

Crystallizations of compact 4-manifolds minimizing combinatorially defined PL-invariants

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Dedicated to our friend Bruno Zimmermann

ABSTRACT. *The present paper is devoted to present a unifying survey about some special classes of crystallizations of compact PL 4-manifolds with empty or connected boundary, called semi-simple and weak semi-simple crystallizations, with a particular attention to their properties of minimizing combinatorially defined PL-invariants, such as the regular genus, the Gurau degree, the gem-complexity and the (gem-induced) trisection genus.*

The main theorem, yielding a summarizing result on the topic, is an original contribution. Moreover, in the present paper the additivity of regular genus with respect to connected sum is proved to hold for all compact 4-manifolds with empty or connected boundary which admit weak semi-simple crystallizations.

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1. Introduction

It is well known that, thanks to a bright idea by Mario Pezzana ([28, 29]), every closed PL n -manifold M can be triangulated by a pseudocomplex K , whose vertices are exactly $n + 1$ (i.e. the minimum possible). If this is the case, K and its dual edge-colored graph are called a *contracted triangulation* and a *crystallization* of M respectively.

More recently, the above result was extended to *singular n -manifolds*, i.e. triangulated polyhedra, whose vertices may have not only spheres, but also closed connected $(n - 1)$ -manifolds as links. In this context, some kind of “minimality” with respect to the number of vertices of the obtained pseudo-

complex can be considered, too. In particular, if M has only one singular vertex, then Pezzana's theorem can be presented exactly in the same form. Hence, each such singular n -manifold may be combinatorially visualized and studied by means of regular graphs of degree $n + 1$ (still called *crystallizations*) whose edges are labelled by $n + 1$ colors and such that the subgraph obtained by deleting all edges of any chosen color is connected ([15, 20]).

Since singular n -manifolds with only one singular vertex are in bijection with manifolds with connected boundary, crystallizations can be thought of as a representation for manifolds with connected (non-spherical) boundary, too. Straightforward generalizations are known for singular n -manifolds with several singular vertices, i.e. for compact manifolds with several boundary components.

The present paper is devoted to present a unifying survey about some special classes of crystallizations of compact PL 4-manifolds with empty or connected boundary, called *semi-simple* and *weak semi-simple crystallizations* (see Section 4 for details), with a particular attention to their properties of minimizing interesting combinatorially defined PL-invariants, such as the *regular genus*, the *Gurau degree* and the *gem-complexity*.

The main achievement is the proof of the following summarizing result, which is an original contribution of the present paper.

THEOREM 1.1 (Main Theorem). *Let M^4 be a compact 4-manifold with empty or connected boundary, and let \widehat{M}^4 be its associated singular manifold; let us assume $rk(\pi_1(M^4)) = m \geq 0$ and $rk(\pi_1(\widehat{M}^4)) = m' \geq 0$. Then:*

(a) *The regular genus $\mathcal{G}(M^4)$ of M^4 satisfies*

$$\mathcal{G}(M^4) \geq 2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4.$$

Moreover, equality holds if and only if M^4 admits a weak semi-simple crystallization.

(b) *The Gurau degree $\mathcal{D}_G(M^4)$ of M^4 satisfies*

$$\mathcal{D}_G(M^4) \geq 12 \left[2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4 \right].$$

Moreover, equality holds if and only if M^4 admits a semi-simple crystallization.

(c) *The gem-complexity $k(M^4)$ of M^4 satisfies*

$$k(M^4) \geq 3\chi(\widehat{M}^4) + 10m - 4(m - m') - 6.$$

Moreover, equality holds if and only if M^4 admits a semi-simple crystallization.

A further original contribution of the paper is Proposition 4.4, yielding a characterization of compact 4-manifolds admitting semi-simple crystallizations, via a relationship between gem-complexity and regular genus.

In Section 5 the relevant¹ problem of the additivity of regular genus with respect to connected sum is studied, and it is proved that the additivity holds for all compact 4-manifolds with empty or connected boundary which admit weak semi-simple crystallizations: see Proposition 5.2.

Further, Section 6 recalls the notion of *gem-induced trisection* (due to [10]), which extends the well-known notion of *trisection* (introduced in 2016 by Gay and Kirby: see [24]) to compact orientable 4-manifolds with connected boundary, whose associated singular manifold is simply-connected. Also in this context, as a particular case of results proved in [10], weak semi-simple crystallizations turn out to have a “minimality property”, which enables to directly relate the so called *gem-induced trisection genus* with the regular genus and/or the Betti numbers of the represented manifold: see Propositions 6.5 and 6.6.

2. Basic elements of crystallization theory

In the present section we will briefly review some basic notions of the so called *crystallization theory*, as a representation tool for piecewise linear (PL) compact manifolds; further details may be found in the quoted papers.

From now on, unless otherwise stated, all spaces and maps will be considered in the PL category, and all manifolds will be assumed to be compact and connected.

DEFINITION 2.1. *An $(n + 1)$ -colored graph $(n \geq 2)$ is a pair (Γ, γ) , where $\Gamma = (V(\Gamma), E(\Gamma))$ is a multigraph (i.e. multiple edges are allowed, while loops are forbidden) which is regular of degree $n + 1$, and γ is an edge-coloration, that is a map $\gamma : E(\Gamma) \rightarrow \Delta_n = \{0, \dots, n\}$ which is injective on adjacent edges.*

For sake of concision, when the coloration is clearly understood, colored graphs are often denoted simply by Γ .

For every $\{c_1, \dots, c_h\} \subseteq \Delta_n$ let $\Gamma_{\{c_1, \dots, c_h\}}$ be the subgraph obtained from Γ by deleting all the edges that are not colored by the elements of $\{c_1, \dots, c_h\}$. In this setting, the complementary set of $\{c\}$ (resp. $\{c_1, \dots, c_h\}$) in Δ_n will be denoted by \hat{c} (resp. $\hat{c}_1 \cdots \hat{c}_h$). The connected components of $\Gamma_{\{c_1, \dots, c_h\}}$ are called $\{c_1, \dots, c_h\}$ -*residues* or *h-residues* of Γ ; their number is denoted by $g_{\{c_1, \dots, c_h\}}$ (or, for short, by g_{c_1, c_2} , g_{c_1, c_2, c_3} and $g_{\hat{c}}$ if $h = 2$, $h = 3$ and $h = n$ respectively).

¹In the closed 4-dimensional case, the problem is strictly related to the Smooth Poincaré Conjecture: see [21] or Section 5.

Each $(n+1)$ -colored graph Γ encodes an associated n -dimensional pseudocomplex $K(\Gamma)$:

- $K(\Gamma)$ contains an n -simplex for each vertex of Γ , and the vertices of any n -simplex are (injectively) labelled by the elements of Δ_n ;
- if two vertices of Γ are c -adjacent ($c \in \Delta_n$), then the corresponding n -simplices of $K(\Gamma)$ are glued along their $(n-1)$ -dimensional faces opposite to the c -labelled vertices, so that equally labelled vertices are identified.

In general $|K(\Gamma)|$ is an n -pseudomanifold and Γ is said to *represent* it.

Via the above construction, it is not difficult to prove that:

- $|K(\Gamma)|$ is a closed n -manifold iff, for each color $c \in \Delta_n$, all \hat{c} -residues of Γ represent the $(n-1)$ -sphere;
- $|K(\Gamma)|$ is a singular² n -manifold iff, for each color $c \in \Delta_n$, all \hat{c} -residues of Γ represent closed connected $(n-1)$ -manifolds.

REMARK 2.2: Note that a bijective correspondence exists between singular n -manifolds and compact n -manifolds with no spherical boundary components. In fact, if N is a singular n -manifold, then a compact n -manifold \check{N} is easily obtained by deleting small open neighbourhoods of its singular vertices: \check{N} turns out to be either closed (in case N itself is a closed manifold, and hence $N = \check{N}$) or with non-empty boundary, without spherical components. Conversely, given a compact n -manifold M without spherical boundary components, a singular n -manifold \widehat{M} can be constructed by capping off each component of ∂M by a cone over it.

For this reason, throughout the present work, we will restrict our attention to compact manifolds without spherical boundary components, and an $(n+1)$ -colored graph Γ will be said to *represent* a compact n -manifold M of this class (or, equivalently, to be a *gem* of M , where *gem* means *Graph Encoding Manifold*: see [26]) if and only if it represents the associated singular manifold \widehat{M} .

A restricted class of graphs gives the name to the whole theory:

DEFINITION 2.3. An $(n+1)$ -colored graph Γ representing a compact n -manifold with empty or connected boundary is said to be a *crystallization* of M if, for each color $c \in \Delta_n$, $\Gamma_{\hat{c}}$ is connected.

²A polyhedron $|K|$ (K being a simplicial complex) is said to be a *singular n -manifold* if the links of the vertices of K are closed connected $(n-1)$ -manifolds. The notion extends also to polyhedra associated to colored graphs: $|K(\Gamma)|$ is said to be a singular n -manifold if the links of vertices of $K(\Gamma)$ in its first barycentric subdivision are closed connected $(n-1)$ -manifolds. In both cases, a vertex whose link is not a $(n-1)$ -sphere is called a *singular vertex*.

The following theorem extends to the boundary case a well-known result - originally due to Pezzana ([28, 29]) - founding the combinatorial representation theory for closed manifolds of arbitrary dimension via colored graphs.

THEOREM 2.4 ([15, 9]). *Any compact orientable (resp. non orientable) n -manifold with no spherical boundary components admits a bipartite (resp. non-bipartite) $(n + 1)$ -colored graph representing it. In particular, any compact n -manifold with empty or connected boundary admits a crystallization representing it.*

The existence of a particular type of embedding of colored graphs into surfaces, is the key result in order to define two of the PL-invariants considered in the present paper.

PROPOSITION 2.5 ([23]). *Let Γ be a bipartite (resp. non-bipartite) $(n + 1)$ -colored graph of order $2p$. Then for each cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$ of Δ_n , up to inverse, there exists a cellular embedding of Γ into an orientable (resp. non-orientable) closed surface $F_\varepsilon(\Gamma)$ whose regions are bounded by the images of the $\{\varepsilon_j, \varepsilon_{j+1}\}$ -colored cycles, for each $j \in \mathbb{Z}_{n+1}$. Moreover, the genus (resp. half the genus) $\rho_\varepsilon(\Gamma)$ of $F_\varepsilon(\Gamma)$ satisfies*

$$2 - 2\rho_\varepsilon(\Gamma) = \sum_{j \in \mathbb{Z}_{n+1}} g_{\varepsilon_j, \varepsilon_{j+1}} + (1 - n)p.$$

DEFINITION 2.6. *Let Γ be an $(n + 1)$ -colored graph. If $\{\varepsilon^{(1)}, \varepsilon^{(2)}, \dots, \varepsilon^{(\frac{n!}{2})}\}$ is the set of all cyclic permutations of Δ_n (up to inverse), $\rho_{\varepsilon^{(i)}}(\Gamma)$ ($i = 1, \dots, \frac{n!}{2}$) is called the regular genus of Γ with respect to the permutation $\varepsilon^{(i)}$. Then, the Gurau degree (or G-degree for short) of Γ , denoted by $\omega_G(\Gamma)$, is defined as*

$$\omega_G(\Gamma) = \sum_{i=1}^{\frac{n!}{2}} \rho_{\varepsilon^{(i)}}(\Gamma)$$

and the regular genus of Γ , denoted by $\rho(\Gamma)$, is defined as

$$\rho(\Gamma) = \min \{ \rho_{\varepsilon^{(i)}}(\Gamma) \mid i = 1, \dots, \frac{n!}{2} \}.$$

As a consequence, focusing on the represented compact n -manifolds, the following combinatorially defined PL-invariants are introduced:

DEFINITION 2.7. *Let M be a compact (PL) n -manifold ($n \geq 2$). The (generalized) regular genus of M is defined as*

$$\mathcal{G}(M) = \min \{ \rho(\Gamma) \mid \Gamma \text{ represents } M \}.$$

and the Gurau degree (or G-degree) of M is defined as

$$\mathcal{D}_G(M) = \min \{ \omega_G(\Gamma) \mid \Gamma \text{ represents } M \}.$$

REMARK 2.8: Note that the (generalized) regular genus is a PL-invariant extending to higher dimension the classical genus of a surface and the Heegaard genus of a 3-manifold. It succeeds in characterizing spheres in arbitrary dimension ([21]), and a lot of classifying results via regular genus have been obtained, especially in dimension 4 and 5 (see [16], [6], [9] and their references). On the other hand, Gurau degree originally arises, within theoretical physics, from the theory of random tensors as an approach to quantum gravity in dimension greater than two ([25]). Also G-degree characterizes spheres in arbitrary dimension and some classifying results via this invariant have recently been obtained in dimension 3 and 4: see [15] for the compact 3-dimensional case, [12] for the closed 4-dimensional case, and [9] for the compact 4-dimensional case.

A further PL-invariant has been - quite naturally - defined within crystallization theory³:

DEFINITION 2.9. *For each compact n -manifold M , its gem-complexity is the non-negative integer $k(M) = p - 1$, where $2p$ is the minimum order of an $(n + 1)$ -colored graph representing M .*

We point out that, for each compact n -manifold with empty or connected boundary, both regular genus and G-degree and gem-complexity are actually realized by a crystallization.

Moreover, if M is a compact n -manifold with empty or connected boundary, it is always possible to assume - up to a permutation of the color set - that any gem (and, in particular, any crystallization) of M has color n as its (unique) possible *singular color*, i.e. that each \hat{c} -residue, with $c \neq n$, represents the $(n - 1)$ -sphere.

In Section 6 a fourth PL-invariant (called *G-trisection genus*) will be combinatorially defined via colored graphs, in the restricted setting of compact 4-manifolds M^4 such that the associated singular manifold \widehat{M}^4 is simply-connected.

3. Computing invariants from crystallizations of compact 4-manifolds

In the present section, M^4 will be a compact 4-manifold with empty or connected boundary, such that $rk(\pi_1(M^4)) = m \geq 0$ and $rk(\pi_1(\widehat{M}^4)) = m' \geq 0$

³Note that a lot of significant classification results have been obtained within crystallization theory with respect to gem-complexity, too: as regards the closed case, see, for example, [1] and [7] for the dimension 3, [8] and [13] for the dimension 4; in the compact case, see [19] for a classification according to gem-complexity for compact orientable 3-manifolds with toric boundary.

(with $m' \leq m$), and Γ will be a 5-colored graph representing M^4 . As pointed out in Section 2, we may assume without loss of generality Γ to be a crystallization (i.e. $\Gamma_{\hat{c}}$ is connected for any $c \in \Delta_4$) and color 4 to be its (unique) possible singular color (i.e. $\Gamma_{\hat{c}}$ represents \mathbb{S}^3 , for any $c \neq 4$). Furthermore, let us denote by \mathcal{P}_4 the set of all cyclic permutations $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ of Δ_4 such that $\varepsilon_4 = 4$.

With the notations settled in Section 2 for the number of residues, [3] and [9] yield, $\forall j, k, l \in \Delta_3$:

$$\begin{aligned} g_{j,k,l} &= 1 + m' + t_{j,k,l}, \quad \text{with } t_{j,k,l} \geq 0 \quad \text{and} \quad \{r, s\} = \Delta_4 - \{j, k, l\}; \\ g_{j,k,4} &= 1 + m + t_{j,k,4}, \quad \text{with } t_{j,k,4} \geq 0 \quad \text{and} \quad \{r, s\} = \Delta_3 - \{j, k\}. \end{aligned}$$

As a consequence:

$$\sum_{i,j,k \in \Delta_4} g_{i,j,k} = 10 + 10m - 4(m - m') + \sum_{i,j,k \in \mathbb{Z}_5} t_{i,j,k} \quad (1)$$

On the other hand, in [9] the following relation is proved to hold for each $i \in \Delta_4$ and for each $\varepsilon \in \mathcal{P}_4$:

$$g_{\widehat{\varepsilon_{i-1}}, \widehat{\varepsilon_{i+1}}} = g_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+3}} = 1 + \rho_{\varepsilon} - \rho_{\widehat{\varepsilon_{i-1}}} - \rho_{\widehat{\varepsilon_{i+1}}} \quad (2)$$

where $\varepsilon_i = (\varepsilon_0, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_4 = 4)$ and $\rho_{\varepsilon}, \rho_{\widehat{\varepsilon_i}}$ respectively denote $\rho_{\varepsilon}(\Gamma)$, $\rho_{\widehat{\varepsilon_i}}(\Gamma_{\widehat{\varepsilon_i}})$.

Therefore:

$$\begin{aligned} g_{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_{i+3}} &= 1 + \rho_{\varepsilon} - \rho_{\varepsilon_i} - \rho_{\widehat{\varepsilon_{i+2}}} = 1 + m' + t_{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_{i+3}} \quad \forall i \in \{2, 4\} \quad \text{and} \\ g_{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_{i+3}} &= 1 + \rho_{\varepsilon} - \rho_{\varepsilon_i} - \rho_{\widehat{\varepsilon_{i+2}}} = 1 + m + t_{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_{i+3}} \quad \forall i \in \{0, 1, 3\}, \end{aligned}$$

which trivially imply

$$\begin{aligned} \rho_{\varepsilon} - \rho_{\varepsilon_i} - \rho_{\widehat{\varepsilon_{i+2}}} - m' &= t_{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_{i+3}} \quad \forall i \in \{2, 4\} \quad \text{and} \\ \rho_{\varepsilon} - \rho_{\varepsilon_i} - \rho_{\widehat{\varepsilon_{i+2}}} - m &= t_{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_{i+3}} \quad \forall i \in \{0, 1, 3\} \end{aligned} \quad (3)$$

where all subscripts are taken in \mathbb{Z}_5 .

Computations regarding the regular genus, the G-degree and the order of Γ , performed in the quoted papers and in [17], allow to prove the following summarizing result, which is an original contribution of the present paper.

PROPOSITION 3.1. *Let Γ be an order $2p$ crystallization of a compact 4-manifold M^4 with empty or connected boundary, with $rk(\pi_1(M^4)) = m \geq 0$ and $rk(\pi_1(\widehat{M}^4)) = m' \geq 0$ ($m' \leq m$). Then:*

$$\begin{aligned}
(a) \quad \rho_\varepsilon(\Gamma) &= 2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4 + \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}}; \\
(b) \quad \omega_G(\Gamma) &= 6 \left[4\chi(\widehat{M}^4) + 10m - 4(m - m') - 8 + \sum_{i, j, k \in \mathbb{Z}_5} t_{i, j, k} \right]; \\
(c) \quad p - 1 &= 3\chi(\widehat{M}^4) + 10m - 4(m - m') - 6 + \sum_{i, j, k \in \mathbb{Z}_5} t_{i, j, k}.
\end{aligned}$$

Proof. In [17], for each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ of Δ_4 , the associated permutation ε' has been defined as $\varepsilon' = (\varepsilon_0, \varepsilon_2, \varepsilon_4, \varepsilon_1, \varepsilon_3)$.⁴ Then, [17, Proposition 7] yields:

$$\chi(N^4) = \left(\rho_\varepsilon(\Gamma) + \rho_{\varepsilon'}(\Gamma) \right) - p + 3 \quad (4)$$

for any order $2p$ crystallization of a singular 4-manifold N^4 with one singular vertex at most.

Moreover, in virtue of [17, Proposition 6(b)],

$$\rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma) = \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}, \varepsilon_{j+2}} - \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+2}, \varepsilon_{j+4}}$$

holds for any 5-colored graph representing a singular 4-manifold N^4 ; hence:

$$\rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma) = \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}} - \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}}. \quad (5)$$

Then, by comparing relations (5) and (4), the following formula follows:

$$\chi(\widehat{M}^4) = 2\rho_\varepsilon(\Gamma) + 3 - p + \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}} - \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} \quad (6)$$

On the other hand, an easy computation (making use of [12, Lemma 21]) yields:

$$\chi(\widehat{M}^4) = 5 - \frac{1}{3} \sum_{i, j, k \in \Delta_4} g_{i, j, k} + \frac{1}{3}p. \quad (7)$$

Hence, by comparison with (6) and by using (1):

$$\chi(\widehat{M}^4) = 2\rho_\varepsilon(\Gamma) + 3 - 3\chi(\widehat{M}^4) + 5 - 10m + 4(m - m') - 2 \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}},$$

⁴Note that, if $\varepsilon \in \mathcal{P}_4$ is assumed (i.e. $\varepsilon_4 = 4$), we can always consider $\varepsilon' = (\varepsilon_1, \varepsilon_3, \varepsilon_0, \varepsilon_2, \varepsilon_4 = 4)$, i.e. $\varepsilon' \in \mathcal{P}_4$, too.

from which

$$\rho_\varepsilon(\Gamma) = 2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4 + \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} \quad (8)$$

easily follows, as well as

$$\rho_{\varepsilon'}(\Gamma) = 2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4 + \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}}. \quad (9)$$

This proves statement (a).

In virtue of [17, Proposition 5],

$$\omega_G(\Gamma) = 6(\rho_\varepsilon(\Gamma) + \rho_{\varepsilon'}(\Gamma))$$

holds for each 5-colored graph (Γ, γ) , and for each pair $(\varepsilon, \varepsilon')$ of associated cyclic permutations of Δ_4 . Hence, by summing relations (8) and (9), statement (b) easily follows:

$$\begin{aligned} \omega_G(\Gamma) &= 6(2\rho_\varepsilon(\Gamma) + (\rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma))) \\ &= 6(4\chi(\widehat{M}^4) + 10m - 4(m - m') - 8 + \sum_{j,k,l \in \Delta_4} t_{j,k,l}). \end{aligned}$$

Finally, in order to prove statement (c), it is sufficient to make use of relation (7), together with relation (1):

$$\begin{aligned} p - 1 &= 3\chi(\widehat{M}^4) - 16 + \sum_{i,j,k \in \Delta_4} g_{i,j,k} \\ &= 3\chi(\widehat{M}^4) - 16 + 10 + 10m - 4(m - m') + \sum_{j,k,l \in \Delta_4} t_{j,k,l}. \end{aligned}$$

□

The following statement, extending [12, Corollary 24] to the connected boundary case, is a direct consequence of Proposition 3.1 (b) and (c):

COROLLARY 3.2. *Let M^4 be a compact 4-manifold M^4 with empty or connected boundary. Then:*

$$\mathcal{D}_G(M^4) = 6[\chi(\widehat{M}^4) - 2 + k(M^4)].$$

4. Weak semi-simple crystallizations of compact 4-manifolds

In [3] and [2] two particular types of crystallizations are introduced and studied, by generalizing the notion of *simple crystallizations* for closed simply-connected 4-manifolds (see [4] and [14]): they are proved to be “minimal” with respect to regular genus, among all graphs representing the same closed 4-manifold.

In [10] these definitions are extended to compact 4-manifolds with empty or connected boundary.

DEFINITION 4.1. *Let M^4 be a compact 4-manifold, with empty or connected boundary. A 5-colored graph Γ representing M^4 is called semi-simple if $g_{j,k,l} = 1 + m' \ \forall j, k, l \in \Delta_3$ and $g_{j,k,4} = 1 + m \ \forall j, k \in \Delta_3$, where $rk(\pi_1(M^4)) = m \geq 0$ and $rk(\pi_1(\widehat{M}^4)) = m' \geq 0$ ($m' \leq m$). Γ is called weak semi-simple with respect to a permutation $\varepsilon \in \mathcal{P}_4$ if $g_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} = 1 + m \ \forall i \in \{0, 2, 4\}$ and $g_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} = 1 + m' \ \forall i \in \{1, 3\}$ (where the additions in subscripts are intended in \mathbb{Z}_5).*

We point out that, as a consequence of the above definition, if Γ is weak semi-simple, then $g_j = 1, \ \forall j \in \Delta_4$, i.e. Γ is a crystallization of M^4 .

In case $m = 0$ (and, hence, $m' = 0$, too), semi-simple (resp. weak semi-simple) crystallizations are said to be *simple* (resp. *weak simple*).

By making use of relations (3), for all $i \in \Delta_4$, it is not difficult to prove the following characterization of weak semi-simple crystallizations:

PROPOSITION 4.2. ([10, Corollary 8]) *Let Γ be a crystallization of a compact 4-manifold M^4 with empty or connected boundary, with $rk(\pi_1(M^4)) = m \geq 0$, $rk(\pi_1(\widehat{M}^4)) = m' \geq 0$ ($m' \leq m$). Then Γ is weak semi-simple with respect to a cyclic permutation $\varepsilon \in \mathcal{P}_4$ if and only if*

$$\rho_{\varepsilon_i} = \frac{1}{2}(\rho_\varepsilon - m) \ \forall i \in \Delta_3 \quad \text{and} \quad \rho_{\varepsilon_4} = \frac{1}{2}(\rho_\varepsilon - m) + (m - m').$$

EXAMPLE 4.3: As concerns the closed case, \mathbb{S}^4 , \mathbb{CP}^2 , $\mathbb{S}^2 \times \mathbb{S}^2$ admit simple crystallizations, while $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3$ (the orientable and non-orientable sphere bundles over \mathbb{S}^1) and \mathbb{RP}^4 admit semi-simple crystallizations. See Figures 1, 2, 3, 4, 5 respectively. Moreover, in [4] a simple (order 134) crystallization of the $K3$ -surface is produced.

In the boundary case, examples of simple crystallizations of $\mathbb{S}^2 \times \mathbb{D}^2$ and ξ_2 - the \mathbb{D}^2 -bundle over \mathbb{S}^2 with Euler number 2, whose boundary is the lens space $L(2, 1)$ - are constructed in [9]: see Figures 6 and 7.

In the same paper, semi-simple crystallizations of \mathbb{Y}_h^4 and $\widetilde{\mathbb{Y}}_h^4$, the genus h orientable and non-orientable 4-dimensional handlebodies, can be found (see Figures 8 and 9, where the orientable cases $h = 1$ and $h = 2$ are depicted), as

well as a weak simple (but not simple!) crystallization of ξ_c ($c \in \mathbb{Z}^+ - \{1, 2\}$), the \mathbb{D}^2 -bundle over \mathbb{S}^2 with Euler number c whose boundary is the lens space $L(c, 1)$: see Figure 10.

Other examples of weak simple crystallizations may be found in the existing catalogue of rigid dipole-free bipartite crystallizations of closed orientable 4-manifolds, up to 20 vertices (see [8]): in particular, all elements with order 16 turn out to be weak simple crystallizations of simply-connected manifolds, whose simple crystallizations appear with less than 16 vertices.

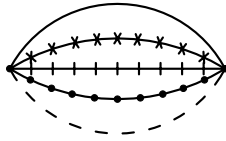


Figure 1: The (unique) simple crystallization of \mathbb{S}^4

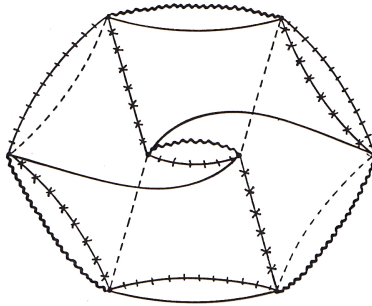


Figure 2: The (unique) simple crystallization of \mathbb{CP}^2

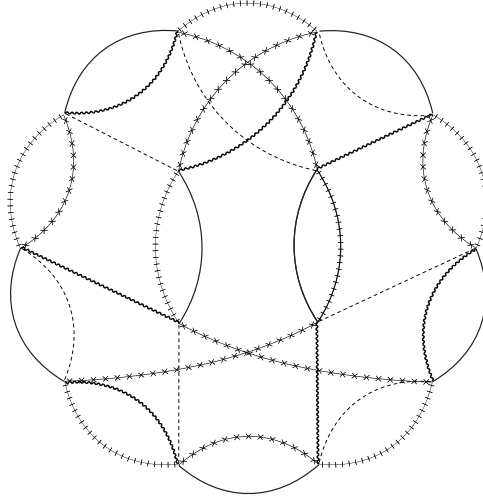
We are now able to prove the Main Theorem, stated in Section 1 .

Proof of the Main Theorem. It is a direct consequence of Proposition 3.1, together with the definitions themselves of semi-simple and weak semi-simple crystallization. In fact:

$$\Gamma \text{ weak semi-simple with respect to } \varepsilon \in \mathcal{P}_4 \iff \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} = 0$$

$$\Gamma \text{ semi-simple} \iff \sum_{i, j, k \in \mathbb{Z}_5} t_{i, j, k} = 0$$

□

Figure 3: A simple crystallization of $\mathbb{S}^2 \times \mathbb{S}^2$

The following statement, characterizing manifolds which admit semi-simple crystallizations via a relationship between gem-complexity and regular genus, is an original contribution of the present paper.

PROPOSITION 4.4. *Let M^4 be a compact 4-manifold with empty or connected boundary, with $rk(\pi_1(M^4)) = m \geq 0$, $rk(\pi_1(\widehat{M}^4)) = m' \geq 0$ ($m' \leq m$). Then:*

$$k(M^4) = \frac{3\mathcal{G}(M^4) + 5m - 2(m - m')}{2} \iff M^4 \text{ admits a semi-simple crystallization.}$$

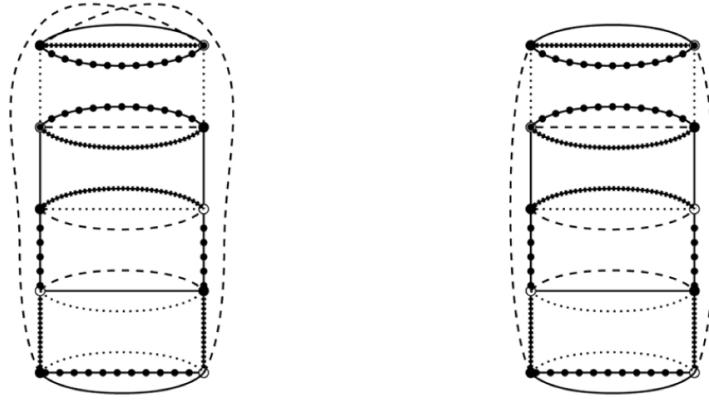
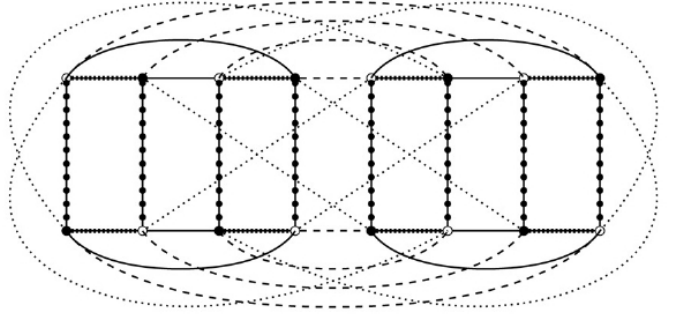
Proof. Let Γ and Γ' be two crystallizations of M^4 and $\varepsilon \in \mathcal{P}_4$ a permutation, such that $\mathcal{G}(M^4) = \rho(\Gamma) = \rho_\varepsilon(\Gamma)$ and $k(M^4) = p' - 1$, $2p'$ being the order of Γ' . Statements (a) and (c) of Proposition 3.1 yield:

$$\begin{aligned} \mathcal{G}(M^4) = \rho_\varepsilon(\Gamma) &= 2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4 + \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} \\ k(M^4) = p' - 1 &= 3\chi(\widehat{M}^4) + 10m - 4(m - m') - 6 + \sum_{i, j, k \in \mathbb{Z}_5} t'_{i, j, k} \end{aligned}$$

where $t_{i, j, k} \geq 0$ (resp. $t'_{i, j, k} \geq 0$) is the difference between the number of $\{i, j, k\}$ -residues in Γ (resp. in Γ') and either $m + 1$ (in case $4 \in \{i, j, k\}$) or $m' + 1$ (in case $4 \notin \{i, j, k\}$).

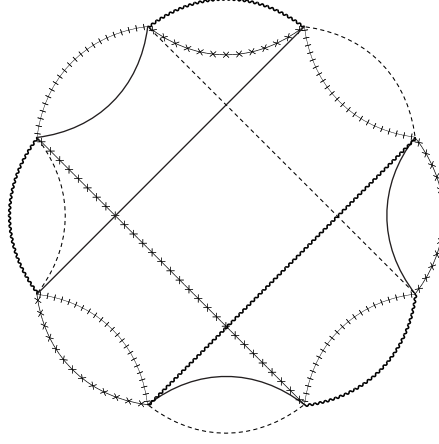
Moreover,

$$\sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} \leq \sum_{i \in \mathbb{Z}_5} t'_{\bar{\varepsilon}_i, \bar{\varepsilon}_{i+2}, \bar{\varepsilon}_{i+4}} \leq \sum_{i \in \mathbb{Z}_5} t'_{\bar{\varepsilon}_i, \bar{\varepsilon}_{i+1}, \bar{\varepsilon}_{i+2}}$$

Figure 4: Semi-simple crystallizations of $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^1 \tilde{\times} \mathbb{S}^3$ Figure 5: The (unique) semi-simple crystallization of \mathbb{RP}^4

holds, for any permutation $\bar{\varepsilon} \in \mathcal{P}_4$ such that $\rho_{\bar{\varepsilon}}(\Gamma') \leq \rho_{\bar{\varepsilon}'}(\Gamma')$, $\bar{\varepsilon}'$ denoting, as in the proof of Proposition 3.1, the permutation of \mathcal{P}_4 which is associated to $\bar{\varepsilon}$. Hence:

$$\begin{aligned}
 & 2k(M^4) - 3\mathcal{G}(M^4) - 5m + 2(m - m') \\
 &= 2 \sum_{i,j,k \in \mathbb{Z}_5} t'_{i,j,k} - 3 \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} \\
 &\geq 2 \sum_{i,j,k \in \mathbb{Z}_5} t'_{i,j,k} - 3 \sum_{i \in \mathbb{Z}_5} t'_{\bar{\varepsilon}_i, \bar{\varepsilon}_{i+2}, \bar{\varepsilon}_{i+4}} \\
 &= \sum_{i \in \mathbb{Z}_5} t'_{\bar{\varepsilon}_i, \bar{\varepsilon}_{i+1}, \bar{\varepsilon}_{i+2}} + \left(\sum_{i \in \mathbb{Z}_5} t'_{\bar{\varepsilon}_i, \bar{\varepsilon}_{i+1}, \bar{\varepsilon}_{i+2}} - \sum_{i \in \mathbb{Z}_5} t'_{\bar{\varepsilon}_i, \bar{\varepsilon}_{i+2}, \bar{\varepsilon}_{i+4}} \right) \geq 0.
 \end{aligned}$$

Figure 6: A simple crystallization of $\mathbb{S}^2 \times \mathbb{D}^2$

This proves that the equality

$$2k(M^4) = 3\mathcal{G}(M^4) + 5m - 2(m - m')$$

holds if and only if

$$\sum_{i \in \mathbb{Z}_5} t'_{\bar{\varepsilon}_i, \bar{\varepsilon}_{i+1}, \bar{\varepsilon}_{i+2}} = 0 \quad \text{and} \quad \sum_{i \in \mathbb{Z}_5} t'_{\bar{\varepsilon}_i, \bar{\varepsilon}_{i+1}, \bar{\varepsilon}_{i+2}} - \sum_{i \in \mathbb{Z}_5} t'_{\bar{\varepsilon}_i, \bar{\varepsilon}_{i+2}, \bar{\varepsilon}_{i+4}} = 0$$

i.e. Γ' is a semi-simple crystallization of M^4 . □

As a consequence of statement (a) of the Main Theorem, together with Proposition 4.2, we have:

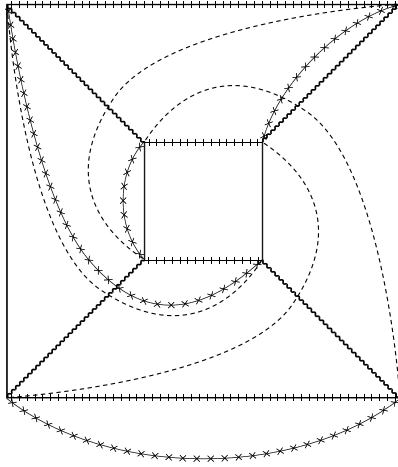
COROLLARY 4.5. *Let Γ be a crystallization of a compact 4-manifold M^4 with empty or connected boundary, with $rk(\pi_1(M^4)) = m \geq 0$ and $rk(\pi_1(\widehat{M}^4)) = m' \geq 0$ ($m' \leq m$). Then, Γ is weak semi-simple with respect to the cyclic permutation $\varepsilon \in \mathcal{P}_4$ if and only if*

$$\mathcal{G}(M^4) = \rho(\Gamma) = \rho_\varepsilon(\Gamma) = 2\chi(\widehat{M}^4) + 5m - 2(m - m') - 4$$

or, equivalently, if and only if

$$\rho_{\varepsilon_i} = \chi(\widehat{M}^4) + m + m' - 2 \quad \forall i \in \Delta_3 \quad \text{and} \quad \rho_{\varepsilon_4} = \chi(\widehat{M}^4) + 2m - 2.$$

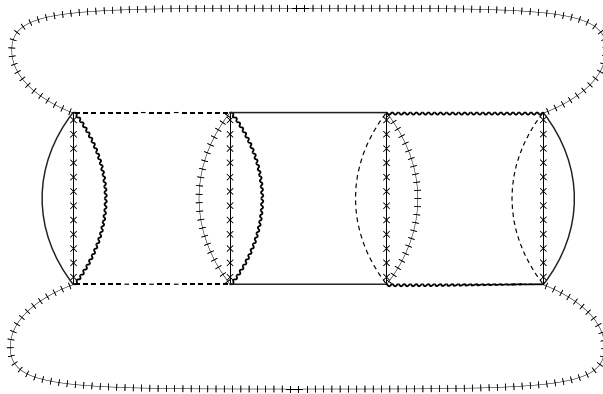
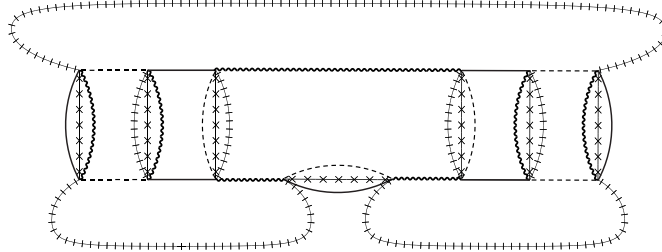
Table 1 summarizes the values of the invariants regular genus, G-degree and gem-complexity for all 4-manifolds considered in Example 4.3, computed via results of the present Section.

Figure 7: A simple crystallization of ξ_2

M^4	$\mathcal{G}(M^4)$	$\mathcal{D}_G(M^4)$	$k(M^4)$	existing minimal crystallizations
\mathbb{S}^4	0	0	0	simple
\mathbb{CP}^2	2	24	3	simple
$\mathbb{S}^2 \times \mathbb{S}^2$	4	48	6	simple
$\mathbb{S}^1 \times \mathbb{S}^3$	1	12	4	semi-simple
$\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3$	1	12	4	semi-simple
$K3$	44	528	66	simple
\mathbb{RP}^4	3	36	7	semi-simple
ξ_2	2	24	3	simple
$\mathbb{S}^2 \times \mathbb{D}^2$	2	24	3	simple
\mathbb{Y}_h^4	h	$12h$	$3h$	semi-simple
$\widetilde{\mathbb{Y}}_h^4$	h	$12h$	$3h$	semi-simple
ξ_c ($c \in \mathbb{Z}^+ - \{1, 2\}$)	2	$\leq 12c$	$\leq 2c - 1$	weak simple

Table 1: invariants for the considered compact 4-manifolds

We conclude the Section by pointing out what it is known about the topological structure of simply-connected compact PL 4-manifolds admitting simple or weak simple crystallizations.

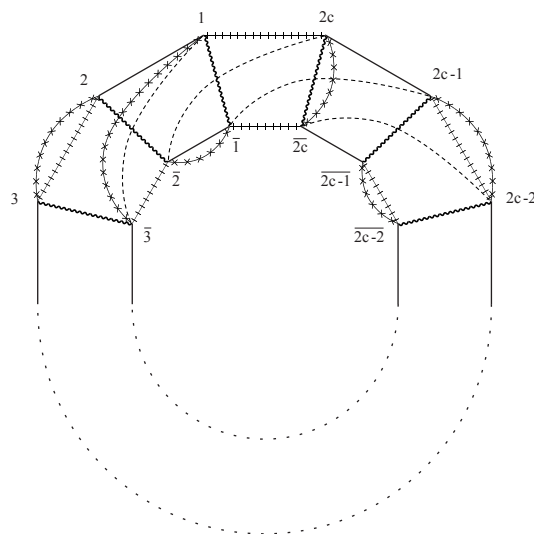
Figure 8: A semi-simple crystallization of \mathbb{Y}_1^4 Figure 9: A semi-simple crystallization of \mathbb{Y}_2^4

Recall that the notions of simple and weak simple crystallizations arise from Definition 4.1, in case of $m = 0$ (and, as a consequence, $m' = 0$, too).

In particular, a simple crystallization (originally defined in [4] only in the closed case) is a 5-colored graph representing a simply connected compact (PL) 4-manifold with the property that the 1-skeleton of the associated triangulation equals the 1-skeleton of a 4-simplex.

In [14], any (simply-connected) closed (PL) 4-manifold M admitting a simple crystallization is proved to admit a *special handlebody decomposition*, i.e. a handle decomposition lacking in 1-handles and 3-handles (see [27, Section 3.3]).

THEOREM 4.6. [14, Theorem 1.1] *Let M^4 be a closed (PL) 4-manifold. If M^4 admits a simple crystallization, then M^4 admits a handle decomposition lacking in 1-handles and 3-handles (or, equivalently, M^4 is represented by a (not dotted) framed link with $\beta_2(M)$ components).*

Figure 10: A weak simple crystallization of ξ_c ($c \geq 3$)

Recently, the same property has been proved to hold also in the compact connected-boundary case, and also for a large class of 4-manifolds, which comprehends those admitting weak simple crystallizations.

THEOREM 4.7 ([11]). *Let M^4 be a compact (PL) 4-manifold, with empty or connected boundary. If M^4 admits a weak simple crystallization, then M^4 admits a handle decomposition lacking in 1-handles and 3-handles (or, equivalently, M^4 is represented by a (not dotted) framed link with $\beta_2(M)$ components).*

REMARK 4.8: Note that - in the closed case - the existence of a special handlebody decomposition is related to Kirby problem n. 50:

“Does every simply-connected closed 4-manifold have a handlebody decomposition without 1-handles? Without 1- and 3-handles?”

On the other hand, since simple crystallizations of TOP-homeomorphic PL manifolds must have the same order, the existence of infinitely many different PL-structures on the same TOP 4-manifold ensures that not all closed simply-connected PL 4-manifolds admit simple crystallizations. Moreover, as a consequence of Theorem 4.6, it may be easily proved that, if an exotic PL-structure on \mathbb{S}^4 (resp. \mathbb{CP}^2) exists, then the corresponding PL 4-manifold does not admit simple crystallizations ([14, Corollary 3.3]).

Hence, Theorem 4.7 may be useful to investigate Kirby problem n. 50, via a class of 5-colored graphs (including weak simple crystallizations) which possibly succeeds in representing all closed simply-connected 4-manifolds.

5. Invariants additivity and related problems

It is well-known that the regular genus is subadditive with respect to connected sum of closed n -manifolds ([22]). This can be checked directly via the so called *graph connected sum construction*, starting from a pair of graphs representing two given closed n -manifolds and realizing their regular genera. For each pair of $(n+1)$ -colored graphs Γ_1, Γ_2 and for each choice of vertices $v_1 \in V(\Gamma_1)$, $v_2 \in V(\Gamma_2)$, the *graph connected sum of Γ_1 and Γ_2 with respect to v_1 and v_2* is the $(n+1)$ -colored graph constructed by deleting v_1 and v_2 from Γ_1 and Γ_2 and welding the “hanging” edges of the same color. The obtained graph has regular genus equal to the sum of the regular genera of Γ_1 and Γ_2 , and - in case of Γ_1, Γ_2 representing two closed n -manifolds M_1, M_2 - it is proved to represent a connected sum of M_1 and M_2 .

On the other hand, the additivity of regular genus under connected sum has been conjectured⁵, and the associated (open) problem is significant, at least in the closed orientable case, and especially in dimension four.

CONJECTURE 5.1 ([21]). *Let M_1^n, M_2^n be two closed (orientable) n -manifolds. Then,*

$$\mathcal{G}(M_1^n \# M_2^n) = \mathcal{G}(M_1^n) + \mathcal{G}(M_2^n).$$

In fact, it is easy to prove that the 4-dimensional case of Conjecture 5.1 implies the 4-dimensional Smooth Poincaré Conjecture, via the well-known Wall Theorem on homotopic 4-manifolds.

However, it is obvious that the construction of graph connected sum can be performed on any pair of $(n+1)$ -colored graphs representing compact n -manifolds and, under suitable conditions, it yields an $(n+1)$ -colored graph representing either an internal or a boundary connected sum of the given manifolds (see [9, Section 4] for details).

Since the present paper focuses on compact manifolds with empty or connected boundary, in the following we will consider only the connected sum constructions that are internal to this class of manifolds.

More precisely, given two compact n -manifolds M_1^n, M_2^n , with empty or connected boundary, $M_1^n \# M_2^n$ will denote an (internal) connected sum of M_1^n and M_2^n if and only if at least one of M_1^n, M_2^n is closed; otherwise it will denote a boundary connected sum of M_1^n and M_2^n . Then, by exploiting the graph connected sum construction, it is not difficult to check that

$$\mathcal{G}(M_1^n \# M_2^n) \leq \mathcal{G}(M_1^n) + \mathcal{G}(M_2^n).$$

⁵Obviously, in the 3-dimensional case, regular genus satisfies the additive property with respect to connected sum, via a classic result on Heegaard genus.

The definition itself of graph connected sum implies that the class of compact 4-manifolds admitting weak simple/semi-simple crystallizations is closed under connected sum. This fact has interesting consequences regarding additivity properties of the PL-invariants regular genus, G-degree and gem-complexity.

PROPOSITION 5.2. *Let M_1 and M_2 be two compact 4-manifolds admitting weak semi-simple (resp. semi-simple) crystallizations. Then, $M_1 \# M_2$ admits weak semi-simple (resp. semi-simple) crystallizations, too.*

As a consequence, additivity of regular genus (resp. of G-degree and gem-complexity) holds within the class of compact 4-manifolds admitting weak semi-simple (resp. semi-simple) crystallizations.

Note that, in general, the additivity of gem-complexity (and of G-degree, too, via Corollary 3.2) cannot hold because of the finiteness of the invariant: it is sufficient to make use of Wall Theorem, together with the existence of infinitely many PL-structures on the same TOP 4-manifold.

Notwithstanding this, Proposition 5.2 enables to compute both the regular genus and the G-degree and the gem-complexity for a large class of compact 4-manifolds, obtained by connected sums of the compact 4-manifolds \mathbb{CP}^2 , $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{S}^1 \tilde{\times} \mathbb{S}^3$, $K3$, \mathbb{RP}^4 , ξ_2 , $\mathbb{S}^2 \times \mathbb{D}^2$, \mathbb{Y}_h^4 , $\tilde{\mathbb{Y}}_h^4$ (which admit semi-simple crystallizations, as shown in Example 4.3).

The following statement extends to compact 4-manifolds a double inequality concerning regular genus, obtained in [13, Proposition 6.5].

PROPOSITION 5.3. *For each compact 4-manifold M^4 with empty or connected boundary, with $rk(\pi_1(M^4)) = m \geq 0$ and $rk(\pi_1(\widehat{M}^4)) = m' \geq 0$ ($m' \leq m$):*

$$2 - 2\mathcal{G}(M^4) \leq \chi(\widehat{M}^4) \leq 2 + \frac{\mathcal{G}(M^4)}{2} - \frac{5m - 2(m - m')}{2}.$$

Proof. It is sufficient to make use of statement (a) of the Main Theorem, together with the following formula, proved in [9, Proposition 13] for any 5-colored graph Γ representing a compact 4-manifold M^4 and for each cyclic permutation ε of Δ_4 :

$$\chi(\widehat{M}^4) = 2 - 2\rho_\varepsilon(\Gamma) + \sum_{i \in \Delta_4} \rho_{\varepsilon_i}(\Gamma_{\widehat{\varepsilon}_i}). \quad (10)$$

□

In [13], by means of the double inequality improved by Proposition 5.3, two classes of closed 4-manifolds were detected, for which additivity of regular genus holds; while the complete identification of the manifolds belonging to one class was performed in the same paper, it was also pointed out that the

problem of completely determining the other class remained open. Now, we can trivially extend the analysis to the compact setting, obtaining the complete characterization of both (extended) classes.

PROPOSITION 5.4. *Let M_1, M_2 be two compact 4-manifolds with empty or connected boundary, with $rk(\pi_1(M_i)) = m_i \geq 0$ and $rk(\pi_1(\widehat{M}_i)) = m'_i \geq 0$ ($m'_i \leq m_i$) for each $i \in \{1, 2\}$.*

(a) *If $\mathcal{G}(M_i) = 1 - \frac{\chi(\widehat{M}_i)}{2}$ for each $i \in \{1, 2\}$, then:*

$$\mathcal{G}(M_1 \# M_2) = \mathcal{G}(M_1) + \mathcal{G}(M_2) \quad \text{and} \quad \mathcal{G}(M_1 \# M_2) = 1 - \frac{\chi(\widehat{M_1 \# M_2})}{2}.$$

(b) *If $\mathcal{G}(M_i) = 2\chi(M_i) + 5m_i - 2(m_i - m'_i) - 4$ for each $i \in \{1, 2\}$, then:*

$$\mathcal{G}(M_1 \# M_2) = \mathcal{G}(M_1) + \mathcal{G}(M_2) \quad \text{and}$$

$$\mathcal{G}(M_1 \# M_2) = 2\chi(\widehat{M_1 \# M_2}) + 5(m_1 + m_2) - 2(m_1 + m_2 - m'_1 - m'_2) - 4.$$

Moreover, a compact 4-manifold M^4 is involved in case (a) (resp. (b)) if and only if it is a connected sum of sphere bundles over \mathbb{S}^1 (resp. if and only if M^4 admits a weak semi-simple crystallization).

Proof. Statements (a) and (b) are direct consequences of the double inequality of Proposition 5.3 by further observing that $\chi(\widehat{M_1 \# M_2}) = \chi(\widehat{M_1}) + \chi(\widehat{M_2}) - 2$. As regards the class of compact 4-manifolds involved in statement (a), note that, in virtue of formula (10), they admit a 5-colored graph, realizing the regular genus (i.e. $\mathcal{G}(M^4) = \rho_\varepsilon(\Gamma)$), such that $\rho_{\varepsilon_i}(\Gamma_{\varepsilon_i}) = 0$ for each $i \in \Delta_4$. Now, [9, Proposition 15] ensures M^4 to be a connected sum of (orientable or non-orientable) sphere bundles over \mathbb{S}^1 .⁶

On the other hand, statement (a) of the Main Theorem easily proves that the class of compact 4-manifolds involved in statement (b) exactly consists in compact 4-manifolds admitting weak semi-simple crystallizations. \square

REMARK 5.5: We point out that the connected sums of (orientable or non-orientable) sphere bundles over \mathbb{S}^1 are the only compact 4-manifolds belonging to both classes involved in Proposition 5.4: they are characterized by the equality between the regular genus and the rank of fundamental group (as proved in [9, Theorem 4]), and hence the double inequality of Proposition 5.3 actually becomes a double equality.

⁶Note that the condition $\rho_{\varepsilon_i}(\Gamma_{\varepsilon_i}) = 0, \forall i \in \Delta_4$ directly implies M^4 to be a closed 4-manifold, since regular genus zero characterizes spheres in any dimension.

6. B-trisections induced by weak semi-simple crystallizations

Throughout this section all manifolds are supposed to be orientable.

The notion of *trisection* of a smooth, oriented closed 4-manifold was introduced in 2016 by Gay and Kirby ([24]), by generalizing the classical idea of Heegaard splitting in dimension 3: the 4-manifold is decomposed into three 4-dimensional handlebodies, with disjoint interiors and mutually intersecting in 3-dimensional handlebodies, so that the intersection of all three “pieces” is a closed orientable surface. The minimum genus of the intersecting surface is called the *trisection genus* of the 4-manifold.

Hass, Bell, Rubinstein and Tillmann in [5] performed an approach to the study of trisections via singular triangulations and their construction was applied by Spreer and Tillmann ([31]) to the case of triangulations induced by crystallizations of closed 4-manifolds. In this setting Spreer and Tillmann succeeded into calculating the trisection genus of all closed standard (PL) 4-manifolds through their simple crystallizations.

The extension to the connected boundary case and to a wider class of edge-colored graphs is presented in [10] following a suggestion in [30]; it relies on the notion of B-trisection.

DEFINITION 6.1. *Let M^4 be a compact 4-manifold with empty (resp. connected) boundary. A B-trisection of M^4 is a triple $\mathcal{T} = (H_0, H_1, H_2)$ of 4-dimensional submanifolds of M^4 , such that:*

- (i) $M^4 = H_0 \cup H_1 \cup H_2$ and H_0, H_1, H_2 have pairwise disjoint interiors;
- (ii) H_1, H_2 are 4-dimensional handlebodies; H_0 is a 4-disk (resp. is (PL) homeomorphic to $\partial M^4 \times [0, 1]$);
- (iii) $H_{01} = H_0 \cap H_1$, $H_{02} = H_0 \cap H_2$ and $H_{12} = H_1 \cap H_2$ are 3-dimensional handlebodies;
- (iv) $\Sigma(\mathcal{T}) = H_0 \cap H_1 \cap H_2$ is a closed connected surface (which is called central surface).

REMARK 6.2: Note that the central surface of a B-trisection $\mathcal{T} = (H_0, H_1, H_2)$ of M^4 is an Heegaard surface for the 3-manifold $\partial H_i = \#_{k_i}(\mathbb{S}^1 \times \mathbb{S}^2)$ ($k_i \geq 0$), for each $i \in \{1, 2\}$, splitting it into the 3-dimensional handlebodies H_{ij} and H_{ik} , with $\{j, k\} = \{0, 1, 2\} - \{i\}$. Moreover, in the closed (resp. boundary) case, $(H_{01}, H_{02}, \Sigma(\mathcal{T}))$ is an Heegaard splitting of $\partial H_0 = \mathbb{S}^3$ (resp. of ∂M^4 , and more precisely of the boundary component of ∂H_0 intersecting $H_1 \cup H_2$). Hence, obviously, we have $k_i \leq \text{genus}(\Sigma(\mathcal{T}))$ for each $i \in \{1, 2\}$, and, in the boundary case, the genus of $\Sigma(\mathcal{T})$ is an upper bound for the Heegaard genus of ∂M^4 .

Moreover, via Seifert-Van Kampen's Theorem, it is not difficult to check that, the simply-connectedness of the singular manifold \widehat{M}^4 is a necessary condition for the existence of a B-trisection of M^4 (see [10]).

In order to construct B-trisections for 4-manifolds with empty or connected boundary, we consider the set, denoted by $G_s^{(4)}$, of all 5-colored graphs having only one $\hat{4}$ -residue and such that all \hat{i} -residues, with $i \in \Delta_3$, represent the 3-sphere. Note that any compact 4-manifold with empty or connected boundary can be represented by an element of this set. Moreover, $G_s^{(4)}$ properly contains (up to permutation of the color set) all weak semi-simple crystallizations of compact 4-manifolds with empty or connected boundary.

The following theorem ensures the existence, for the whole class of compact 4-manifolds with empty or connected boundary, of a triple of submanifolds satisfying "almost all" conditions required by a B-trisection.

THEOREM 6.3 ([10]). *Let M^4 be a compact 4-manifold with empty or connected boundary. For each 5-colored graph $(\Gamma, \gamma) \in G_s^{(4)}$ representing M^4 and for each $\varepsilon \in \mathcal{P}_4$, a triple $\mathcal{T}(\Gamma, \varepsilon) = (H_0, H_1, H_2)$ of submanifolds of M^4 is obtained, satisfying properties (i), (ii) and (iv) of Definition 6.1, and such that $H_{01} = H_0 \cap H_1$ and $H_{02} = H_0 \cap H_2$ are 3-dimensional handlebodies. Moreover, the central surface $\Sigma(\mathcal{T}(\Gamma, \varepsilon)) = H_0 \cap H_1 \cap H_2$ is a closed connected surface of genus $\rho_{\varepsilon_4}(\Gamma_{\varepsilon_4})$.*

The notions of gem-induced trisection and G-trisection genus arise quite naturally from the above result.

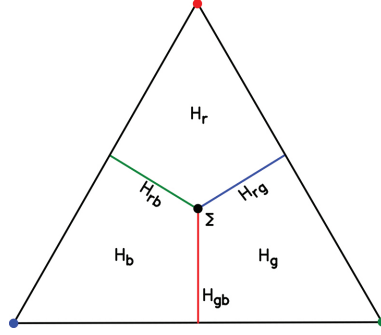
DEFINITION 6.4. *Let M^4 be a compact 4-manifold M^4 with empty or connected boundary. If the triple $\mathcal{T}(\Gamma, \varepsilon) = (H_0, H_1, H_2)$ of M^4 , associated to a 5-colored graph $\Gamma \in G_s^{(4)}$ and a permutation $\varepsilon \in \mathcal{P}_4$, is a B-trisection (i.e. if $H_{12} = H_1 \cap H_2$ is a 3-dimensional handlebody, too), then it is called a gem-induced trisection of M^4 .*

The gem-induced trisection genus - or G-trisection genus for short - $g_{GT}(M^4)$ of M^4 is the minimum genus of the central surface of any gem-induced trisection $\mathcal{T}(\Gamma, \varepsilon)$ of M^4 :

$$g_{GT}(M^4) = \min\{\text{genus}(\Sigma(\mathcal{T}(\Gamma, \varepsilon))) \mid \mathcal{T}(\Gamma, \varepsilon) \text{ B-trisection of } M^4\}.$$

As a direct consequence of Theorem 6.3, if Γ is a crystallization of a closed simply-connected 4-manifold M^4 admitting a gem-induced trisection - actually a trisection -, then the trisection genus of M^4 is less or equal to $\rho_{\varepsilon_4}(\Gamma_{\varepsilon_4})$.

Detecting classes of 5-colored graphs inducing triples $\mathcal{T}(\Gamma, \varepsilon)$, and possibly B-trisections, with minimal genus of their central surface is thus a relevant problem. The problem is faced in [10], proving in particular that weak semi-simple

Figure 11: The 2-simplex σ

crystallizations guarantee the minimality of the genus of the central surface of a gem-induced trisection, if any. For this reason, we will now briefly sketch the construction of the triple $\mathcal{T}(\Gamma, \varepsilon)$ of Theorem 6.3 for the particular case of a crystallization Γ that is weak semi-simple with respect to the permutation $\varepsilon \in \mathcal{P}_4$.

Let us denote by σ the standard 2-simplex, by v_b, v_r, v_g its vertices and by σ' its first barycentric subdivision. Following [5] and [31], let us consider the following partition of $\Delta_4 : b = \{4\}$, $g = \{\varepsilon_1, \varepsilon_3\}$, $r = \{\varepsilon_0, \varepsilon_2\}$. In the following, for sake of simplicity, we suppose $\varepsilon = (0, 1, 2, 3, 4)$.

Let $\mu : K(\Gamma) \rightarrow \sigma$ be the simplicial map sending all vertices of $K(\Gamma)$, whose colors belong to the same partition class, to one vertex of the standard 2-simplex; then H_b (resp. H_r) (resp. H_g) is the preimage by μ of the star of v_b (resp. v_r) (resp. v_g) in σ' . Therefore it is easy to see that H_b is a regular neighbourhood of the (unique) 4-colored vertex of $K(\Gamma)$ and precisely it is the cone over its disjoint link. On the other hand H_r (resp. H_g) is a regular neighbourhood of the 1-dimensional subcomplex $K_{02}(\Gamma)$ (resp. $K_{13}(\Gamma)$) of $K(\Gamma)$ generated by its i -colored vertices, with $i \in \{0, 2\}$ (resp. $i \in \{1, 3\}$) and it is not difficult to see that H_r (resp. H_g) is a 4-dimensional handlebody of genus $g_{1,3,4} - 1 = m$ (resp. $g_{0,2,4} - 1 = m$), where $m = rk(\pi_1(M^4))$.

The bijection between M^4 and \widehat{M}^4 described in Remark 2.2 allows to prove (see [10] for details) that (H_b, H_r, H_g) defines a triple $\mathcal{T}(\Gamma, \varepsilon) = (H_0, H_1, H_2)$ satisfying Theorem 6.3, by setting $H_1 = H_r$, $H_2 = H_g$ and $H_0 = H_b$, if $\partial M^4 = \emptyset$, or H_0 to be a collar of ∂M^4 obtained by removing from H_b a suitable neighbourhood of the singular vertex, if $\partial M^4 \neq \emptyset$.

In fact, $H_{rb} = H_r \cap H_b$ (resp. $H_{gb} = H_g \cap H_b$) is the preimage under μ of the edge of σ' depicted in Figure 11 as the “green” (resp. “red”) edge, and turns out to be always an handlebody. Moreover, both in the closed and connected boundary case, the central surface $\Sigma = H_0 \cap H_1 \cap H_2 = H_b \cap H_r \cap H_g$ is proved

to be a closed connected surface of genus $\rho_{\varepsilon_4}(\Gamma_{\varepsilon_4})$. With regard to H_{rg} , this complex is the preimage under μ of the edge of σ' depicted in Figure 11 as the “blue” edge.

By Definition 6.4, the triple $\mathcal{T}(\Gamma, \varepsilon) = (H_0, H_1, H_2)$ of M^4 is a gem-induced trisection of M^4 if H_{rg} collapses to a graph.

The following proposition states, for weak semi-simple crystallizations, a “minimality property” regarding the genus of the associated central surface, which is actually proved in [10, Proposition 14] for a larger class of 5-colored graphs.

PROPOSITION 6.5 ([10]). *Let M^4 be a compact 4-manifold with empty or connected boundary and let Γ be a weak semi-simple crystallization of M^4 with respect to $\varepsilon \in \mathcal{P}_4$. Let $\mathcal{T}(\Gamma, \varepsilon) = (H_0, H_1, H_2)$ be the triple of submanifolds of M^4 associated to Γ and ε ; then*

$$\text{genus}(\Sigma(\mathcal{T}(\Gamma, \varepsilon))) \leq \text{genus}(\Sigma(\mathcal{T}(\bar{\Gamma}, \bar{\varepsilon})))$$

for all $\bar{\Gamma} \in G_s^{(4)}$ such that $|K(\Gamma)| \cong |K(\bar{\Gamma})|$ and for all $\bar{\varepsilon} \in \mathcal{P}_4$.

It is pointed out in [18] that, if (H_0, H_1, H_2) is a trisection of a closed 4-manifold M^4 with central surface Σ , then $g(\Sigma) \geq \beta_1(M^4) + \beta_2(M^4)$.

A more general formula ([10, Proposition 18]) extends the above to a wider class of compact 4-manifolds with empty or connected boundary. By applying it to the case of weak semi-simple crystallizations, and by making use also of Proposition 4.2, we have:

PROPOSITION 6.6 ([10]). *Let M^4 be a compact 4-manifold with empty or connected boundary which admits a weak semi-simple crystallization Γ giving rise to a gem-induced trisection $\mathcal{T}(\Gamma, \varepsilon)$. Then,*

$$g_{GT}(M^4) = \frac{1}{2}(\rho_\varepsilon(\Gamma) + m) = \beta_2(M^4) + \beta_1(M^4) + 2(m - \beta_1(M^4)),$$

with $m = rk(\pi_1(M^4))$.

In particular,

$$g_{GT}(M^4) = \frac{1}{2}\rho_\varepsilon(\Gamma) = \beta_2(M^4)$$

for each compact (simply-connected) 4-manifold M^4 , with empty or connected boundary, which admits a weak simple crystallization Γ giving rise to a gem-induced trisection $\mathcal{T}(\Gamma, \varepsilon)$.

In this case, if M^4 is closed, its G -trisection genus (equal to the second Betti number $\beta_2(M^4)$) coincides with its trisection genus.

Finally, given a 5-colored graph $\Gamma \in G_s^{(4)}$ representing a compact 4-manifold M^4 with empty or connected boundary, a sufficient condition is known for $\mathcal{T}(\Gamma, \varepsilon)$ to be a B-trisection of M^4 for each cyclic permutation $\varepsilon \in \mathcal{P}_4$. It makes use of the existence of a presentation of $\pi_1(\widehat{M}^4)$ whose generators are in bijection with the 4-colored edges of Γ and whose relations are in bijection with the bicolored cycles of Γ involving color 4: see [9] for details.

PROPOSITION 6.7 ([10]). *Let $\langle X, R \rangle$ be the presentation of $\pi_1(\widehat{M}^4)$ with $X = \{x_1, \dots, x_p\}$ in bijection with the 4-colored edges of Γ and $R = \{r_1, \dots, r_q\}$ in bijection with the $\{4, i\}$ -cycles of Γ , for each $i \in \Delta_3$.*

If the presentation $\langle X, R \rangle$ collapses to the trivial one through a finite sequence of moves of the following type:

*if $r_j = x_s$ ($j \in \mathbb{N}_q$, $s \in \mathbb{N}_p$), then delete x_s from the generator set,
and from each relation containing it, too,*

then M^4 admits a gem-induced trisection $\mathcal{T}(\Gamma, \varepsilon)$, for each $\varepsilon \in \mathcal{P}_4$.

All simple, semi-simple or weak semi-simple crystallizations depicted in Figures 1–3 and 6–10 turn out to satisfy the sufficient condition of Proposition 6.7; the same happens with the simple crystallization of the surface K3 presented in [4]. Therefore, all of them give rise to gem-induced trisections, that, by Proposition 6.6, realize the G-trisection genus of the represented manifolds (see Tables 2, where only manifolds M^4 such that $\pi_1(\widehat{M}^4) = 0$ are taken into account).

Proposition 6.6 also ensures that, for the closed simply-connected 4-manifolds of Tables 1 and 2, the described crystallizations turn out to realize also the trisection genus.

M^4	$g_{GT}(M^4)$
\mathbb{S}^4	0
\mathbb{CP}^2	1
$\mathbb{S}^2 \times \mathbb{S}^2$	2
$K3$	22
$\mathbb{S}^2 \times \mathbb{D}^2$	1
\mathbb{Y}_h^4	h
ξ_c ($c \geq 2$)	1

Table 2: computation of the G-trisection genus.

The following proposition shows that the G-trisection genus has the same behaviour as the regular genus with respect to connected sums (compare with Proposition 5.2):

PROPOSITION 6.8 ([10]). *Let M_1^4, M_2^4 be two compact 4-manifolds with empty or connected boundary admitting gem-induced trisections. Then $M_1^4 \sharp M_2^4$ admits gem-induced trisections, too, and*

$$g_{GT}(M_1^4 \sharp M_2^4) \leq g_{GT}(M_1^4) + g_{GT}(M_2^4).$$

Furthermore, equality holds if M_1^4 and M_2^4 admit B-trisections induced by weak semi-simple crystallizations.

REMARK 6.9: As a consequence of the previous proposition, for any compact 4-manifold M^4 , with empty or connected boundary, that is a connected sum of the manifolds in Table 2, the equality

$$g_{GT}(M^4) = \beta_2(M^4) + \beta_1(M^4)$$

holds. In particular, for all closed simply-connected “standard” 4-manifolds, Propositions 6.6 and 6.8 ensure that the trisection genus equals the second Betti number, as proved by Spreer and Tillmann in [31] by using simple crystallizations.

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