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Realizations of certain odd-degree surface branch data

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Dedicated to my colleague and friend Bruno Zimmermann

Abstract. *We consider surface branch data with base surface the sphere, odd degree d, three branching points, and partitions of d of the form*

$$(2,\ldots,2,1) \quad (2,\ldots,2,2h+1) \quad \pi$$

with π having length ℓ. This datum satisfies the Riemann-Hurwitz necessary condition for realizability if $h-\ell$ is odd and at least -1. For several small values of h and ℓ (namely, for $h+\ell \leqslant 5$) we explicitly compute the number ν of realizations of the datum up to the equivalence relation given by the action of automorphisms (even unoriented ones) of both the base and the covering surface. The expression of ν depends on arithmetic properties of the entries of π. In particular we find that in the only case where ν is 0 the entries of π have a common divisor, in agreement with a conjecture of Edmonds-Kulkarny-Stong and a stronger one of Zieve.

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1. Introduction

In this introduction we first review the notion of surface branched cover and branch datum, and we define the *weak Hurwitz number* of a branch datum (*i.e.*, the number of its realizations up to a certain "weak equivalence" relation). We then state the new results established in the rest of the paper, concerning the exact computation of this number for branch data of a specific type, and we comment on the connections of these results with an old conjecture of Edmonds-Kulkarny-Stong and a recent stronger one of Zieve.

Surface branched covers. A surface branched cover is a continuous function $f:\widetilde{\Sigma}\to\Sigma$ where $\widetilde{\Sigma}$ and Σ are closed, orientable and connected surfaces and f is locally modeled on maps of the form

$$(\mathbb{C},0) \ni z \mapsto z^m \in (\mathbb{C},0).$$

If $m>1$ the point 0 in the target $\mathbb{C}$ is called a *branching point*, and m is called the local degree at the point 0 in the source $\mathbb{C}$. There are finitely many branching points, removing which, together with their pre-images, one gets a genuine cover of some degree d. If there are n branching points, the local degrees at the points in the pre-image of the j-th one form a partition π_j of d of some length ℓ_j, and the following Riemann-Hurwitz relation holds:

$$\chi\left(\widetilde{\Sigma}\right) - (\ell_1 + \ldots + \ell_n) = d\left(\chi\left(\Sigma\right) - n\right).$$

Let us now call *branch datum* an array of the form

$$\left(\widetilde{\Sigma}, \Sigma, d, n, \pi_1, \ldots, \pi_n\right)$$

with $\widetilde{\Sigma}$ and Σ orientable surfaces, d and n positive integers, and π_j a partition of d for $j = 1,\ldots,n$. We say that a branch datum is *compatible* if it satisfies the Riemann-Hurwitz relation. (Note that $\widetilde{\Sigma}$ and Σ are orientable by assumption; see [3] for a definition of compatibility in a non-orientable context.)

The Hurwitz problem. The very old *Hurwitz problem* asks which compatible branch data are *realizable* (namely, associated to some existing surface branched cover) and which are *exceptional* (non-realizable). Several partial solutions to this problem have been obtained over the time, and we quickly mention here the fundamental [3], the survey [16], and the more recent [2, 13, 14, 15, 20]. In particular, for an orientable Σ the problem has been shown to have a positive solution whenever Σ has positive genus. When Σ is the sphere S, many realizability and exceptionality results have been obtained (some of experimental nature), but the general pattern of what data are realizable remains elusive. One guiding conjecture [3] in this context is that *a compatible branch datum is always realizable if its degree is a prime number*. It was actually shown in [3] that proving this conjecture in the special case of 3 branching points would imply the general case. This is why many efforts have been devoted in recent years to investigating the realizability of compatible branch data with base surface Σ the sphere S and having $n = 3$ branching points. See in particular [14, 15] for some evidence supporting the conjecture.

Hurwitz numbers. Two branched covers $f_1 : \widetilde{\Sigma} \to \Sigma$ and $f_2 : \widetilde{\Sigma} \to \Sigma$ are said to be *weakly equivalent* if there exist homeomorphisms $\widetilde{g} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ and $g : \Sigma \to \Sigma$ such that $f_1 \circ \widetilde{g} = g \circ f_2$, and *strongly equivalent* if the set of branching points in Σ is fixed once and forever and one can take $g = \mathrm{id}_\Sigma$. The *(weak or strong) Hurwitz number* of a compatible branch datum is the number of (weak or strong) equivalence classes of branched covers realizing it. So the Hurwitz problem can be rephrased as the question whether a Hurwitz number is positive or not (a weak Hurwitz number can be smaller than the corresponding strong one, but they can only vanish simultaneously). Long ago Mednykh in [10, 11] gave some formulae for the computation of the strong Hurwitz numbers, but the actual implementation of these formulae is rather elaborate in general. Several results were also obtained in more recent years in [4, 7, 8, 9, 12]. Some remarks on the different ways of counting the realizations of a branch datum are contained in [19].

Computations. In this paper we consider branch data of the form

$$(\heartsuit) \qquad \left(\widetilde{\Sigma}, S, 2k+1, 3, [2,\ldots,2,1], [2,\ldots,2,2h+1], \pi = [d_i]_{i=1}^{\ell}\right)$$

for $h \geqslant 0$. Here we employ square brackets to denote an unordered array of integers with repetitions. A direct calculation shows that such a datum is compatible for $h - \ell = 2g - 1$, where g is the genus of $\widetilde{\Sigma}$. So $h - \ell$ should be odd and at least -1, and $g = \frac{1}{2}(h - \ell + 1)$. We compute the weak Hurwitz number of the datum for $h + \ell \leqslant 5$, namely for the following values of (g, h, ℓ):

$$(0,0,1) \quad (0,1,2) \quad (1,2,1) \quad (0,2,3) \quad (1,3,2) \quad (2,4,1).$$

Organizing the statements according to g and denoting by T the torus and by $2T$ the genus-2 surface, these are the results we prove in this article:

Theorem 1.1.

- $(g = 0,\ h = 0,\ \ell = 1)$ *The branch datum*

$$(S, S, 2k+1, 3, [2,\ldots,2,1], [2,\ldots,2,1], [2k+1])$$

always has a unique realization up to weak equivalence.

- $(g = 0,\ h = 1,\ \ell = 2)$ *The branch datum*

$$(S, S, 2k+1, 3, [2,\ldots,2,1], [2,\ldots,2,3], [p,q])$$

always has a unique realization up to weak equivalence.

- $(g = 0,\ h = 2,\ \ell = 3)$ *The number ν of weakly inequivalent realizations of*

$$(S, S, 2k+1, 3, [2,\ldots,2,1], [2,\ldots,2,5], [p,q,r])$$

is as follows:

– $\nu = 0$ *if* $p = q = r$*;*
– $\nu = 1$ *if two of* p, q, r *are equal to each other but not all three are;*
– $\nu = 2$ *if* p, q, r *are all different from each other and one of them is greater than* k*;*
– $\nu = 3$ *if* p, q, r *are all different from each other and all less than or equal to* k*.*

THEOREM 1.2. • $(g = 1,\ h = 2,\ \ell = 1)$ *The number of weakly inequivalent realizations of*

$$(T, S, 2k + 1, 3, [2, \ldots, 2, 1], [2, \ldots, 2, 5], [2k + 1])$$

is $\left[\left(\frac{k}{2}\right)^2\right]$.

• $(g = 1,\ h = 3,\ \ell = 2)$ *The number of weakly inequivalent realizations of*

$$(T, S, 2k + 1, 3, [2, \ldots, 2, 1], [2, \ldots, 2, 7], [p, q])$$

with $p > q$ *is is always positive and given by*

$$\left[\left(\frac{1}{2}\left(k - \left[\frac{p+1}{2}\right]\right)\right)^2\right] + \left[\left(\frac{1}{2}\left[\frac{p-1}{2}\right]\right)^2\right] + \left[\frac{p}{2}\right]^2$$
$$-(p-1)\cdot\left[\frac{p}{2}\right] + \left[\left(\frac{p}{2}\right)^2\right] + k^2 - k(p-1) + \frac{1}{2}(p-1)(p-4)$$

except for $k = 4$ *and* $p = 7$ *where this formula turns the value* 6 *but the correct one is* 5*.*

THEOREM 1.3. $(g = 2,\ h = 4,\ \ell = 1)$ *The number of weakly inequivalent realizations of*

$$(2T, S, 2k + 1, 3, [2, \ldots, 2, 1], [2, \ldots, 2, 9], [2k + 1])$$

is 10 *for* $k = 4$ *and otherwise positive and given by*

$$\frac{k}{16}(7k^3 - 42k^2 + 72k - 37) + \frac{5}{8}(2k-3)\left[\frac{k}{2}\right].$$

The prime-degree conjecture. As already mentioned, it was conjectured in [3] that any compatible branch datum with prime degree is actually realizable, and it was shown in the same paper that establishing the conjecture with $n = 3$ branching points would suffice to prove the general case. More recently, Zieve [21] conjectured that an arbitrary compatible branch datum

$$\left(\widetilde{\Sigma}, \Sigma, d, n, \pi_1, \ldots, \pi_n\right)$$

is realizable provided that

- $\mathrm{GCD}(\pi_j) = 1$ for $j = 1, \ldots, n$ and
- $\sum_{j=1}^{n} \left(1 - \frac{1}{\mathrm{lcm}(\pi_j)}\right) \neq 2.$

As one easily sees, the compatible branch data with $\sum_{j=1}^{n} \left(1 - \frac{1}{\mathrm{lcm}(\pi_j)}\right) = 2$ are precisely those whose associated candidate orbifold cover (see [14]) is of Euclidean type. These branch data were fully analyzed in [14], where it was shown that indeed some are exceptional (even with $\mathrm{GCD}(\pi_j) = 1$ for $j = 1, \ldots, n$ in some cases). So an equivalent way of expressing Zieve's conjecture is to say that a branch datum is realizable if $\mathrm{GCD}(\pi_j) = 1$ for $j = 1, \ldots, n$ and the datum is not one of the exceptional ones found in [14]. This would imply the prime-degree conjecture, because:

- If one of the π_i reduces to $[d]$ only then the branch datum is realizable by [3];
- All the exceptional data of [14] occur when the degree is composite.

We can now remark that our results are in agreement with Zieve's conjecture, because the only branch datum for which we compute the weak number Hurwitz number to be 0 comes from the first case in the last item of Theorem 1.1, namely for a branch datum of the form

$$(S, S, 3p, 3, [2, \ldots, 2, 1], [2, \ldots, 2, 5], [p, p, p])$$

for odd $p \geqslant 3$, and $d = 3p$ is composite in this case.

2. Weak Hurwitz numbers and dessins d'enfant

In the previous papers [17, 18] we have carried out the computation of weak Hurwitz numbers for different (even-degree) branch data, but the machine we will employ here is the same used in [17, 18]. We quickly recall it to make the present paper self-contained (but we omit the proofs). Our techniques are based on the notion of dessin d'enfant, popularized by Grothendieck in [5] (see also [1]), but actually known before his work and already exploited to give partial answers to the Hurwitz problem (see [6, 16] and the references quoted therein). Here we explain how to use dessins d'enfant to compute weak Hurwitz numbers. Let us fix until further notice a branch datum

$$(\diamondsuit) \qquad \left(\widetilde{\Sigma}, S, d, 3, \pi_1 = [d_{1i}]_{i=1}^{\ell_1}, \pi_2 = [d_{2i}]_{i=1}^{\ell_2}, \pi_3 = [d_{3i}]_{i=1}^{\ell_3}\right).$$

A graph Γ is *bipartite* if it has black and white vertices, and each edge joins black to white. If Γ is embedded in $\widetilde{\Sigma}$ we call *region* a component R of $\widetilde{\Sigma} \backslash \Gamma$, and *length* of R the number of white (or black) vertices of Γ to which R is incident (with multiplicity). A pair (Γ,σ) is called *dessin d'enfant* representing $(\diamondsuit)$ if $\sigma\in\mathfrak{S}_3$ and $\Gamma\subset\widetilde{\Sigma}$ is a bipartite graph such that:

- The black vertices of Γ have valence $\pi_{\sigma(1)}$;
- The white vertices of Γ have valence $\pi_{\sigma(2)}$;
- The regions of Γ are discs with lengths $\pi_{\sigma(3)}$.

We will also say that Γ *represents* $(\diamondsuit)$ *through* σ.

Remark 2.1: Let $f:\widetilde{\Sigma}\to S$ be a branched cover matching $(\diamondsuit)$ and take $\sigma\in\mathfrak{S}_3$. If α is a segment in S with a black and a white end at the branching points corresponding to $\pi_{\sigma(1)}$ and $\pi_{\sigma(2)}$, then $\big(f^{-1}(\alpha),\sigma\big)$ represents $(\diamondsuit)$, with vertex colours of $f^{-1}(\alpha)$ lifted via f.

Reversing the construction described in the previous remark one gets the following:

Proposition 2.2. *To a dessin d'enfant* (Γ,σ) *representing* $(\diamondsuit)$ *one can associate a branched cover* $f:\widetilde{\Sigma}\to S$ *realizing* $(\diamondsuit)$*, well-defined up to equivalence.*

We next define an equivalence relation $\sim$ on dessins d'enfant as that generated by:

- $(\Gamma_1,\sigma_1)\sim(\Gamma_2,\sigma_2)$ if $\sigma_1=\sigma_2$ and there is an automorphism $\widetilde{g}:\widetilde{\Sigma}\to\widetilde{\Sigma}$ such that $\Gamma_1=\widetilde{g}\,(\Gamma_2)$ matching colours;
- $(\Gamma_1,\sigma_1)\sim(\Gamma_2,\sigma_2)$ if $\sigma_1=\sigma_2\circ(1\,2)$ and $\Gamma_1=\Gamma_2$ as a set but with vertex colours switched;
- $(\Gamma_1,\sigma_1)\sim(\Gamma_2,\sigma_2)$ if $\sigma_1=\sigma_2\circ(2\,3)$ and Γ_1 has the same black vertices as Γ_2 and for each region R of Γ_2 we have that $R\cap\Gamma_1$ consists of one white vertex and disjoint edges joining this vertex to the black vertices on the boundary of R.

Theorem 2.3. *The branched covers associated as in Proposition 2.2 to two dessins d'enfant are equivalent if and only if the dessins are related by* $\sim$.

When the partitions π_1,π_2,π_3 in the branch datum $(\diamondsuit)$ are pairwise distinct, to compute the corresponding weak Hurwitz number one can stick to dessins d'enfant representing the datum through the identity, namely one can list up to automorphisms of $\widetilde{\Sigma}$ the bipartite graphs with black and white vertices of valence π_1 and π_2 and regions of length π_3. When the partitions are not distinct, however, it is essential to take into account the other moves generating $\sim$. In any case we will henceforth omit any reference to the permutations in $\mathfrak{S}_3$.

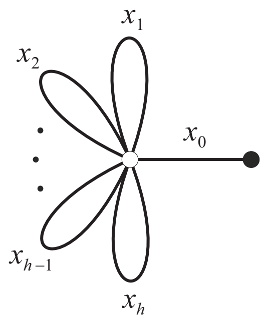

Figure 1: The abstract dessin d'enfant Γ.

Relevant data and repeated partitions. We now specialize again to a branch datum of the form ($\heartsuit$). We will compute its weak Hurwitz number ν by enumerating up to automorphisms of $\widetilde{\Sigma}$ the dessins d'enfant Γ representing it through the identity, namely the bipartite graphs Γ with black vertices of valence $[2,\ldots,2,1]$, the white vertices of valence $[2,\ldots,2,2h+1]$, and the regions of length π. Ignoring the embedding in $\widetilde{\Sigma}$, such a Γ is abstractly always as shown in Fig. 1, where x_0 stands for x_0 alternating black and white 2-valent vertices, while x_i stands for x_i+1 black and x_i white alternating 2-valent vertices for $i>0$. Counting the white vertices we get

$$k-h+1=1+\sum_{i=0}^{h}x_i \ \Rightarrow\ \sum_{i=0}^{h}x_i=k-h$$

with of course $x_i \geqslant 0$ for all i, and no other restriction. Enumerating these Γ's up to automorphisms of $\widetilde{\Sigma}$ already gives the right value of ν except if two of the partitions of d in coincide, and we have:

Proposition 2.4. *In a branch datum of the form* ($\heartsuit$) *with* $h+\ell\leqslant 5$ *two of the partitions of* d *coincide precisely in the following cases:*

- $(S,S,2k+1,3,[2,\ldots,2,1],[2,\ldots,2,1],[2k+1])$;
- $(S,S,5,3,[2,2,1],[2,3],[2,3])$;
- $(S,S,9,3,[2,2,2,2,1],[2,2,5],[2,2,5])$;
- $(T,S,5,3,[2,2,1],[5],[5])$;
- $(T,S,9,3,[2,2,2,2,1],[2,7],[2,7])$;
- $(2T,S,9,3,[2,2,2,2,1],[9],[9])$.

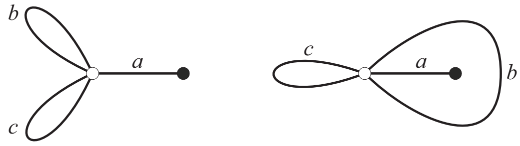

Figure 2: Embeddings of Γ in S for $h=2$ and $\ell=3$.

Proof. The lengths of the partitions π_1, π_2, π in ($\heartsuit$) are $\ell_1 = k+1$, $\ell_2 = k-h+1$ and $\ell = h+1-2g$.

We have $\pi_1 = \pi_2$ only for $h=0$, $\ell=1$ and $g=0$, whence the first listed item.

We have $\pi_1 = \pi$ only for $k+1 = h+1-2g$, whence $h-k = 2g \geqslant 0$, but of course $h \leqslant k$, so $h=k$ and the first listed item again.

We have $\pi_2 = \pi$ only for $k-h+1 = \ell$, so $k = h+\ell-1$, whence in particular $k \leqslant 4$, and listing the relevant cases is straightforward. $\square$

While proving our results, for the first four data of the previous statement we will find that there is a unique Γ up to automorphisms of $\widetilde{\Sigma}$ giving a realization. In these cases, we will not need to consider the second and third generating moves of $\sim$, but for the last two data we will have to do this, actually getting a correction to the computation.

3. Genus 0

In this section we prove Theorem 1.1.

For $h=0$ and $\ell=1$ the graph Γ of Fig. 1 reduces to a segment, so of course it has a unique embedding in S and the conclusion is obvious.

For $h=1$ and $\ell=2$ the embedding is again unique, and it realizes $[2x_0+x_1+2, x_1+1]$. Assuming $p>q$, namely $k+1 \leqslant p \leqslant 2k$ and $q = 2k+1-p$, we get the unique realization of the datum choosing $x_0 = p-k-1$ and $x_1 = 2k-p$.

Turning to the case $h=2$ and $\ell=3$, we now have two embeddings of Γ in S, shown in Fig. 2 and denoted by $\mathrm{I}(a,b,c)$ and $\mathrm{I\!I}(a,b,c)$ —for the sake of simplicity we use from now on letters such as a,b,c instead of $x_0,\ldots,x_h$. These graphs realize $[2a+b+c+3, b+1, c+1]$ and $[2a+b+2, b+c+2, c+1]$ respectively. Moreover $\mathrm{I}(a,b,c)$ has a symmetry switching b and c, while $\mathrm{I\!I}(a,b,c)$ has no symmetries. Let us now assume $p \geqslant q \geqslant r$.

Claim I: The number of realizations of $[p,q,r]$ through $\mathrm{I}(a,b,c)$ is 1 if $p>k$ and 0 otherwise.

Proof of Claim I: Since $2a+b+c+3$ is greater than $b+1$ and $c+1$, we can realize $[p,q,r]$ only with

$$\begin{cases} p=2a+b+c+3 \\ q=b+1 \\ r=c+1 \end{cases}$$

(for $q>r$ we might as well choose $q=c+1$ and $r=b+1$, but the $b\leftrightarrow c$ symmetry of $\mathrm{I}(a,b,c)$ makes this alternative immaterial). Noting that $q+r=2k+1-p$ one sees that the system as unique solution

$$\begin{cases} a=p-k-1 \\ b=q-1 \\ c=r-1 \end{cases}$$

which is acceptable precisely for $p>k$.

Before proceeding with another claim we note that we can split the possibilities for $[p,q,r]$ in 6 mutually exclusive cases IJ/M, where

- $I,J\in\{G,E\}$ with G standing for $>$ and E standing for $=$
- $M\in\{G,L\}$ with G standing for $>$ and L standing for $\leqslant$
- $IJ/M=\{[p,q,r]:\ p\,I\,j\,J\,q,\ p\,M\,k\}$
- $EE/G=EG/G=\emptyset$, so we write EE and EG instead of EE/L and EG/L.

So Claim I states that there is one realization through $\mathrm{I}(a,b,c)$ in cases GG/G and GE/G and none in the other cases.

Claim II: The number of realizations of $[p,q,r]$ through $\mathrm{II}(a,b,c)$ is as follows:

0 in cases EE and GE/G;

1 in cases EG, GE/L and GG/G;

3 in case GG/L.

Proof of Claim II: Since $b+c+2>c+1$ case EE cannot be realized. For $p=q>r$ (case EG) we can only have

$$\begin{cases} p=2a+b+2 \\ p=b+c+2 \\ r=c+1 \end{cases} \Leftrightarrow \begin{cases} a=k-p \\ b=p-r-1 \\ c=r-1 \end{cases}$$

and the solution is acceptable because $p \leqslant k$. For $p > q = r$ (case GE) we can only have

$$\left\{\begin{array}{l} p = b + c + 2 \\ q = 2a + b + 2 \\ q = c + 1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a = k - p \\ b = p - q - 1 \\ c = q - 1 \end{array}\right.$$

which is acceptable precisely for $p \leqslant k$, so there is no realization in case GE/G and one in case GE/L. For $p > q > r$ (case GG) there are three possibilities:

$$\left\{\begin{array}{l} p = 2a + b + 2 \\ q = b + c + 2 \\ r = c + 1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a = k - q \\ b = q - r - 1 \\ c = r - 1 \end{array}\right.$$

which is always acceptable, and

$$\left\{\begin{array}{l} p = b + c + 2 \\ q = 2a + b + 2 \\ r = c + 1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a = k - p \\ b = p - r - 1 \\ c = r - 1 \end{array}\right.$$

$$\left\{\begin{array}{l} p = b + c + 2 \\ q = c + 1 \\ r = 2a + b + 2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a = k - p \\ b = p - q - 1 \\ c = q - 1 \end{array}\right.$$

which are acceptable for $p \leqslant k$, whence 1 realization in case GG/G and 3 in case GG/L.

Conclusion: The number of realizations of $[p, q, r]$ through $\mathrm{I} + \mathrm{II}$ is $0 + 0 = 0$ in case EE, $1 + 0 = 1$ in case GE/G, $0 + 1 = 1$ in case GE/L, $0 + 1 = 1$ in case EG, $1 + 1 = 2$ in case GG/G and $0 + 3 = 3$ in case GG/L.

Note that for the branch datum

$$(S, S, 9, 3, [2, 2, 2, 2, 1], [2, 2, 5], [2, 2, 5])$$

from Proposition 2.4 we have found $\nu = 1$ already, so we do not have to worry about the repetitions in the partitions. The proof is complete.

4. Genus 1

In this section we prove Theorem 1.2.

For $h = 2$ and $\ell = 1$ the graph Γ of Fig. 1 has a unique embedding in T with a single disc as a complement, as shown in Fig. 3. This graph is subject to the symmetry $b \leftrightarrow c$, so the number of realizations of the branch datum equals the number of expressions $k - 2$ as $a + b + c$ with $a, b, c \geqslant 0$ up to $b \leftrightarrow c$, namely

$$\sum_{a=0}^{k-2} \left(\left[\frac{k - 2 - a}{2} \right] + 1 \right) = \sum_{a=0}^{k-2} \left[\frac{k - a}{2} \right] = \sum_{n=2}^{k} \left[\frac{n}{2} \right] = \sum_{n=0}^{k} \left[\frac{n}{2} \right] = \left[\left(\frac{k}{2} \right)^2 \right].$$

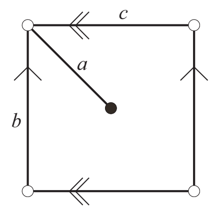

Figure 3: Embedding of Γ in T for $h = 2$ and $\ell = 1$.

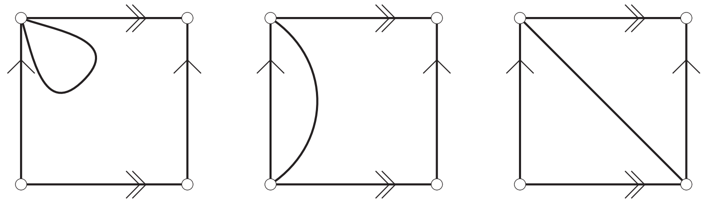

Figure 4: A bouquet of 3 circles in T with 2 discs as regions.

For $h = 3$ and $\ell = 2$ we first determine the embeddings in T of the bouquet B of 3 circles with two discs as regions. Of course at least a circle of B is non-trivial on T, so its complement is an annulus. Then another circle must join the boundary components of this annulus, so we can assume two circles of B form a standard meridian-longitude pair on T. Then the possibilities for B up to automorphisms of T are as in Fig. 4. Note that these embeddings have respectively a $\mathbb{Z}/_2$, a $\mathbb{Z}/_2 \times \mathbb{Z}/_2$ and a $\mathfrak{S}_3 \times \mathbb{Z}/_2$, symmetry. It easily follows that the relevant embeddings in T of Γ are up to automorphisms those shown in Fig. 5. Note that we have a symmetry switching c and d in cases I, IV, V, VII, and no other one. Moreover the different embeddings of Γ realize the following π's:

- $\mathrm{I}(a, b, c, d) \longrightarrow (2a + b + 2, b + 2c + 2d + 5)$
- $\mathrm{II}(a, b, c, d) \longrightarrow (b + 1, 2a + b + 2c + 2d + 6)$
- $\mathrm{III}(a, b, c, d) \longrightarrow (b + 1, 2a + b + 2c + 2d + 6)$
- $\mathrm{IV}(a, b, c, d) \longrightarrow (b + 1, 2a + b + 2c + 2d + 6)$
- $\mathrm{V}(a, b, c, d) \longrightarrow (2a + c + d + 3, 2b + c + d + 4)$
- $\mathrm{VI}(a, b, c, d) \longrightarrow (2a + 2b + c + d + 5, c + d + 2)$
- $\mathrm{VII}(a, b, c, d) \longrightarrow (2a + b + c + d + 4, b + c + d + 3)$.

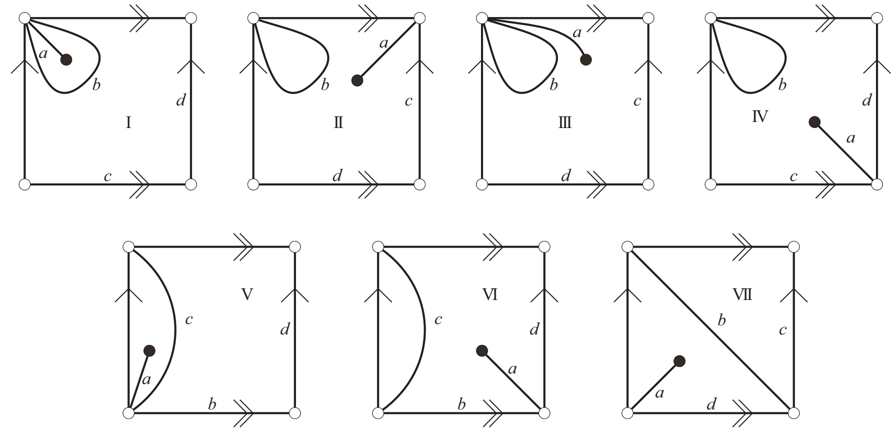

Figure 5: Embeddings in T of Γ with 2 discs as regions.

We will count the realizations of $\pi = [p, q]$ assuming $p > q$, namely $p > k$ and $q = 2k + 1 - p$, and analyzing case after case the contribution of each of the graphs I to VII. Along the way we will discuss all the cases where the contribution is null, which will only happen when p is close to its lower bound $k + 1$ or upper bound $2k$. Occasionally, to be completely precise, we would need to discuss separately some small values of p (and hence k), for which the contribution is also null, but as a matter of fact all these cases are included in the general ones, as the reader can easily check.

Claim I: The number of realizations of $[p, q]$ through $\mathrm{I}(a, b, c, d)$ is

$$\left[\left(\frac{1}{2}\left(k - \left[\frac{p+1}{2}\right]\right)\right)^2\right] + \left[\left(\frac{1}{2}\left[\frac{p-1}{2}\right]\right)^2\right] - \left[\left(\frac{p-k-1}{2}\right)^2\right]. \qquad (1)$$

Proof of Claim I: We first count the non-negative solutions a, b, c, d up to the symmetry $c \leftrightarrow d$ of the system

$$\begin{cases} 2a + b + 2 = p \\ b + 2c + 2d + 5 = 2k + 1 - p. \end{cases}$$

To begin, we state that (a, b, c, d) solve the system if and only if they satisfy the conditions

$$\begin{cases} c, d \geqslant 0 \\ c + d \leqslant k - 2 - \left[\frac{p+1}{2}\right] \\ a = p - k + 1 + c + d \\ b = 2k - 4 - p - 2c - 2d. \end{cases}$$

In fact, if (a,b,c,d) solve the system then (from the second equation)

$$\begin{aligned} & c+d=\frac{1}{2}(2k-p-4-b)=k-2-\frac{1}{2}(p+b) \\ \Rightarrow\quad & c+d\leqslant k-2-\frac{p}{2}\ \Leftrightarrow\ c+d\leqslant k-2-\left[\frac{p+1}{2}\right] \end{aligned}$$

and the expression of a,b in terms of p,k,c,d is readily derived. Conversely we must show that if $c,d\geqslant 0$ and $c+d\leqslant k-2-\frac{p}{2}$ then the expressions

$$a=p-k+1+c+d \qquad b=2k-4-p-2c-2d$$

turn non-negative values. For a, this is true because $p>k$ (so actually $a\geqslant 2$) and for b it is true because $c+d\leqslant k-2-\frac{p}{2}$. The statement implies that the number of solutions is 0 for $k-2-\frac{p}{2}<0$, namely for $p>2k-4$, while otherwise it is

$$\sum_{n=0}^{k-2-\left[\frac{p+1}{2}\right]}\left(\left[\frac{n}{2}\right]+1\right)=\left[\left(\frac{1}{2}\left(k-2-\left[\frac{p+1}{2}\right]\right)\right)^2\right]+k-1-\left[\frac{p+1}{2}\right] \tag{2}$$

but a straight-forward argument shows that the expression on the right-hand side of (2) gives the correct value 0 also for $2k-4<p\leqslant 2k$. We next count the non-negative solutions a,b,c,d up to the symmetry $c\leftrightarrow d$ of the system

$$\begin{cases} 2a+b+2=2k+1-p \\ b+2c+2d+5=p \end{cases}$$

and we state that (a,b,c,d) solve the system if and only if they satisfy the conditions

$$\begin{cases} c,d\geqslant 0 \\ p-k-2\leqslant c+d\leqslant\left[\frac{p-1}{2}\right]-2 \\ a=k-p+2+c+d \\ b=p-2c-2d-5. \end{cases}$$

In fact, if (a,b,c,d) solve the system then (from the second equation)

$$c+d=\frac{1}{2}(p-5-b)\ \Rightarrow\ c+d\leqslant\frac{p-1}{2}-2\ \Leftrightarrow\ c+d\leqslant\left[\frac{p-1}{2}\right]-2.$$

Moreover the expressions of a,b in terms of p,k,c,d are readily obtained, and that of a implies that $c+d\geqslant p-k-2$. Conversely, for $c,d\geqslant 0$ and $p-k-2\leqslant c+d\leqslant\frac{p-1}{2}-2$ we see that $a=k-p+2+c+d$ and $b=p-2c-2d-5$ are non-negative. Now recall that $p>k$, so $p-k-2<0$ only for $p=k+1$, in which case the number of solutions is

$$\sum_{n=0}^{\left[\frac{k}{2}\right]-2}\left(\left[\frac{n}{2}\right]+1\right)=\left[\left(\frac{1}{2}\left[\frac{k}{2}\right]-1\right)^2\right]+\left[\frac{k}{2}\right]-1=\left[\left(\frac{1}{2}\left[\frac{k}{2}\right]\right)^2\right]. \tag{3}$$

Moreover we have

$$\left[\frac{p-1}{2}\right]-2<p-k-2 \ \Leftrightarrow\ \frac{p-1}{2}<p-k \ \Leftrightarrow\ p>2k-1 \ \Leftrightarrow p=2k$$

in which case there are no solutions. For $k+1<p<2k$ we have instead

$$\begin{aligned}&\sum_{n=p-k-2}^{\left[\frac{p-1}{2}\right]-2}\left(\left[\frac{n}{2}\right]+1\right)\\ =\ &\left[\left(\frac{1}{2}\left[\frac{p-1}{2}\right]-1\right)^2\right]-\left[\left(\frac{p-k-3}{2}\right)^2\right]+\left[\frac{p-1}{2}\right]-p+k+1\end{aligned} \tag{4}$$

but the expression on the right-hand side of (4) is seen to coincide with (3) for $p=k+1$ and to vanish for $p=2k$. To conclude we must check that the sum of the two expressions on the right-hand sides of (2) and (4) give the claimed value (1), which only requires a little manipulation that we omit here.

Before turning to the next case, we note that the number of realizations of (p,q) through I is always positive except for $p=2k$ (this follows from the proof of formula (1) rather than from its expression).

Claim $\mathbb{II}+\mathbb{III}$: The number of realizations of $[p,q]$ through each of $\mathbb{II}(a,b,c,d)$ and $\mathbb{III}(a,b,c,d)$ is

$$\frac{1}{2}(p-k-1)(p-k-2). \tag{5}$$

Proof of Claim $\mathbb{II}+\mathbb{III}$: Since $2a+b+2c+2d+6>b+1$ the only realizations come from the solutions of

$$\begin{cases}2a+b+2c+2d+6=p\\ b+1=2k+1-p\end{cases}$$

(and, as a matter of fact, there are no solutions if $p-(2k+1-p)<5$, namely for $p\leqslant k+2$). The solutions we seek come with

$$b=2k-p \qquad a+c+d=p-k-3$$

so there are

$$\sum_{a=0}^{p-k-3}(p-k-3-a+1)=\frac{1}{2}(p-k-1)(p-k-2)$$

of them, and this expression is correct also for $p=k+1$ and $p=k+2$ (which are the only cases where there are no realizations).

Claim IV: The number of realizations of $[p,q]$ through IV(a,b,c,d) is

$$\left[\left(\frac{p-k-1}{2}\right)^2\right]. \tag{6}$$

Proof of Claim IV: The situation is identical to the previous one, except that now we have the symmetry $c \leftrightarrow d$ to take into account, so the number of realizations is

$$\sum_{a=0}^{p-k-3}\left[\frac{p-k-3-a+2}{2}\right]=\sum_{n=2}^{p-k-1}\left[\frac{n}{2}\right]=\sum_{n=0}^{p-k-1}\left[\frac{n}{2}\right]=\left[\left(\frac{p-k-1}{2}\right)^2\right]$$

which again is correct also for $p=k+1$ and $p=k+2$ (only cases where there are no realizations).

Claim V: The number of realizations of $[p,q]$ through V(a,b,c,d) is

$$\left[\frac{p}{2}\right]^2-(p-1)\cdot\left[\frac{p}{2}\right]-k(p-k)+\frac{1}{2}p(p-1). \tag{7}$$

Proof of Claim V: We first count the non-negative integer solutions (a,b,c,d) up to the $c \leftrightarrow d$ symmetry of

$$\begin{cases} 2a+c+d+3=p \\ 2b+c+d+4=2k+1-p \end{cases}$$

noting that there is none if $2k+1-p \leqslant 3$, namely for $p \geqslant 2k-2$, so we assume $p \leqslant 2k-3$. We first state that (a,b,c,d) is a solution if and only if

$$\begin{cases} p-k \leqslant a \leqslant \left[\frac{p-1}{2}\right]-1 \\ c+d=p-2a-3 \\ b=k-p+a \end{cases}$$

and $0 \leqslant p-k \leqslant \left[\frac{p-1}{2}\right]-1$. The last assertion is easy since $p>k$, and $p-k$ equals $\left[\frac{p-1}{2}\right]-1$ precisely for $p=2k-3$ and $p=2k-4$, while it is strictly less for smaller p. Now if (a,b,c,d) is a solution we have

$$\begin{cases} a=\frac{p-3-c-d}{2} \leqslant \frac{p-3}{2} \ \Rightarrow\ a \leqslant \left[\frac{p-3}{2}\right]=\left[\frac{p-1}{2}\right]-1 \\ c+d=p-2a-3 \\ b=\frac{2k+1-p-p+2a+3-4}{2}=k-p+a \ \Rightarrow\ a \geqslant p-k. \end{cases}$$

The sufficiency of these conditions for (a,b,c,d) to be a solution is proved very similarly. We then have the count

$$\begin{aligned}
& \sum_{a=p-k}^{\left[\frac{p-1}{2}\right]-1}\left(\left[\frac{p-2a-3}{2}\right]+1\right) \\
= & \sum_{a=p-k}^{\left[\frac{p-1}{2}\right]-1}\left(\left[\frac{p-1}{2}\right]-a-1+1\right) \\
= & \left[\frac{p-1}{2}\right]\cdot\left(\left[\frac{p-1}{2}\right]-1-(p-k)+1\right) \\
& -\frac{1}{2}\left(\left[\frac{p-1}{2}\right]-1\right)\cdot\left[\frac{p-1}{2}\right]+\frac{1}{2}(p-k-1)(p-k) \\
= & \frac{1}{2}\left[\frac{p-1}{2}\right]^2-\frac{1}{2}\left[\frac{p-1}{2}\right](2p-2k-1)+\frac{1}{2}(p-k-1)(p-k) \qquad (8)
\end{aligned}$$

which is readily seen to give the correct value 0 also for $2k-2 \leqslant p \leqslant 2k$. The argument for the system

$$\begin{cases} 2a+c+d+3=2k+1-p \\ 2b+c+d+4=p \end{cases}$$

is very similar. There are solutions for $p \leqslant 2k-2$ and they correpond to

$$\begin{cases} p-k-1 \leqslant b \leqslant \left[\frac{p}{2}\right]-2 \\ c+d=p-2b-4 \\ a=k-p+b+1 \end{cases}$$

so there are

$$\begin{aligned}
& \sum_{b=p-k-1}^{\left[\frac{p}{2}\right]-2}\left(\left[\frac{p-2b-4}{2}\right]+1\right) \\
= & \sum_{b=p-k-1}^{\left[\frac{p}{2}\right]-2}\left(\left[\frac{p}{2}\right]-b-2+1\right) \\
= & \left(\left[\frac{p}{2}\right]-1\right)\cdot\left(\left[\frac{p}{2}\right]-2-(p-k-1)+1\right) \\
& -\frac{1}{2}\left(\left[\frac{p}{2}\right]-2\right)\cdot\left(\left[\frac{p}{2}\right]-1\right)+\frac{1}{2}(p-k-2)(p-k-1) \\
= & \frac{1}{2}\left[\frac{p}{2}\right]^2-\frac{1}{2}\left[\frac{p}{2}\right](2p-2k-1)+\frac{1}{2}(p-k-1)(p-k) \qquad (9)
\end{aligned}$$

of them, and the formula is correct also for $2k-1 \leqslant p \leqslant 2k$. To conclude we must now show that summing (8) and (9) we get (7), which is proved with a little patience noting that $\left[\frac{p-1}{2}\right]+\left[\frac{p}{2}\right]=p-1$.

Claim VI: The number of realizations of $[p,q]$ through $\mathrm{VI}(a,b,c,d)$ is

$$(2k-p)(p-k-1). \tag{10}$$

Proof of Claim VI: We must count the solutions of

$$\begin{cases} 2a+2b+c+d+5=p \\ c+d+2=2k+1-p. \end{cases}$$

For $p=k+1$ and $p=2k$ there is no solution, otherwise the solutions (a,b,c,d) are the 4-tuples such that

$$c+d=2k-1-p \qquad a+b=p-k-2$$

so there are $(2k-p)(p-k-1)$ of them as claimed, and the formula is correct for $p=k+1$ and $p=2k$ as well.

Claim VII: The number of realizations of $[p,q]$ through $\mathrm{VII}(a,b,c,d)$ is

$$\left[\left(\frac{p}{2}\right)^2\right]-k(p-k). \tag{11}$$

Proof of Claim VII: We must count the solutions of

$$\begin{cases} 2a+b+c+d+4=p \\ b+c+d+3=2k+1-p \end{cases}$$

up to $c \leftrightarrow d$, and there is none for $p \geqslant 2k-1$. Otherewise the system is equivalent to

$$b+c+d=2k-p-2 \qquad a=p-k-1$$

so the number of solutions is

$$\begin{aligned}
\sum_{b=0}^{2k-p-2}\left(\left[\frac{2k-p-2-b}{2}\right]+1\right) &= \sum_{n=0}^{2k-p-2}\left(\left[\frac{n}{2}\right]+1\right) \\
&= \left[\left(\frac{2k-p-2}{2}\right)^2\right]+2k-p-1 \\
&= \left[\left(k-1-\frac{p}{2}\right)^2\right]+2k-p-1 \\
&= k^2-2k+1-(k-1)p+\left[\left(\frac{p}{2}\right)^2\right]+2k-p-1 \\
&= \left[\left(\frac{p}{2}\right)^2\right]-k(p-k)
\end{aligned}$$

which turns the right value 0 also for $p \geqslant 2k-1$.

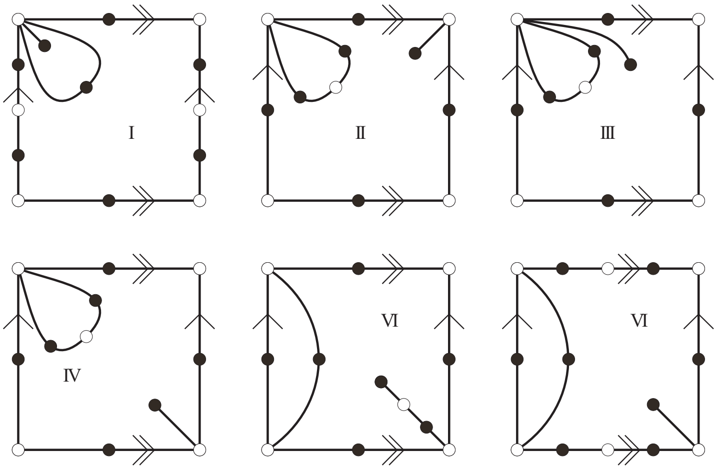

Figure 6: Graphs in T realizing $(T, S, 9, 3, [2, 2, 2, 2, 1], [2, 7], [2, 7])$.

Summing the contributions from I to VII the expression in the statement of Theorem 1.2 is now easily obtained, but we still have to worry about the penultimate item in Proposition 2.4. The above discussion or a direct inspection show that this datum is realized by the graphs

$$\begin{array}{ccc} \mathrm{I}(0,0,1,0) & \mathrm{II}(0,1,0,0) & \mathrm{III}(0,1,0,0) \\ \mathrm{IV}(0,1,0,0) & \mathrm{VI}(1,0,0,0) & \mathrm{VI}(0,1,0,0) \end{array}$$

shown in Fig. 6. Since the second and third partition of the datum coincide, all we have to do is to check whether any of these graphs are dual to each other under the last transformation generating the equivalence $\sim$ of Theorem 2.3. This is done in Figg. 7 to 11, and the conclusion is that the number of inequivalent realizations of the datum is 5 rather than 6, as in the statement of Theorem 1.2.

5. Genus 2

The proof of Theorem 1.3 employs that of Theorem 0.1 in [17]. In fact, it readily follows from [17] (see Fig. 12 there) that the embeddings in $2T$ of the graph Γ of Fig. 1 are up to symmetry the 13 ones described in Fig. 12. This figure contains four pictures showing an octagon whose edges should be paired according to the labels, so that the octagon becomes $2T$ and its edges become a bouquet B of 4 circles, which is part of the embedding of Γ in $2T$. For each of the four embeddings I to IV of B in Γ, the extra leg $\Gamma \setminus B$ of Γ can be embedded

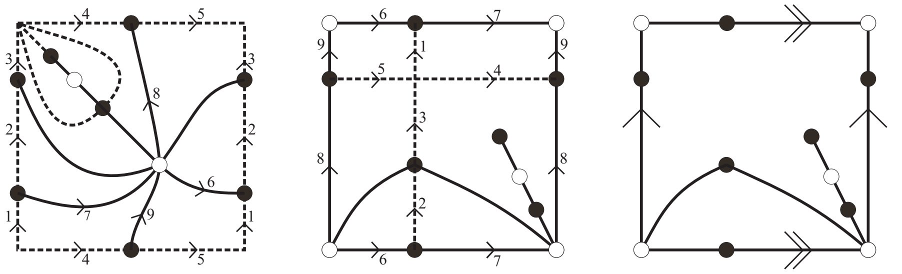

Figure 7: The dual to $\mathrm{I}(0,0,1,0)$ is $\mathrm{VI}(1,0,0,0)$. On the left the dotted lines are the edges of the original graph and the solid lines are the edges of the dual graph; some edges of the original graph are oriented and numbered from 1 to 5 to encode the way they should be identified to each other; some edges of the dual graph are also oriented and numbered from 6 to 9; in the center we show the result of cutting along the edges from 6 to 9 and gluing along those from 1 to 5; on the right we show the same figure as in the center but deleting the original graph, from which one easily sees the type of the dual graph. Similar explanations apply to the next four figures.

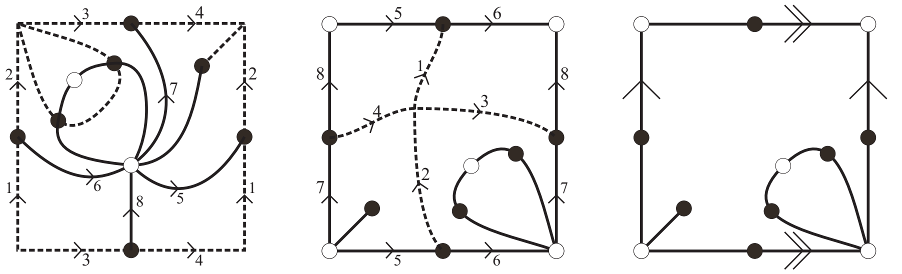

Figure 8: $\mathrm{II}(0,1,0,0)$ is self-dual.

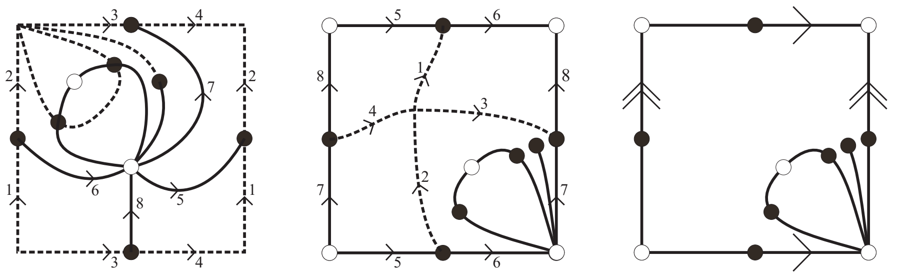

Figure 9: $\mathrm{III}(0,1,0,0)$ is self-dual.

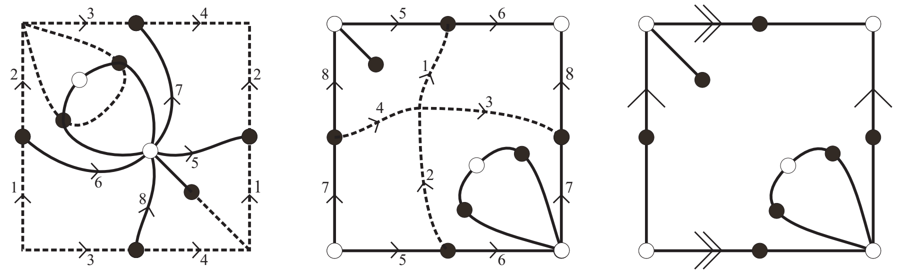

Figure 10: $\mathrm{IV}(0,1,0,0)$ is self-dual.

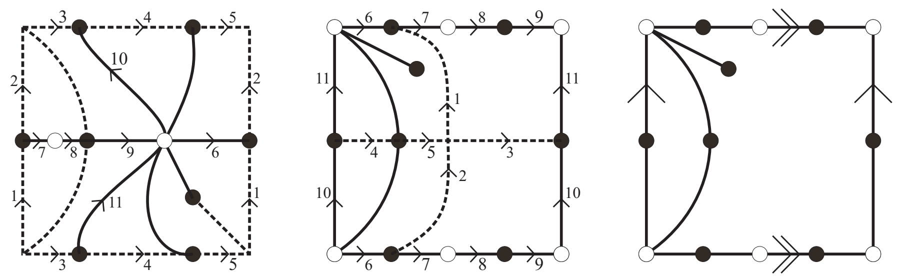

Figure 11: $\mathrm{VI}(0,1,0,0)$ is self-dual.

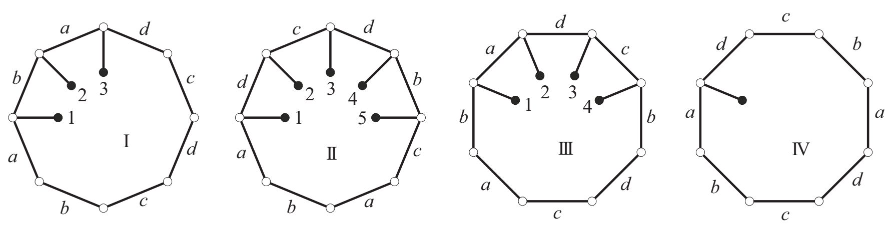

Figure 12: Inequivalent embeddings of Γ in $2T$ with a single disc as a complement.

in several inequivalent ways, namely 3 ways for I, 5 ways for II, 4 ways for III, and only 1 for IV, whence the 13 possibilities. Let us denote by e the label of the extra leg. Note that there is a symmetry $(a,b,c,d,e) \leftrightarrow (b,a,d,c,e)$ in case I.1, a (combinatorially equivalent) symmetry $(a,b,c,d,e) \leftrightarrow (d,c,b,a,e)$ in cases I.3, II.2, II.5 and IV, and no other one. It follows that the number $\nu(k)$ of realizations of $[2k+1]$ is 8 times

- the number of ways of expressing $k-4$ as $a+b+c+d+e$ with integer $a,b,c,d,e \geqslant 0$

plus 5 times

- the number of ways of expressing $k-4$ as $a+b+c+d+e$ with integer $a,b,c,d,e \geqslant 0$ up to the symmetry $(a,b,c,d,e) \leftrightarrow (b,a,d,c,e)$.

Replacing each of these integers with itself plus 1, we see that $\nu(k)$ is 8 times

- the number of ways of expressing $k+1$ as $a+b+c+d+e$ with integer $a,b,c,d,e \geqslant 1$,

namely $\binom{k}{4}$, plus 5 times

- the number of ways of expressing $k+1$ as $a+b+c+d+e$ with integer $a,b,c,d,e \geqslant 0$ up to the symmetry $(a,b,c,d,e) \leftrightarrow (b,a,d,c,e)$.

With this interpretation, in [17] it was shown that $\nu(k-1)$ is given by

$$\frac{k-1}{16}(7k^3-63k^2+197k-208)+\frac{5}{8}(5-2k)\left[\frac{k}{2}\right].$$

Replacing k by $k+1$ in this expression and noting that $\left[\frac{k+1}{2}\right] = k - \left[\frac{k}{2}\right]$ we get

$$\nu(k) = \frac{k}{16}(7k^3-42k^2+72k-37)+\frac{5}{8}(2k-3)\left[\frac{k}{2}\right].$$

This value is correct except for the case $k=4$, where we have to take into account the datum with repeated partitions in the last item of Proposition 2.4, and we must analyze whether any of these 13 embeddings are dual to each other under the last transformation generating the equivalence $\sim$ of Theorem 2.3. This is done in Figg. 13 to 16, where it is shown that each of the graphs

I.1 I.2 I.3 II.4 III.1 III.3 IV

is self-dual, while we have the following dualities:

II.1 $\leftrightarrow$ II.5 II.2 $\leftrightarrow$ II.3 III.2 $\leftrightarrow$ III.4

Therefore for $k=4$ we have $\nu = 10$ rather than $\nu = 13$.

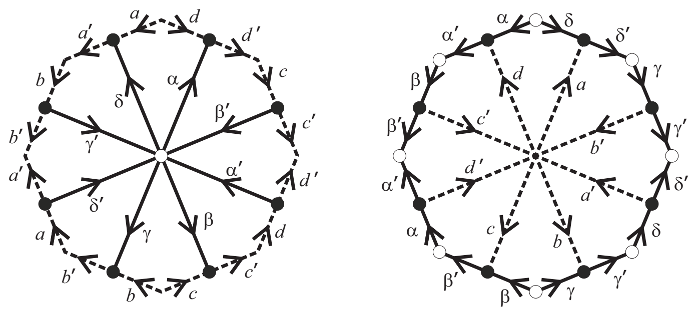

Figure 13: Duals of the graphs of type I.∗. On the left we show by dashed lines the edges of the original graph giving a bouquet of 4 circles (so only the free leg of the graph is missing) and by solid lines those of the dual graph (again, excluding the free leg). The original edges are oriented and labelled as $a, a', \ldots, d, d'$ according to the way they must be identified. The new edges are also oriented and labelled as $\alpha, \alpha', \ldots, \delta, \delta'$. On the right we show the result of cutting along the $\alpha, \alpha', \ldots, \delta, \delta'$ and gluing along the $a, a', \ldots, d, d'$. Since the new pattern of identifications is identical to the original one (with Latin and Greek letters switched), we first of all can conclude that the dual to any graph of type I.∗ is also of type I.∗. More exactly, we see that the original extra leg of the graph I.1 is contained in the quadrilateral with boundary $a'b'^{-1}\gamma'\delta'^{-1}$ on the left, hence the extra leg of the dual is contained in the same quadrilateral on the right, which shows that the dual is also of type I.1 (the position of the leg is not the same but it is combinatorially equivalent). For I.2 the extra leg is in the quadrilateral $a'b\gamma'\delta$, so I.2 is self-dual (different but equivalent position of the extra leg). Finally, for I.3 it is in $a\delta^{-1}\alpha d^{-1}$, so also I.3 is self-dual (same position of the extra leg).

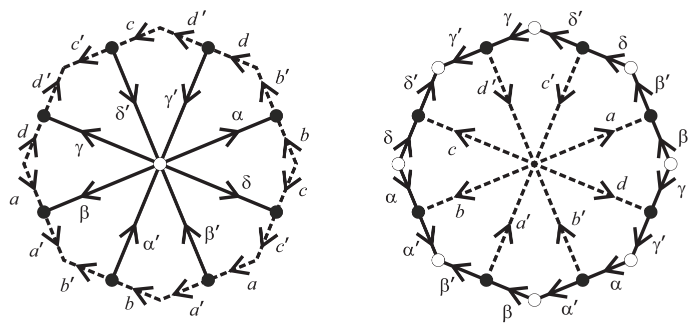

Figure 14: Duals of the graphs of type II.∗. These two images are explained as in the previous figure and show that any graph of type II.∗ is dual to another one of type II.∗. More precisely, the original extra leg of II.1 lies in the quadrilateral $a\beta^{-1}\gamma d^{-1}$, so the dual is II.5. For II.2 it lies in $c'd'^{-1}\gamma^{-1}\delta'^{-1}$, so the dual is II.3, while for II.4 it lies in $b'd\gamma'\alpha$, so II.4 is self-dual (after duality the new position of the leg is different but combinatorially equivalent).

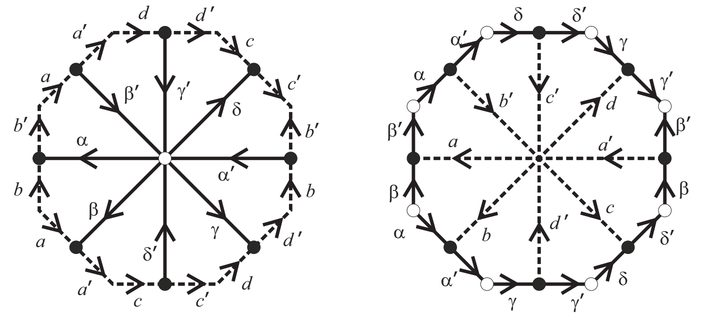

Figure 15: Duals of the graphs of type III.∗. Again these pictures show as above that they can only be dual to each other. For III.1 the extra leg is in $a\beta'\alpha b'$, so III.1 is self-dual. For III.2 it is in $a'd\gamma'\beta'^{-1}$, so the dual is III.4. Finally, for III.3 it is in $c\delta^{-1}\gamma'^{-1}d'$, so III.3 is self-dual (different but equivalent position of the leg).

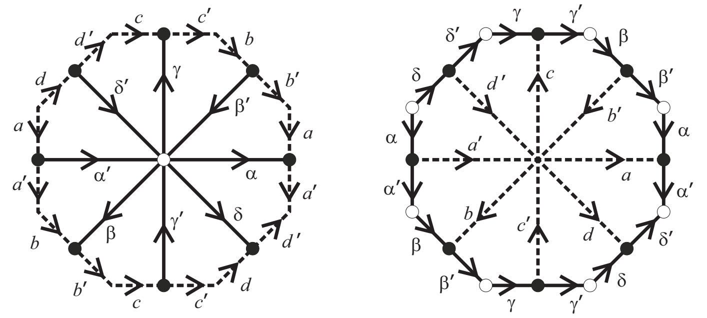

Figure 16: The graph IV is self-dual (the position of the extra leg is immaterial).

Acknowledgements

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