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Two Moore's theorems for graphs

Alexander Mednykh and Ilya Mednykh

Devoted to the 70-th anniversary of our friend and colleague
Bruno Zimmermann

Abstract. *Let X be a finite connected graph, possibly with loops and multiple edges. An automorphism group of X acts purely harmonically if it acts freely on the set of directed edges of X and has no invertible edges. Define a genus g of the graph X to be the rank of the first homology group. A finite group acting purely harmonically on a graph of genus g is a natural discrete analogue of a finite group of automorphisms acting on a Riemann surface of genus g. In the present paper, we investigate cyclic group $\mathbb{Z}_n$ acting purely harmonically on a graph X of genus g with fixed points. Given subgroup $\mathbb{Z}_d < \mathbb{Z}_n$, we find the signature of orbifold $X/\mathbb{Z}_d$ through the signature of orbifold $X/\mathbb{Z}_n$. As a result, we obtain formulas for the number of fixed points for generators of group $\mathbb{Z}_d$ and for genus of orbifold $X/\mathbb{Z}_d$. For Riemann surfaces, similar results were obtained earlier by M. J. Moore.*

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1. Introduction

Let X be a finite connected graph, possibly with loops and multiple edges. We provide each edge of X (including loops) by two possible orientations. Define the genus g of the graph X to be the rank of its first homology group. An automorphism group of a graph is said to act *harmonically* if it acts freely on the set of its directed edges and *purely harmonically* if it also has no invertible edges. By [2] and [4], a finite group acting harmonically on a graph of genus g is a natural discrete analogue of a finite group of homeomorphisms acting on a closed orientable topological surface of genus g. In particular, in papers [4, 5], a discrete version of the classical $84(g-1)$ Hurwitz theorem is established. Also discrete versions of the Oikawa [15] and the Arakawa [1] theorems which sharpen the Hurwitz upper bound for various classes of groups where found in [9, 13].

An automorphism of a graph X is said to be *harmonic* if it generates a cyclic group acting harmonically on X. In paper [12], the authors have found a discrete analogue of the Wiman theorem by which the order of a harmonic automorphism of a graph X of genus $g \geq 2$ does not exceed $2g+2$ and this bound is achieved for any even g. However, in contrary to the Riemann surface case considered by Wiman, an automorphism of maximal order acts without fixed points. The size of cyclic group acting harmonically on X with given number of fixed points was estimated from the above in [6].

In the present paper, we investigate cyclic group $\mathbb{Z}_n$ acting purely harmonically on a graph X of genus g with fixed points. Given subgroup $\mathbb{Z}_d < \mathbb{Z}_n$, we find the signature of orbifold $X/\mathbb{Z}_d$ through the signature of $X/\mathbb{Z}_n$. To do this, we use an approach developed by the first named author [10] for the case of Riemann surfaces. As a result, we obtain formulas for the number of fixed points for generators of group $\mathbb{Z}_d$ and for genus of orbifold $X/\mathbb{Z}_d$. For Riemann surfaces, similar results were obtained earlier by M. J. Moore [14].

2. Basic definitions and preliminary facts

In this paper, a graph X is a finite connected multigraph, possibly with loops. We provide each edge of X including loops, by the two possible orientations. Denote by $V(X)$ the set of vertices and by $E(X)$ the set of directed edges of X. Given $e \in E(X)$, by $\bar{e}$ we denote edge e taking with the opposite orientation. Let $G \leq \mathrm{Aut}(X)$ be a group of automorphisms of a graph X. An edge $e \in E(X)$ is called *invertible* if there is $\varphi \in G$ such that $\varphi(e) = \bar{e}$. Let G act without invertible edges. Define the quotient graph X/G so that its vertices and edges are G-orbits of the vertices and edges of X. Note that if the endpoints of an edge $e \in E(X)$ lie in the same G-orbit then the G-orbit of e is a loop in the quotient graph X/G. We say that the group G acts *harmonically* on a graph X if it acts freely on the set of directed edges $E(G)$ which simply means that, each element of G that fixes an edge $e \in E(G)$ is the identity. If G acts harmonically and without invertible edges, we say that G acts *purely harmonically* on X.

Let G be a finite group acting purely harmonically on a graph X. For every $\tilde{v} \in V(X)$ denote by $G_{\tilde{v}}$ the stabilizer of $\tilde{v}$ in the group G and by $|G_{\tilde{v}}|$ its order. Next to each vertex $v \in V(X/G)$ we prescribe the number $m_v = |G_{\tilde{v}}|$, where $\tilde{v} \in \varphi^{-1}(v)$. Since G acts transitively on each fibre of φ, these numbers are well-defined. The point v, for which $m_v \geq 2$, will be called *branch point of order m_v*. Defining the *genus* of a graph as its cyclomatic number or Betti number (equivalently, rank of the first homology group) we have the following version of the Riemann-Hurwitz formula that can be found in [2, 4, 11].

Proposition 2.1. *Let G be a finite group acting purely harmonically on a graph*

X of genus g. Denote by γ genus of the factor graph X/G. Then

$$g-1=|G|\left(\gamma-1+\sum_{v\in V(X/G)}\left(1-\frac{1}{m_v}\right)\right). \quad (1)$$

Observe that actually v in the above sum run over the branch points of X/G.

We prefer to look on the quotient graph X/G as on a one-dimensional orbifold. In this case, the notion of signature is very important. If the group G acts purely harmonically on X, the signature is defined as the sequence $(\gamma; m_1, \ldots, m_r)$, where γ is the genus of X/G and $m_1, m_2, \ldots, m_r$ are branch orders used in Proposition 2.1. In the case of repetition, we will use power mark to indicate the number of equal entries. For example, we write $(3; 1^3, 2^2, 3^1)$ instead of $(3; 1, 1, 1, 2, 2, 3)$.

3. Finite group acting purely harmonically on a graph

The main technique of the present paper is the uniformization theory of graphs and their coverings [7, 12, 9].

Let G be a finite group acting purely harmonically on a graph X of genus g with the factor space X/G of signature $(\gamma; m_1, \ldots, m_r)$. In what follows, we suppose that all vertex stabilizers of G on graph X are cyclic groups.

Let $\mathcal{X}$ be a graph of groups with a trivial group assigned to each vertex and each edge of X. Consider a graph of groups $\mathcal{Y}$ obtained by prescribing the respective group Z_{m_i}, $i = 1, \ldots, r$ to each of r points of the branch set and trivial groups to all other vertices and edges of Y. Then the map $\varphi : X \to Y$ can be naturally extended to a covering $\Phi : \mathcal{X} \to \mathcal{Y}$ of graph of groups. Denote by $\Delta = \pi_1(\mathcal{X})$ and $\Gamma = \pi_1(\mathcal{Y})$ the fundamental groups and by $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$ the universal covering trees of graphs of groups $\mathcal{X}$ and $\mathcal{Y}$ respectively. By the Bass uniformization theorem [3, Proposition 2.4], there exists a lift of Φ to an isomorphism $\widetilde{\Phi} : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{Y}}$ between covering trees equivariant under the action of Δ and Γ on $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$ respectively. We note that $\mathcal{X} \cong \widetilde{\mathcal{X}}/\Delta$ and $\mathcal{Y} \cong \widetilde{\mathcal{Y}}/\Gamma$. Identifying $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$ via isomorphism $\widetilde{\Phi}$ we replace the covering $\Phi : \mathcal{X} \to \mathcal{Y}$ by the covering $\widetilde{\mathcal{X}}/\Delta \to \widetilde{\mathcal{X}}/\Gamma$ induced by a group inclusion $H \lhd \Gamma$ with $\Gamma/\Delta \cong G$.

By [3, p. 7], Δ is a free group of the rank g and $\Gamma = \mathbb{Z}^{*\gamma} * \mathbb{Z}_{m_1} * \ldots * \mathbb{Z}_{m_r}$, where $\mathbb{Z}^{*\gamma}$ is a free product of γ copies of $\mathbb{Z}$. Let $\widetilde{X}$ be the universal covering tree of the graph X. Note that $\widetilde{X}$ is the underlying graph of the graph of group $\widetilde{\mathcal{X}}$. Following traditions in the Riemann surface theory (see, for example, [8]), one can refer to Γ as a *universal covering group* of the orbifold X/G.

4. Cyclic group action on Riemann surfaces and graphs

First of all, we are going to find a discrete version of the following result established earlier in [10]. Then, we will use the obtained result to establish graph-theoretical versions of two Moore's theorems proved in [14].

THEOREM 4.1 (A. D. Mednykh, 1980). *Let S be a compact Riemann surface and $\mathbb{Z}_n$ be a cyclic group of conformal automorphisms of S. Denote by $(\gamma; m_1, \ldots, m_r)$ the signature of orbifold $S/\mathbb{Z}_n$. Let $\mathbb{Z}_d$ be a subgroup of $\mathbb{Z}_n$ of order d. Then the orbifold $S/\mathbb{Z}_d$ has the signature*

$$\Big(\gamma_d; (m_1, d)^{n/[m_1,d]}, \ldots, (m_r, d)^{n/[m_r,d]}\Big),$$

where $(m_i, d) = \gcd(m_i, d)$, $[m_i, d] = \operatorname{lcm}(m_i, d)$, $i = 1, 2, \ldots, r$, while γ and γ_d are genera of the respective orbifolds.

We prove the following theorem.

THEOREM 4.2. *Let X be a finite connected graph and $\mathbb{Z}_n$ be a cyclic group acting purely harmonically on X. Denote by $(\gamma; m_1, \ldots, m_r)$ the signature of orbifold $X/\mathbb{Z}_n$. Let $\mathbb{Z}_d$ be a subgroup of $\mathbb{Z}_n$ of order d. Then the orbifold $X/\mathbb{Z}_d$ has the signature*

$$\Big(\gamma_d; (m_1, d)^{n/[m_1,d]}, \ldots, (m_r, d)^{n/[m_r,d]}\Big),$$

where $(m_i, d) = \gcd(m_i, d)$, $[m_i, d] = \operatorname{lcm}(m_i, d)$, $i = 1, 2, \ldots, r$, while γ and γ_d are genera of the respective orbifolds.

Proof. Let $\widetilde{X}$ be the universal covering graph of orbifold $\mathbb{O} = X/\mathbb{Z}_n$. See [12] for detailed definition. Then there is an action of the group $\Gamma = \mathbb{Z}^{*\gamma} * \mathbb{Z}_{m_1} * \ldots * \mathbb{Z}_{m_r}$ on $\widetilde{X}$ such that the factor graph $\widetilde{X}/\Gamma$ is isomorphic to $\mathbb{O}$. That is, Γ is the universal covering group of $\mathbb{O}$. Moreover, there exists an order preserving epimorphism $\theta : \Gamma \to \mathbb{Z}_n$ whose kernel is a free group of rank g, where g is genus of graph X.

Denote by H preimage $\theta^{-1}(\mathbb{Z}_d)$. Then H is the universal covering group of orbifold $\mathbb{O}_d = X/\mathbb{Z}_d$. Identifying $\mathbb{O}_d$ with $\widetilde{X}/H$ and $\mathbb{O}$ with $\widetilde{X}/\Gamma$ we have the sequence of orbifold coverings $\widetilde{X} \xrightarrow{p} \mathbb{O}_d \xrightarrow{q} \mathbb{O}$ induced by group inclusions $\mathrm{I} < H < \Gamma$. Since θ is order preserving we have $|\Gamma : H| = |\mathbb{Z}_n : \mathbb{Z}_d| = n/d$. This number coincides with multiplicity of covering q. That is, each edge of $\mathbb{O}$ has exactly n/d preimages in $\mathbb{O}_d$. Consider a vertex $x \in \mathbb{O}_d$. By [12, Example 1], branch order of x in orbifold $\mathbb{O}_d$ is equal to the size of stabilizer $|H_{\tilde{x}}|$ of group H in any preimage $\tilde{x} \in p^{-1}(x)$. In the same time, branch order of $y = q(x)$

in orbifold $\mathbb{O}$ is equal to the size of stabilizer $|\Gamma_{\tilde{x}}|$. Avoiding the points with trivial stabilizer, we can assume that $|\Gamma_{\tilde{x}}| = m_j$ for some $j = 1, 2. \ldots, r$ and y is a branch point of orbifold $\mathbb{O}$. Recall [12] that $\Gamma_{\tilde{x}}$ is a cyclic group of order m_j. The stabilizer $H_{\tilde{x}}$ is formed by those elements of $\Gamma_{\tilde{x}} = \mathbb{Z}_{m_j}$ whose images under epimorphism θ belong to $\mathbb{Z}_d$. Since θ preserves the order of elements, we have $H_{\tilde{x}} = \mathbb{Z}_{(m_j,d)}$, where (m_j, d) is the greatest common divisor of m_j and d. So, each element in the fiber $q^{-1}(y)$ of the n/d-fold covering q has branch order (m_j, d). Notice there are exactly $(n/d) : (m_j/(m_j, d)) = n(m_j, d)/(m_j d) = n/[m_j, d]$ of them. This gives the proof of the theorem. □

As a consequence of the above result, we obtain the following version of the Moore formula for the number of fixed points well known in the Riemann surface theory [8, 14]. An alternative proof of this result can be obtained by making use of discrete version of Macbeath's formula given in [6].

Theorem 4.3 (Moore's formula for graphs). *Let $\mathbb{Z}_n$ be a cyclic group acting purely harmonically on a graph X and h be an element of order d, $d > 1$ in the group $\mathbb{Z}_n$. Denote by $(\gamma; m_1, \ldots, m_r)$ the signature of the orbifold $X/\mathbb{Z}_n$. Then the number of fixed points of h is given by the formula*

$$\sum_{d\,|\,m_i} \frac{n}{m_i}.$$

Proof. Consider the canonical map $\varphi : X \to X/\mathbb{Z}_d$, where $\mathbb{Z}_d$ is a cyclic group generated by h. By Theorem 4.2, the signature of orbifold $\mathbb{O}_d = X/\mathbb{Z}_d$ is equal to

$$\Big(\gamma_d; (m_1, d)^{n/[m_1,d]}, \ldots, (m_r, d)^{n/[m_r,d]}\Big),$$

where γ_d is genus of $\mathbb{O}_d$.

Let $x \in \mathbb{O}_d$ and $F_x = \varphi^{-1}(x)$ be the fiber of x. Since the covering φ is regular, the group $\mathbb{Z}_d$ acts transitively of the set F_x. So, the fiber F_x contains a fixed point of h if and only if it consists of one element. If x is an ordinary point of orbifold $\mathbb{O}_d$ (that is, branch point of order 1), the fiber F_x consists of $d > 1$ elements and has no fixed points of h. According to the signature of $\mathbb{O}_d$, for any $i = 1, 2, \ldots, r$ we have $n/[m_i, d]$ branch points of order (m_i, d). The fiber of such a point has length one if and only if $d/(m_i, d) = 1$. The latter is equivalent to $d|m_i$. As a result, the number of fixed points of h is given by

$$\sum_{d\,|\,m_i} \frac{n}{[m_i, d]} = \sum_{d\,|\,m_i} \frac{n}{m_i}.$$

□

As a direct consequence of Theorem 4.3 we have the following statement.

Corollary 4.4. *Let $\mathbb{Z}_n$ be a cyclic group acting purely harmonically on a graph X and denote by $(\gamma; m_1, \ldots, m_r)$ the signature of the orbifold $X/\mathbb{Z}_n$. Then the number of fixed points of a generator of $\mathbb{Z}_n$ coincides with the number of entities m_i in the signature which are equal to n.*

Now our aim is to obtain Moore's formula for the genus of $X/\mathbb{Z}_d$. The original result [14, Theorem 4], rewritten in terms of orbifolds, is given by the theorem below. The proof of this theorem in the paper [14] is quite complicated. Our approach, based on Theorems 4.1 and 4.2, allows to get the result quickly.

Theorem 4.5. *Let G be a cyclic group of automorphisms of order n acting on a Riemann surface S of genus g at least two. Suppose that the signature of orbifold S/G has r_b periods b, for each b dividing n, and, for $d|n$, let G_d be the subgroup of G of order d. Then the orbit space S/G_d has genus γ_d, given by*

$$\gamma_d = 1 + \frac{1}{d}(g-1) - \frac{1}{2d}\sum_{bb'=n} b' r_b\big((b,d)-1\big),$$

where (b,d) denotes the greatest common divisor of b and d.

The following theorem a discrete version of the above Moore's theorem for cyclic orbifold $\mathbb{O}_d = X/\mathbb{Z}_d$.

Theorem 4.6. *Let X be a finite graph of genus g and $\mathbb{Z}_n$ be a cyclic group acting purely harmonically on X. Denote by $(\gamma; m_1, \ldots, m_r)$ be the signature of orbifold $X/\mathbb{Z}_n$. Let $\mathbb{Z}_d$ be a subgroup of $\mathbb{Z}_n$ of order d. Then genus g of graph X and genus γ_d of orbifold $\mathbb{O}_d = X/\mathbb{Z}_d$ are related by the formula*

$$g - 1 = d(\gamma_d - 1) + \sum_{i=1}^{r} \frac{n}{m_i}\big((m_i, d) - 1\big).$$

where $(m_i, d) = \gcd(m_i, d)$.

Proof. By Theorem 4.2, orbifold $\mathbb{O}_d = X/\mathbb{Z}_d$ has the following signature

$$\Big(\gamma_d; (m_1, d)^{n/[m_1,d]}, \ldots, (m_r, d)^{n/[m_r,d]}\Big).$$

By the Riemann-Hurwitz formula (1) we obtain

$$g - 1 = d\left(\gamma_d - 1 + \sum_{i=1}^{r} \frac{n}{[m_i, d]}\left(1 - \frac{1}{(m_i, d)}\right)\right). \tag{2}$$

We note that $[m_i, d](m_i, d) = m_i\, d$ and

$$
\begin{aligned}
d\left(\frac{n}{[m_i,d]}\left(1-\frac{1}{(m_i,d)}\right)\right) &= d\left(\frac{n}{[m_i,d]}-\frac{n}{[m_i,d](m_i,d)}\right)=\frac{n\,d}{[m_i,d]}-\frac{n\,d}{m_i\,d}\\
&=\frac{n}{m_i}\left(\frac{m_i\,d}{[m_i,d]}-1\right)=\frac{n}{m_i}\big((m_i,d)-1\big).
\end{aligned}
$$

Hence,

$$
d\sum_{i=1}^{r}\frac{n}{[m_i,d]}\left(1-\frac{1}{(m_i,d)}\right)=\sum_{i=1}^{r}\frac{n}{m_i}\big((m_i,d)-1\big)
$$

and the result follows from (2). □

Assuming that signature $(\gamma; m_1, \ldots, m_r)$ has r_b periods $m_j = b$, for each b dividing n, we restate the main result of Theorem 4.6 in the form

$$
\gamma_d = 1+\frac{1}{d}(g-1)-\frac{1}{d}\sum_{bb'=n} b' r_b\big((b,d)-1\big).
$$

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Authors' addresses:

Alexander Mednykh
Sobolev Institute of Mathematics
pr. Koptyuga 4, Novosibirsk, 630090, Russia
Novosibirsk State University
Pirogova 1, Novosibirsk, 630090, Russia
E-mail: smedn@mail.ru

Ilya Mednykh
Sobolev Institute of Mathematics
pr. Koptyuga 4, Novosibirsk, 630090, Russia
Novosibirsk State Universit
Pirogova 1, Novosibirsk, 630090, Russia
E-mail: ilyamednykh@mail.ru

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