r) \rightarrow G(r')$, described at the beginning of the proof, is the upper meridian pair of $G(r')$. Thus the composition $G(\tilde{r}) \rightarrow G(r) \cong G(r')$ is an upper-meridian-pair-preserving epimorphism. Hence the result (B) implies that \tilde{r} satisfies the condition (2). This completes the proof of Theorem 8.1. \square

REMARK 8.2: (1) We can describe all epimorphisms between given two hyperbolic 2-bridge knot groups as follows. For each $r \in \hat{\mathbb{Q}}$, fix a one-relator presentation $G(r) = \langle a, b \mid u_r \rangle$ as in [33, Section 3], where the ordered pair (a, b) represents the upper-meridian pair. If \tilde{r} belongs to the $\hat{\Gamma}_r$ -orbit of r or ∞ , then the identity map on the free group $F(a, b)$ with free basis $\{a, b\}$ descends to an (upper-meridian-pair-preserving) epimorphism $G(\tilde{r}) \rightarrow G(r)$: we call it the *ORS epimorphism* from $G(\tilde{r})$ to $G(r)$ (cf. [38, Proof of Theorem 1.1 in p.428]). By the proof of Theorem 8.1 and [33, Proof of Main Theorem 2.4 in pp.364–365], we can see that any epimorphism $G(\tilde{r}) \rightarrow G(r)$ between given two hyperbolic 2-bridge knot groups is equal to a composition

$$G(\tilde{r}) \cong G(\tilde{r}^*) \rightarrow G(r^*) \cong G(r)$$

of two isomorphisms between 2-bridge knot groups and an ORS epimorphism $G(\tilde{r}^*) \rightarrow G(r^*)$.

- (2) The above conclusion also holds for any epimorphism between hyperbolic 2-bridge link groups which are induced by a non-zero degree map between the exteriors of the links, because such an epimorphism maps the upper-meridian pair to a parabolic generating pair and so we can apply Theorem 1.2(1) as in the proof of Theorem 8.1.
- (3) The results of González-Acuña and Ramírez [21, 22, 23] imply that the conclusion in (1) also holds when the target is a non-hyperbolic 2-bridge knot $K(r)$ with $r = 1/p$ ($p > 1$ odd). To see this, note that $G(1/p)$ is the pullback (or the fiber product) of the diagram $\mathbb{Z} \rightarrow \mathbb{Z}_{2p} \leftarrow \mathbb{Z}_2 * \mathbb{Z}_p$ (see

[28, Example 2]). In fact, this diagram together with the abelianization $\xi : G(1/p) \rightarrow \mathbb{Z}$ and the natural epimorphism $\rho : G(1/p) \rightarrow \mathbb{Z}_2 * \mathbb{Z}_p$ form a pullback diagram. Now let $\varphi : G(\tilde{r}) \rightarrow G(1/p)$ be an epimorphism. For simplicity, we assume that \tilde{r} belongs to the $\hat{\Gamma}_r$ -orbit of r or ∞ , and let $\varphi_0 : G(\tilde{r}) \rightarrow G(1/p)$ be the ORS epimorphism. Set $\tilde{\xi} = \xi \circ \varphi$, $\tilde{\xi}_0 = \xi \circ \varphi_0$, $\tilde{\rho} = \rho \circ \varphi$, $\tilde{\rho}_0 = \rho \circ \varphi_0$. Then φ and φ_0 are uniquely determined by the pairs $(\tilde{\xi}, \tilde{\rho})$ and $(\tilde{\xi}_0, \tilde{\rho}_0)$, respectively. By the argument in [10, the last part of the proof of Corollary 1.3] based on [22, Theorem 18], we see either (i) $\tilde{\xi} = \tilde{\xi}_0$ and $\tilde{\rho} = \tilde{\rho}_0$ modulo post composition of an inner-automorphism of $\mathbb{Z}_2 * \mathbb{Z}_p$ or (ii) $\tilde{\xi} = h_1 \circ \tilde{\xi}_0$ and $\tilde{\rho} = h_2 \circ \tilde{\rho}_0$ modulo post composition of an inner-automorphism of $\mathbb{Z}_2 * \mathbb{Z}_p$, where $h_1 = -1_{\mathbb{Z}}$ is the automorphism of \mathbb{Z} defined by $h_1(z) = -z$ and h_2 is the automorphism of $\mathbb{Z}_2 * \mathbb{Z}_p$ defined by $h_2 = 1_{\mathbb{Z}_2} * (-1_{\mathbb{Z}_p}) = (-1_{\mathbb{Z}_2}) * (-1_{\mathbb{Z}_p})$. In the first case, $\varphi = \varphi_0$ modulo post composition of an inner-automorphism of $G(1/p)$. In the second case, $\varphi = h \circ \varphi_0$ modulo post composition of an inner-automorphism of $G(1/p)$, where h is the automorphism of $G(1/p)$ induced by the strong inversion of $K(1/p)$.

We do not know if Theorem 8.1 holds for 2-bridge links, because we do not know if the result (C) holds for 2-component 2-bridge links. In fact, there are epimorphisms between 2-bridge link groups which are induced by degree 0 maps between the link complements (see [38, Remark 6.3 and Proof of Corollary 8.1]), in contrast to [10, Corollary 1.3]. However, we can obtain a condition for the existence of a degree one map between hyperbolic 2-bridge link complements. (We thank the referee for suggesting this to us.) From now on, we work with link exteriors instead of link complements, and by a *degree one map* between the exteriors of two links L_1 and L_2 in S^3 , we mean a map $f : (E(L_1), \partial E(L_1)) \rightarrow (E(L_2), \partial E(L_2))$ such that $f_* : H_3(E(L_1), \partial E(L_1)) \rightarrow H_3(E(L_2), \partial E(L_2))$ carries the fundamental class $[E(L_1), \partial E(L_1)]$ to $\pm[E(L_2), \partial E(L_2)]$. To describe the result, recall that any ORS epimorphism $G(\tilde{r}) \rightarrow G(r)$ is induced by (the restriction to the link exteriors of) a “branched fold map” between 2-bridge links (see [38, Theorem 6.1]), whose degree, $\deg(\tilde{r}, r)$, is calculated from continued fraction expansions of \tilde{r} and r (see [38, Proposition 6.2(2) and Remark 6.3(1)]).

THEOREM 8.3. *There is a degree one map $f : E(K(\tilde{r})) \rightarrow E(K(r))$ with $r = q/p$ between hyperbolic 2-bridge link exteriors, if and only if one of the following conditions holds.*

1. \tilde{r} or $\tilde{r} + 1$ belongs to the $\hat{\Gamma}_r$ -orbit of r or ∞ , and $\deg(\tilde{r}, r)$ or $\deg(\tilde{r} + 1, r)$, accordingly, is equal to ± 1 .
2. \tilde{r} or $\tilde{r} + 1$ belongs to the $\hat{\Gamma}_{r'}$ -orbit of r' or ∞ , where $r' = q'/p$ with $qq' \equiv 1 \pmod{p}$, and $\deg(\tilde{r}, r')$ or $\deg(\tilde{r} + 1, r')$, accordingly, is equal to ± 1 .

Proof. The if part follows from the description above and [38, Theorem 6.1 and Proposition 6.2]. To prove the only if part, let $f : E(K(\tilde{r})) \rightarrow E(K(r))$ be a degree one map between hyperbolic 2-bridge link exteriors. Then f induces an epimorphism $f_* : G(\tilde{r}) \rightarrow G(r)$, and it maps the upper meridian pair of $G(\tilde{r})$ to a parabolic generating pair of $G(r)$. Thus, as in the proof of Theorem 8.1, we see by using Theorem 1.2 that one of the conditions (1) and (2) in Theorem 1.2 holds. Moreover, as noted in Remark 8.2(2), the epimorphism f_* is equal to an ORS epimorphism modulo pre/post-compositions of isomorphisms of 2-bridge link groups. Since $E(K(r))$ is aspherical and since the peripheral subgroups are malnormal, the map f is properly homotopic to the branch-fold map given by [38, Theorem 6.1], modulo pre/post-compositions of homeomorphisms between 2-bridge link exteriors. Hence the degree of f is equal to $\pm \deg(\tilde{r}, r)$, $\pm \deg(\tilde{r} + 1, r)$, $\pm \deg(\tilde{r}, r')$ or $\pm \deg(\tilde{r} + 1, r')$ accordingly. Since $\deg(f) = \pm 1$, one of the conditions of Theorem 8.3 holds. \square

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Authors' addresses:

Shunsuke Aimi
Yonago-Nishi High School
Ohtani-cho 200, Yonago, Tottori, 683-8512, Japan
E-mail: 1991.aimi@gmail.com

Donghi Lee
Department of Mathematics
Pusan National University
San-30 Jangjeon-Dong, Geumjung-Gu, Pusan, 609-735, Korea
E-mail: donghi@pusan.ac.kr

Shunsuke Sakai
Department of Mathematics
Hiroshima University
Higashi-Hiroshima, 739-8526, Japan
E-mail: d180102@hiroshima-u.ac.jp

Makoto Sakuma
Advanced Mathematical Institute
Osaka City University
3-3-138, Sugimoto, Sumiyoshi, Osaka City 558-8585, Japan
and
Department of Mathematics
Hiroshima University
Higashi-Hiroshima, 739-8526, Japan
E-mail: sakuma@hiroshima-u.ac.jp

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