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A collocation-spectral method to solve the bi-dimensional degenerate diffusive logistic equation with spatial heterogeneities in circular domains

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"Cambia, todo cambia... Pero no cambia mi amor." Marcela.

"Profesor Julián, usted me enseñó que la enseñanza es más que impartir conocimiento, es inspirar el cambio. Por esta razón, le dedico este artículo. Felicidades en su sexágesimo aniversario." Frank.

ABSTRACT. In this paper we simulate positive solutions, large solutions and metasolutions of the heterogeneous logistic equation in a disk and an annulus. The numerical methods introduced in this paper are extremely innovative because they make unnecessary determining any previous lifting and solving any decoupled system of ordinary differential equations. Moreover, they can be used to solve non-radially symmetric problems. The models are of a huge interest in Spatial Ecology because they enable us to analyse the effects of the spatial heterogeneity on the evolution of the terrestrial ecosystems. The large solutions and the metasolutions have been computed by the first time in this paper.

Keywords: Heterogeneous logistic equation, unequal distribution of resources, spectral methods, collocation methods, numerical simulation of large solutions and metasolutions.

MS Classification 2010: 35J61, 35J70, 65N35, 65P30, 92D25.

1. Introduction

As a consequence of the unequal distribution of resources, populations distribute themselves in habitats of different size and quality. Algae, cyanobacteria and mountain pine beetles, see [1, 17, 29], grow and reproduce rapidly in some concrete habitats, having extraordinary and dramatic impact in some ecosystems, as changing food webs, decreasing biodiversity and altering ecosystem conditions. Inspired by Section 1.2 of López-Gómez [21], we propose the diffusive heterogeneous logistic equation to model the disproportionate growth of a population.

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Definitely, modelling the heterogeneous distribution of populations in patches of the landscape with different population densities is crucial in conservation planning. Using mathematical models where the habitat is assumed to be spatially homogeneous becomes a tight restriction that leads too often to numerical results that do not match up with the collected field data. At the same time, modelling with reaction-diffusion systems with constant coefficients may also result in inaccurate predictions. The key issue is to implement variable coefficients in reaction-diffusion equations.

Moreover, it is incredibly important to assign correct values to the parameters, in this case, the proliferation rate λ that depends on the size of the patches. There are critical values of this parameter for which the species can survive and grow in each patch as we are going to see below in this paper.

In contrast to spatial structure population models, we use a simpler model that is more tractable and easier to interpret. We solve numerically for the first time the following master equation in Spatial Ecology in an habitat Ω to be considered circular, in the presence of spatial heterogeneity,

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$
(1)

where Δ is the Laplacian, $\Omega \in \{B_R((0,0)), A(R_0, R_1)\}$, with

$$B_R((x_0, y_0)) := \{(x, y) \in \mathbb{R}^2 : ||(x - x_0, y - y_0)|| < R\},\$$
$$A(R_0, R_1) := \{(x, y) \in \mathbb{R}^2 : 0 < R_0 < ||(x, y)|| < R_1\},\$$

and either

$$\mathcal{B}u = \mathcal{D}u = u - f$$

(general Dirichlet boundary conditions), whith $f \ge 0$ or

$$\mathcal{B}u = \frac{\partial u}{\partial \eta} = 0$$

(homogeneous Neumann boundary conditions), where η stands for the outward unit normal vector-field on $\partial\Omega$, $\lambda \in \mathbb{R}$ is a constant, f are the prescribed values of u along the boundary $\partial\Omega$, and $m \geq 0$, $m \neq 0$, is a function of class $\mathcal{C}^{\mu}(\overline{\Omega})$, for some $\mu \in (0, 1]$, satisfying the following hypotheses:

(A) The set

$$\Omega_+ := \{ x \in \Omega : m(x, y) > 0 \}$$

is a subdomain of Ω with $\overline{\Omega}_+ \subset \Omega$, whose boundary, $\partial \Omega_+$, is of class \mathcal{C}^3 , and the open set

$$\Omega_0 := \Omega \backslash \Omega_+$$

consists of two components $\Omega_{0,i}$, $i \in \{1,2\}$, such that

and

$$\overline{\Omega}_{0,1} \cap \overline{\Omega}_{0,2} = \emptyset,$$

$$\lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}] < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}].$$
(2)

Throughout this paper, for any given regular subdomain D of Ω , we denote by $\lambda_1[-\Delta, \mathcal{D}, D]$ the principal eigenvalue of $-\Delta$ in D under homogeneous Dirichlet boundary conditions. As a consequence of the *Maximum Principle*,

$$\lambda_1[-\Delta, \mathcal{D}, D_2] < \lambda_1[-\Delta, \mathcal{D}, D_1] \text{ if } D_1 \subsetneq D_2$$

(see [20] for any further required details). So, roughly speaking, (2) entails $\Omega_{0,1}$ to be larger than $\Omega_{0,2}$, but not exactly, as the principal eigenvalue also dependes on some hidden geometrical properties of the underlying domains. Figure 1 shows some of spatial configurations of m(x, y) treated in this paper. Problem (1) is considered degenerate, always that $\Omega_0 \neq \emptyset$.



Figure 1: Spatial configuration of m(x, y) in the disk $B_R((0, 0))$ and the annulus A(10, 100).

This problem is used in Spatial Ecology to model the evolution of the distribution of a single species, u, randomly dispersed in the inhabiting area, Ω . In this context, it is very important to obtain the solutions of (1) because, at least in case f = 0, they provide us with the limiting profiles as $t \to \infty$ of the solutions of the parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - m(x, y)u^2 & \text{in } \Omega \times (0, \infty), \\ \mathcal{B}u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 > 0 & \text{in } \Omega. \end{cases}$$
(3)

From the point of view of the applications, allowing m(x, y) to vanish on a subdomain of Ω , enables us to model the three different possible behaviors of the solution of (1), with f = 0 and $\mathcal{B} = \mathcal{D}$, according to three distinct ranges of the parameter λ . Precisely, according to López-Gómez [18]:

- The inhabiting region Ω cannot support the species u if $\lambda \leq \lambda_1[-\Delta, \mathcal{D}, \Omega]$.
- The species u grows according to the Verhulst law if $\lambda_1[-\Delta, \mathcal{D}, \Omega] < \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}].$
- The species u grows according to the Malthus law in $\overline{\Omega}_{0,1}$, while it has a logistic behavior in $\Omega \setminus \overline{\Omega}_{0,1}$ if $\lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}] \leq \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}]$.
- The species u grows according to the Verhulst law in Ω_+ , while it exhibits Malthusian growth in $\Omega \setminus \overline{\Omega}_+$ if $\lambda \ge \lambda_1 [-\Delta, \mathcal{D}, \Omega_{0,2}]$.

Therefore, as the previous results establish that, for the appropriate ranges of values of the parameter λ , the metasolutions provide us with the limiting profiles of the positive solutions of the evolution problem, from the point of view of the applications it is imperative to design efficient numerical algorithms to compute all the solutions and *metasolutions* of (1). A function $\mathcal{M} : \Omega \to [0, \infty]$ is said to be a metasolution of (1) supported in $D, D \in {\Omega \setminus \overline{\Omega}_{0,1}, \Omega_+}$ if there exists a solution (*large solution*) L of

$$\begin{cases} -\Delta L = \lambda L - m(x, y)L^2 & \text{in } D, \\ L = 0 & \text{on } \partial D \cap \partial \Omega, \end{cases}$$

satisfying

$$\lim_{{\rm dist}((x,y),\partial D \backslash \partial \Omega) \downarrow 0} L(x,y) = \infty,$$

for which

$$\mathcal{M} = \begin{cases} \infty & \text{in } \Omega \backslash D, \\ L & \text{in } D. \end{cases}$$

Computing the positive solutions, the large solutions and the metasolutions is the main goal of this paper, where, for the first time, the degenerate logistic equation in circular domains, without radial symmetries on the coefficient m(x, y), has been solved numerically. Our numerical schemes and methods enjoy a great versatility, as it will become apparent later.

From the point of view of numerical analysis, our main contribution here consists in developing a number of, really necessary, algebraic manipulations on the differentiation matrix L_{Δ} of the Laplace operator in polar coordinates in order to impose either general inhomogeneous Dirichlet boundary conditions, or homogeneous Neumann ones, both in arbitrary disks and circular annuli. From a theoretical point of view, in the problem with $\Omega = B_1((0,0))$ the most common pseudo-spectral method available is based on the expansion of uin terms of eigenfunctions of the Laplace operator and it can be expressed as

$$u(r,\theta) \approx \sum_{m=0}^{M} \sum_{n=1}^{N} a_{mn} J_m(\sqrt{\lambda_{mn}}r) \cos(m\theta) + \sum_{m=1}^{M} \sum_{n=1}^{N} b_{mn} J_m(\sqrt{\lambda_{mn}}r) \sin(m\theta),$$

where N, M are positive integers, the J_m 's denote the Bessel functions of first kind, λ_{mn} are the eigenvalues of $-\Delta$ in $B_1((0,0))$ under Dirichlet boundary conditions, and a_{mn} , b_{mn} are the (unknown) coefficients of the expansion, that are determined in this paper through the collocation points, (r_i, θ_j) , which are the Chebyshev–Gauss–Lobatto points in the *r*-direction and the equidistant spaced points in the θ -direction. Unfortunately, in the case of the logistic equation, this paradigmatic classical scheme becomes unstable for

$$\lambda > \lambda[-\Delta, \mathcal{D}, B_1((0,0))] + \epsilon$$

if $\epsilon > 0$, being precisely this range of values of λ the one for which the large solutions and metasolutions of the model play a significant role in describing the dynamics of the evolution problem (3).

As a by-product, during the last several years a variety of methods have been developed to approximate the solutions of the Poisson equation in a disk. The monograph of Boyd and Fu Yu [3] collects a rather complete review of them comparing some of the main available schemes to solve the Poisson equation in a disk through the Zernike and the Logan–Shepp ridge polynomials, the Chebyshev–Fourier series, the cylindrical Robert functions, the Bessel–Fourier expansions, the square-to-disk conformal mapping, and the radial basis functions. But yet none of these schemes can be directly applied to compute the large solutions and the metasolutions of our problem. Very recently, the authors obtained in [24] the differentiation matrices of the Laplace equation in polar coordinates subjected to non homogeneous Robin boundary conditions and also, the differentiation matrix of the biharmonic equation subjected to nonhomogeneous boundary conditions. More references concerning pseudospectral methods in the disk can also be found in [24]. The paradigmatic monographs of e.g., Gottlieb–Orszag [13], Fornberg [10], Boyd [2], Peyret [26], Canuto et al [6], and Shen–Tang–Wang [32] reveal the great importance of using spectral and pseudo-spectral methods to solve a huge variety of partial differential equations.

Although some sophisticated numerical calculations of radially symmetric classical solutions for (1), as well as some explosive solutions that do not belong to $\bigcup_{p=1}^{\infty} L^p_{loc}(\Omega)$, were carried out by Gómez-Reñasco and López-Gómez [12], this paper solves for the first time (1) without imposing any radial symmetry on the coefficients. Actually, the numerics of [12] where utterly one-dimensional using ODE's techniques.

Collocation-spectral methods are some of the most versatile methods for treating non-linear problems as well as simulating solutions of partial differential equations with variable coefficients, as it will be seen in this work. Furthermore, solving problem (1) in the unit disk is the first necessary step to solve the same problem on a more complicated geometry via a conformal mapping. But this analysis will be accomplished in an upcoming work and will appear elsewhere.

The organization of this paper is as follows. In Section 2 we apply the underlying collocation spectral method to simulate numerically the classical positive solutions, large solutions and metasolutions of the heterogeneous logistic equation in the unit disk and in a circular annulus for both the Dirichlet and the Neumann problems. In Appendixes A and B we obtain the discretization matrices of the Laplace operator in polar coordinates for homogeneous and inhomogeneous Dirichlet boundary conditions, as well as for homogeneous Neumann boundary conditions.

2. The Logistic Equation with Spatial Heterogeneity.

In this section, we apply the collocation spectral method developed in the Appendix to approximate the positive solutions of (1). As a consequence of the presence of the weight function m(x, y) in front of the non-linear term, the richness of the set of positive solutions of (1) increases extraordinarily. Actually the model can exhibit classical positive solution, large positive solutions and metasolutions of (1). Subsequently, we will compute all these types of solutions.

It should be emphasized that, without a deep previous knowledge of the analytical results of López-Gómez [18] and [21], the numerical resolution of (1) would be an extremely hard task, by the lack of a priori bounds in L^{∞} for the gradients of all these classical and non-classical solutions, which might become infinity even in some open sub-domains of the underlying domain. When necessary, we will refer to [18] for the available theoretical results about (1).

2.1. Classical solutions and metasolutions in $B_1((0,0))$ under Dirichlet boundary conditions

In this section we consider the problem (1) with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } B_1((0, 0)), \\ u = 0 & \text{on } \partial B_1((0, 0)), \end{cases}$$
(4)

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where $m: B_1((0,0)) \to [0,\infty)$ is given by:

$$m(x,y) = \begin{cases} -\left(\sqrt{x^2 + y^2} - 0.5\right)\left(\sqrt{x^2 + y^2} - 0.3\right) & \text{if } (x,y) \in A(0.3,0.5), \\ 0 & \text{otherwise.} \end{cases}$$
(5)

Figure 2 shows a plot of m(x, y) for this choice.



Figure 2: Plot of m(x, y) for the choice (5).

In polar coordinates, the problem (4) becomes into

$$\begin{cases} -\frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \lambda u - m(r,\theta) u^2 & \text{in } (0,1) \times [0,2\pi) \\ u(1,\theta) = 0 & \text{on } [0,2\pi), \\ u(r,\theta) = u(r,\theta+\pi) & \text{in } [0,1] \times (-\infty,\infty), \end{cases}$$
(6)

where $m: [0,1] \times [0,2\pi) \to [0,\infty)$ is given by

$$m(r,\theta) = \begin{cases} -(r-0.5)(r-0.3) & \text{if } (r,\theta) \in (0.3, 0.5) \times [0, 2\pi), \\ 0 & \text{otherwise.} \end{cases}$$
(7)

Naturally, this model fits into the abstract setting of this paper with

$$\Omega_{+} = A(0.3, 0.5),$$
 $\Omega_{0} = B_{0.3}((0, 0)) \cup A(0.5, 1)$

Table 1 collects the theoretical and numerical values of the principal eigenvalue λ_1 of $-\Delta$ in the most relevant subdomains of Ω from the point of view of describing the dynamics of (3). Namely, Ω and each of the two components of

 Ω_0 . We call theoretical λ_1 the value of the approximation obtained by using Bessel function and computed λ_1 the value calculated through the Inverse Power Method applied to the differentiation matrices approximating the Laplace operator with N + 1 nodes in the r-direction and N_{θ} nodes in the θ -direction.

Subdomain	Theoretical λ_1	Computed λ_1	Computed λ_1
		$N=17, N_{\theta}=40$	$N=42, N_{\theta}=40$
$B_1((0,0))$	5.783185962946	5.783185962959	5.783185962956
A(0.5,1)	39.013288499083	39.013288499012	39.013288498923
$B_{0.3}((0,0))$	64.257621810519	64.257621810502	64.257621810334

Table 1: The principal eigenvalues in the relevant subdomains.

Thanks to Table 1, if we take

$$\Omega_{0,1} = A(0.5, 1), \qquad \Omega_{0,2} = B_{0.3}((0, 0)),$$

then $\lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}] < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}]$. The existence of classical positive solutions of (1) is guaranteed from the following theorem borrowed from [18]. As all the remaining results going back to [18] and [12], it is collected here by the sake of completeness.

THEOREM 2.1. Suppose m(x, y) satisfies (A). Then,

1. The problem (4) possesses a classical positive solution if, and only if,

$$\lambda_1[-\Delta, \mathcal{D}, \Omega] < \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}].$$
(8)

Moreover, it is unique if it exists.

2. Suppose (8) and let θ_{λ} denote the unique classical positive solution of (4). Then

$$\lim_{\lambda \downarrow \lambda_1[-\Delta, \mathcal{D}, \Omega]} ||\theta_\lambda||_{L^{\infty}(\Omega)} = 0,$$
(9)

and

$$\lim_{\lambda \uparrow \lambda_1 [-\Delta, \mathcal{D}, \Omega_{0,1}]} ||\theta_\lambda||_{L^{\infty}(\Omega)} = \infty$$
(10)

uniformly in $(\Omega_{0,1} \cup \Omega_{0,2}) \setminus \partial \Omega$.

3. The mapping $\lambda \to \theta_{\lambda}$ is point-wise increasing and, if we regard to it as a mapping from $(\lambda_1[-\Delta, \mathcal{D}, \Omega], \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}])$ into $C^{1,\nu}(\Omega), \ 0 < \nu < 1$, then it is differentiable and $\frac{\partial \theta_{\lambda}}{\partial \lambda} \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all p > 1.

In order to compute some distinguished solutions along the global curve of classical positive solutions of (4) we apply the collocation spectral method already described in Appendix A to obtain a nonlinear system of equations that we solve using the Newton method. Succeeding in the choice of an appropriate initial data for the Newton method is utterly based on a good knowledge of the available analytical results.

Figure 3 shows some of the classical positive solutions that we have computed with our method. The value $\lambda_1[-\Delta, \mathcal{D}, B_1((0,0))] = 5.783185$ is the unique value of λ for which bifurcation to positive solutions from u = 0 occurs. It should be noted how these solutions grow in $\overline{\Omega_{0,1}}$, while, in strong apparent contrast, they stabilize in $B_1((0,0)) \setminus \overline{\Omega_{0,1}}$, as λ increases.

As λ moves up from $\lambda_1[-\Delta, \mathcal{D}, B_1((0,0))] = 5.783185$, the principal eigenvalue of the linearization around the positive solutions grows from zero up to reach its maximum value critical λ , where it becomes decreasing for any further value λ up to approach the critical value where the bifurcation from infinity takes place, where it converges to zero. As this feature, was not previously observed in the specialized literature, we conjecture that

$$\lim_{\lambda \uparrow \lambda_1 [-\Delta, \mathcal{D}, \Omega_{0,1}]} \lambda_1 [-\Delta + 2 m(x, y) \theta_{\lambda} - \lambda, \mathcal{D}, \Omega] = 0.$$

Table 2 collects some representative values of λ together with the L^{∞} -norms of the corresponding positive solutions and the principal eigenvalues of their linearizations (p.e.l.).

Value of λ	p.e.l.	$ u _{\infty}$
30.0	1.9279	2.3520e + 005
32.5	1.0511	1.3294e + 006
33.6	0.7035	5.0819e + 006
34.0	0.4480	1.0785e+007

Table 2: The principal eigenvalues of the linearizations.

Now, we will show the results of our numerical experiments for computing the metasolutions of (4). First, we need to introduce some concepts going back to [12].

DEFINITION 2.2. Consider the problem

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } D, \\ u = \infty & \text{on } \partial D, \end{cases}$$
(11)

where D is un proper subdomain of Ω . A function $u \in C^{2+\mu}(D)$ is said to be a large (or explosive) solution of (11) if it satisfies the differential equation in



Figure 3: Plots of the classical positive solutions of (4) for $\lambda \in \{6, 13, 22, 29, 34\}$.

 $D, u = 0 \text{ on } \partial D \cap \partial \Omega, \text{ and }$

$$\lim_{\operatorname{dist}((x,y),\partial D\setminus\partial\Omega)\downarrow 0} u(x,y) = \infty.$$

DEFINITION 2.3. Consider (11) with $D \in \{\Omega \setminus \overline{\Omega_{0,1}}, \Omega_+\}$. Then, a function $\mathcal{M} : \Omega \to [0, \infty]$ is said to be a metasolution of (11) supported in D if there exists a large solution L of (11) in D for which

$$\mathcal{M} = \begin{cases} \infty & in \ \Omega \backslash D, \\ L & in \ D. \end{cases}$$
(12)

According to López-Gómez [18] and [21], it is known that:

- If $\lambda_1[-\Delta, \mathcal{D}, \Omega] \leq \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$, the problem (4) admits a classical positive solution.
- If $\lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}] \leq \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}]$, the problem (4) admits a metasolution supported in $\Omega \setminus \overline{\Omega_{0,1}}$.
- If λ ≥ λ₁[−Δ, D, Ω_{0,2}], the problem (4) admits a metasolution supported in Ω₊.

Moreover, the minimal metasolutions in these ranges describe the limiting profiles of all positive solutions of the evolution problem (3), when the initial data u_0 is a subsolution of problem (1), see Theorem 5.2 in [21]. So, the importance of computing them from the point of view of the design, or restoration, of spatially heterogeneous ecosystems. According to the previous analytical results, (1) possesses a metasolution supported in $\Omega \setminus \overline{\Omega_{0,1}}$ if

$$\lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}] \simeq 39.013288 \le \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}] \simeq 64.257622.$$
(13)

To compute this metasolution, we first computed the large solution u of

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } B_{0.5}((0, 0)), \\ u = \infty & \text{on } \partial B_{0.5}((0, 0)). \end{cases}$$
(14)

The most natural strategy to approximate the large solution of (14) is to compute the unique positive solution of

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } B_{0.5}((0, 0)), \\ u = \beta & \text{on } \partial B_{0.5}((0, 0)), \end{cases}$$
(15)

for sufficiently large β . Figure 4 shows some numerical solutions of (15) with $\beta = 3 * 10^5$. Our numerics reveal that the metasolutions supported in $\Omega \setminus \overline{\Omega_{0,1}}$

are point-wise increasing in $\Omega \setminus \overline{\Omega_{0,1}}$ with respect to λ . They grow at a faster rate in A(0.5, 1), where m = 0, than in $B_{0.5}((0, 0))$, where m > 0. Each of these metasolutions takes the value β on $\partial B_{0.5}((0, 0))$. As $\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}]$, the corresponding metasolution exhibits a complete blow-up in $B_{0.3}((0, 0))$, while it stabilizes in A(0.3, 0.5). Finally, to obtain the metasolution supported in Ω_+ for $\lambda \geq \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}]$, we computed the large solution of

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } A(0.3, 0.5), \\ u = \infty & \text{on } \partial A(0.3, 0.5), \end{cases}$$
(16)

approximating it by the unique solution of

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } A(0.3, 0.5), \\ u = \beta & \text{on } \partial A(0.3, 0.5), \end{cases}$$
(17)

for β sufficiently large. Figure 5 shows some plots of these metasolutions.

Since the problems (14) and (17) are radially symmetric, the positive large solution of each of these problems is unique, by, e.g., Theorem 7.1 of J. López-Gómez [21] (see also [19]). Moreover, due to Theorem 4.7 of [21], we already know that the positive solutions of (14) and (17) approximate these unique large solutions as $\beta \uparrow \infty$. For uniqueness results in more general settings, the reader is sent to the more recent paper of J. López-Gómez and L. Maire [22]. Figure 6 shows a zoom of the profiles of the positive solutions of (15) for $\lambda = 40$, as well as the profiles of the positive solutions of (17) for $\lambda = 70$ and $\beta \in \{3 * 10^5, 4 * 10^5, 5 * 10^5\}$.

2.2. Classical positive solutions in $A(R_0, R_1)$ under Dirichlet boundary conditions

In this subsection, we compute numerically some classical positive solutions of

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } A(10, 100), \\ u = 0 & \text{on } \partial A(10, 100), \end{cases}$$
(18)

where $m: A(10, 100) \rightarrow [0, \infty)$ is defined by:

$$m(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A(95,100), \\ 10^{-11}p(x,y)(x^2+y^2-10^2)(95^2-x^2-y^2) & \text{if } (x,y) \in A(10,95) \setminus B_6((30,40)), \\ 0 & \text{if } (x,y) \in B_6((30,40)). \end{cases}$$
(19)

where $p(x, y) = (x - 30)^2 + (y - 40)^2 - 36$. Figure 7 shows a plot of m(x, y) defined in (19).



Figure 4: Plots of the solutions of (15) in $B_1(0)\setminus \overline{A(0.5,1)}$ for $\lambda \in \{40, 48, 55, 60, 64\}$.



Figure 5: Plots of the solutions of (17) in A(0.3, 0.5) for $\lambda \in \{70, 100\}$.



Figure 6: Profiles approximating the large solution of (14) for $\lambda = 40$ and, of (16) for $\lambda = 70$.



Figure 7: Plots of m(x, y) and its contour lines for the choice (19).

Consequently, the problem is far from being radially symmetric. The existence of classical positive solutions of (18) is guaranteed by Theorem 2.1. Problem (18) can be rewritten as:

$$\begin{cases} -\frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \lambda u - m(r,\theta) u^2 & \text{in } (10,100) \times [0,2\pi) \\ u(10,\theta) = 0 & \text{on } [0,2\pi) \\ u(100,\theta) = 0 & \text{on } [0,2\pi) \\ u(r,\theta) = u(r,\theta+\pi) & \text{in } [10,100] \times (-\infty,\infty). \end{cases}$$
(20)

In Table 3 we are giving the theoretical and numerical values of the principal eigenvalue of $-\Delta$ in some of the relevant subdomains of Ω . The theoretical value is calculated from the estimate 2.4048 for the first zero of the Bessel function J_0 .

Subdomain	Theoretical λ_1	Computed λ_1
A(10,100)	0.001097	0.001098
$B_6((30,40))$	0.160640	0.160644
A(95,100)	0.394757	0.394757

Table 3: The principal eigenvalues in some relevant subdomains.

The corresponding model fits into the general setting of this paper with

$$\Omega_{+} = A(10,95) \setminus B_6((30,40)), \quad \Omega_{0,1} = B_6((30,40)) \quad \Omega_{0,2} = A(95,100).$$

Figure 8 shows some of the classical positive solutions that we have computed. These solutions grow in $\overline{B_6((30, 40))}$, while they stabilize in the set $A(10, 100) \setminus \overline{B_6((30, 40))}$, as λ increases. As λ increases from 0.001098 approximating the principal eigenvalue in $B_6((30, 40))$, which is given by 0.160644, the solutions blow up in $B_6((30, 40))$ as $\lambda \uparrow 0.160644$.

2.3. Classical positive solutions in $B_1((0,0))$ under Neumann conditions

In this subsection we compute the classical positive solution of

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } B_1((0, 0)), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial B_1((0, 0)), \end{cases}$$
(21)

using the collocation spectral method described in the Appendixes. Here, η stands for the outward unit normal along $\partial B_1((0,0))$. So, $\eta(x,y) = (x,y)$



Figure 8: Plots of the classical solutions of (18) in A(10, 100) for $\lambda \in \{0.004, 0.03, 0.06\}$.

for all $(x, y) \in \partial B_1((0, 0))$. The existence of solutions of (21) for any domain $\Omega \subset \mathbb{R}^2$ is guaranteed by the next theorem going back to Ouyang [25]. The case of general boundary operators on $\partial\Omega$ was first considered by J. M. Fraile et al. [11], where the open set Ω_0 consists of a single component with $\overline{\Omega}_0 \subset \Omega$. Nevertheless, in this paper we will investigate numerically some cases where Ω_0 consists of two disjoint components. Our numerical experiments show that the positive classical solutions of (21) tend to infinity in $\Omega_{0,1}$ as $\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$.

THEOREM 2.4. Assume that $m \ge 0 \ (\not\equiv 0)$ is a smooth function in Ω .

- 1. If $\Omega_0 = \emptyset$, then for every $\lambda > 0$ there exists a unique solution $u(\lambda)$ of problem (21).
- 2. If $\Omega_0 \neq \emptyset$, then for any $\lambda \in (0, \lambda_1[-\Delta, \mathcal{D}, \Omega_0])$ there exists a unique solution of (21), whereas (21) cannot admit a positive solution if $\lambda \geq \lambda_1[-\Delta, \mathcal{D}, \Omega_0]$.

Moreover

$$\lim_{\lambda \uparrow \lambda_1 [-\Delta, \mathcal{D}, \Omega_0]} ||u(\lambda)||_{L^2(\Omega)} = \infty.$$
(22)

Note that problem (21) can be written as:

$$\begin{cases} -\frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \lambda u - m(r, \theta) u^2 & \text{in } (0, 1) \times [0, 2\pi), \\ \frac{\partial u}{\partial r} (1, \theta) = 0 & \text{on } [0, 2\pi), \\ u(r, \theta) = u(r, \theta + \pi) & \text{in } [0, 1] \times (-\infty, \infty). \end{cases}$$

$$(23)$$

In order to show the excellent accuracy of the numerical method, we are taking $m \equiv 1$ in the first simulation. In this case, the corresponding model fits into the abstract setting of Theorem 2.4, with $\Omega_+ = B_1((0,0))$ and $\Omega_0 = \emptyset$. Thus, the problem (21) has a unique positive solution for all $\lambda > 0$. In this case, we know that the solution of (21) is $u \equiv \lambda$. Figure 9 shows the plots of some classical positive solutions computed through the spectral collocation method introduced in this paper, and the distribution of the error $E(x, y) = |u(x, y) - \lambda|$ in $B_1((0,0))$ for $\lambda = 8$ and $\lambda = 100$. Note that the maximum value of the error is of order 10^{-13} for $\lambda = 8$ and 10^{-12} for $\lambda = 100$.

Finally, for the last simulation, we take m as in (5). It should be remember that for this choice the model fits into the abstract setting of this paper with $\Omega_+ = A(0.3, 0.5), \ \Omega_0 = B_{0.3}((0,0)) \cup A(0.5,1), \ \Omega_{0,1} = A(0.5,1)$ and $\Omega_{0,2} = B_{0.3}((0,0)$. In this case, combining the abstract theory of Fraile et al. [11] with López-Gómez [21, Ch. 4], it becomes apparent that (21) has a classical positive solution if, and only if, $0 < \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$. Actually, this is a rather direct consequence of Daners and López-Gómez [7, Th. 1.1].

Figure 9: Numerical solution of (21) for $\lambda \in \{8, 100\}$ with $m \equiv 1$ and the corresponding errors E(x, y).

Figure 10 shows the plots of the numerical solutions of (21) for $\lambda \in \{0.0003, 5\}$. Although it is well known that the solutions are point-wise increasing in Ω with respect to λ , our experiments suggest that they grow at a faster rate on $\Omega_{0,1}$. Actually, these solutions grow up to infinity on $\overline{\Omega}_{0,1}$ as $\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$.

2.4. Case Neumann II : Numerical Computation of Classical Positive Solutions in the circular annulus $\Omega = A(R_0, R_1)$

Firstly, we consider the problem

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } A(4, 10), \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial A(4, 10), \end{cases}$$
(24)

where η is the unit outward vector on $\partial A(4, 10)$ and $m \equiv 1$. The existence of solutions of (24) is guaranteed by Theorem 2.4. The problem (24) in polar

Figure 10: Plots of the classical solutions of (23) with m as in (5) for $\lambda \in \{0.0003, 5\}$.

coordinates can be rewritten as:

$$\begin{cases} -\frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \lambda u - m(r,\theta) u^2 & \text{in } (4,10) \times [0,2\pi), \\ -\frac{\partial u}{\partial r} (4,\theta) = 0 & \text{on } [0,2\pi), \\ \frac{\partial u}{\partial r} (10,\theta) = 0 & \text{on } [0,2\pi), \\ u(r,\theta) = u(r,\theta+\pi) & \text{in } [4,10] \times (-\infty,\infty). \end{cases}$$
(25)

Since $\Omega_+ = A(4, 10)$ and $\Omega_0 = \emptyset$, by the Theorem 2.4, there exists a classical positive solution for all $\lambda > 0$. Naturally, as in the previous section, the solutions of (24) must be $u \equiv \lambda$. Figure 11 shows some of the numerical solutions that we computed.

Figure 11: Numerical solution of (24) in A(4, 10) for $\lambda \in \{4, 15\}$ with $m \equiv 1$.

To end this paper, we consider (24) in A(4, 10) with two different coefficients $m: A(4, 10) \to [0, \infty)$ defined by

$$m(x,y) = \begin{cases} (\sqrt{x^2 + y^2} - \gamma)(9 - (\sqrt{x^2 + y^2}) & \text{if } (x,y) \in A(\gamma,9), \\ 0 & \text{if } (x,y) \in A(4,\gamma) \cup A(9,10). \end{cases}$$
(26)

where $\gamma \in \{4.9, 5.5\}$. Figure 12 shows a plot of m(x, y) for $\gamma = 5.5$. For this choice, $\Omega_+ = A(\gamma, 9)$ and $\Omega_0 = A(4, \gamma) \cup A(9, 10)$. Table 4 provides the numerical values of the principal eigenvalues of $-\Delta$ in some relevant subdomains of Ω . These values have been computed applying the Inverse Power Method to the discretization matrix of the Laplace operator, taking 85 nodes in the *r*-direction and 60 nodes in the θ -direction.

Although it is possible to give a theoretical value for the underlying principal eigenvalues as in the tables above, in this occasion it is much faster and versatile to compute them through the Inverse Power Method applied to the

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Figure 12: Plot of m(x, y) for the choice (26) with $\gamma = 5.5$.

corresponding differentiation matrix. Actually, our method might be far more accurate than using the available tables.

Subdomain	A(4,10)	A(4,5.5)	A(9,10)	A(4, 4.9)
Computed λ_1	0.268642	4.375300	9.866831	12.172021

	Table	4:	The	principal	eigenvalue	of some	relevant	subdomains.
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Thanks to the values given in Table 4, we have that $\Omega_{0,1} = A(4, 5.5)$ and $\Omega_{0,2} = A(9, 10)$ if $\gamma = 5.5$, since

$$\lambda_1[-\Delta, \mathcal{D}, A(4, 5.5)] < \lambda_1[-\Delta, \mathcal{D}, A(9, 10)],$$

whereas $\Omega_{0,1} = A(9, 10)$ and $\Omega_{0,2} = A(4, 4.9)$ if $\gamma = 4.9$, because in such case

$$\lambda_1[-\Delta, \mathcal{D}, A(9, 10)] < \lambda_1[-\Delta, \mathcal{D}, A(4, 4.9)]$$

So, the relative position of these principal eigenvalues have inter-exchanged.

Figures 13 and 14 show the plots of some positive solutions of (24) with m(x, y) defined by (26); these plots were computed for $\gamma = 5.5$ and $\gamma = 4.9$, respectively. In both cases, as predicted by the theory, the solutions are pointwise increasing with respect to λ . However, these solutions grow faster in $\overline{A}(4, 5.5)$ than in $\overline{A}(9, 10)$ if $\gamma = 5.5$, while they grow faster in $\overline{A}(9, 10)$ than in $\overline{A}(4, 5.5)$ if $\gamma = 4.9$, as expected from the existing theory.

Actually, these solutions grow to infinity in $\bar{A}(4, 5.5)$ as $\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$ if $\gamma = 5.5$, stabilizing to some fixed profile in $\bar{A}(9, 10)$, whereas they grow-up to infinity in $\bar{A}(9, 10)$ as $\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$ if $\gamma = 4.9$, staying bounded in its complement.

Figure 13: Numerical solution of (24) in A(4, 10) for $\lambda \in \{1, 1.1\}$ with m(x, y) given by (26) with $\gamma = 5.5$.

2.5. Final remarks

Table 5 collects the number of collocation points used in the simulations presented in this paper. When the domain is a disk, it equals $\left(\frac{N_r+1}{2}\right)N_{\theta}$, while it is given by $(N_r + 1)N_{\theta}$ if, instead, it is an annulus.

As illustrated in Table 5, for obtaining Figure 8, in order to capture the fastest growth of the solution in $\Omega_{0,1} = B_6((30, 40))$, we had to increase the number of collocations points up to 2300.

In the simulations sketched by Figure 5, we have taken more collocations points than in the simulations of Figures 3-4 to approximate the growth of the solution on $\partial A(0.3, 0.5)$. Finally, note that, in order to get Figures 13 and 14, where $\Omega_{0,1} \neq \emptyset$, we have used more collocation points than in the simulations necessary to get Figure 11, where $\Omega_0 = \emptyset$.

Appendix

A. Construction of the differentiation matrices in the unit disk

The main goal of this appendix is to discretize the Laplace operator in polar coordinates in the unit disk $B_1((0,0))$ in order to impose inhomogeneous Dirichlet and homogeneous Neumann conditions. First, we will discretize the disk spectrally by taking a periodic Fourier grid in θ and a nonperiodic Chebyshev grid in r. Note that, when performing the radial interpolation, as the

Figure 14: Numerical solution of (24) in A(4, 10) for $\lambda \in \{1, 1.8\}$ with m(x, y) given by (26) with $\gamma = 4.9$.

radius is positive, the collocation points (r_i, θ_j) with negative r_i , correspond to those which have the same radius and θ increased by π (see [4, 9, 15, 23]). The collocation points are $(r_i, \theta_j) = \left(\cos\left(\frac{(i-1)\pi}{N_r}\right), \frac{2\pi j}{N_{\theta}}\right)$ for $1 \le i \le N + 1$ and $1 \le j \le N_{\theta}$, where $N = (N_r - 1)/2$. To avoid the inherent loss of regularity at the origin, the grid parameter N_r in the r-direction is taken to be odd, and N_{θ} must be even to be able to apply the symmetry properties in θ .

Some pioneer results about Chebyshev-Fourier expansion can be found in [2, 5, 8, 28, 30, 31]. In Gottlied, Hussaini and Orszag [14] it was shown that the trigonometric interpolant of a smoothly differentiable function with period

Figure	Domain	N_r	N_{θ}	Total
3	$B_1((0,0))$	55	30	990
4	$B_{0.5}((0,0))$	55	30	990
5	A(0.3, 0.5)	45	30	1380
8	A(10, 100)	45	50	2300
9	$B_1((0,0))$	35	30	540
11	A(4, 10)	40	30	1230
13	A(4, 10)	61	30	1830
14	A(4, 10)	61	30	1830

Table 5: Total number of collocations points.

 $2\pi, g(\theta)$ can be written as

$$g(\theta) = \sum_{l=1}^{N_{\theta}} g(\theta_l) S_{N_{\theta}}(\theta - \theta_l)$$

where $S_{N_{\theta}}$ is the periodic sinc function:

$$S_{N_{\theta}}(\theta) = \frac{\sin\left(\frac{N_{\theta}\theta}{2}\right)}{N_{\theta}\tan\left(\frac{\theta}{2}\right)}.$$

Thus, let us consider

$$u_{N+1,N_{\theta}}(r,\theta) = \sum_{k=1}^{N_r} L_k(r) P_k(\theta)$$

where L_k 's are the Lagrange polynomials $L_k(r) = \prod_{i \neq k} (r - r_i) / (r_k - r_i)$ and

$$P_k(\theta) = \sum_{l=1}^{N_{\theta}} a_{k,l} S_{N_{\theta}}(\theta - \theta_l)$$

is the trigonometric interpolant of $u(r_k, \theta)$ in the points $\theta_l, l = 1, \ldots, N_{\theta}$. Then

$$u_{N+1,N_{\theta}}(r,\theta) = \sum_{k=1}^{N_{r}+1} \sum_{l=1}^{N_{\theta}} a_{k,l} S_{N_{\theta}}(\theta - \theta_{l}) L_{k}(r).$$
(27)

Note that the approximate solution used in Huang and Sloan [16] coincides with the expression in (27) but there, the collocation points in the radial direction are of the form $\frac{1-\cos\left(\frac{(i-1)\pi}{N_r}\right)}{2}$ for $1 \le i \le N+1$.

Taking into account that

$$r_{N_r+2-i} = -r_i$$
 and $\theta_{j+\frac{N_\theta}{2}} = \theta_j + \pi$, for $1 \le j \le \frac{N_\theta}{2}$ and $1 \le i \le \frac{N_r+1}{2}$,

we can conclude that

$$u(r_{N_r+2-i}\,\cos\theta_j, r_{N_r+2-i}\,\sin\theta_j) = u(r_i\,\cos\theta_{j+\frac{N_\theta}{2}}, r_i\,\sin\theta_{j+\frac{N_\theta}{2}}).$$
(28)

Since

$$a_{N_r+2-i,j}=u(r_{N_r+2-i},\theta_j), \qquad a_{i,j+\frac{N_\theta}{2}}=u(r_i,\theta_{j+\frac{N_\theta}{2}}),$$

from (28) we finally obtain that

$$a_{N_r+2-i,j} = a_{i,j+\frac{N_{\theta}}{2}}, \quad 1 \le j \le \frac{N_{\theta}}{2}, \quad 1 \le i \le \frac{N_r+1}{2}.$$
 (29)

Very recently, using (29), the authors proved in [24] that $u_{N+1,N_{\theta}}(r,\theta)$ can be rewritten as:

$$u_{N+1,N_{\theta}}(r,\theta) = \sum_{k=1}^{\frac{N_{r}+1}{2}} \sum_{l=1}^{N_{\theta}} a_{k,l} \left[S_{N_{\theta}}(\theta-\theta_{l}) L_{k}(r) + S_{N_{\theta}}\left(\theta-\theta_{l+\frac{N_{\theta}}{2}}\right) L_{N_{r}+2-k}(r) \right]$$

where $a_{k,l} = u(r_k, \theta_l)$.

Therefore, there exist $\left(\frac{N_r+1}{2}\right) N_{\theta}$ unknowns in $u_{N+1,N_{\theta}}(r,\theta)$. Thus, the associated matrix to the Laplacian in polar coordinates on the full grid is an $(N+1)N_{\theta} \times (N+1)N_{\theta}$ matrix consisting of Kronecker products where $N = (N_r - 1)/2$. Let us define the differentiation matrices D_1 , D_2 , E_1 , E_2 and $D_{\theta}^{(2)}$ by

$$(E_{1})_{i,j} = L'_{j}(r_{i}); \qquad 1 \le i, j \le N+1, (E_{2})_{i,j} = L'_{N_{r}+2-j}(r_{i}); \qquad 1 \le i, j \le N+1, (D_{1})_{i,j} = L''_{j}(r_{i}); \qquad 1 \le i, j \le N+1, (D_{2})_{i,j} = L''_{N_{r}+2-j}(r_{i}); \qquad 1 \le i, j \le N+1, (D_{\theta}^{(2)})_{k,l} = S''_{N_{\theta}}(\theta_{k} - \theta_{l}); \qquad 1 \le l, k \le N_{\theta}.$$

$$(30)$$

Consequently, the discretization matrix of the Laplacian in polar coordinates, denoted by L_{Δ} , takes the following form:

$$L_{\Delta} = (D_1 + RE_1) \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + (D_2 + RE_2) \otimes \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + R^2 \otimes D_{\theta}^{(2)}$$

where I stands for the identity of order $\frac{N_{\theta}}{2} \times \frac{N_{\theta}}{2}$ and R is the diagonal matrix $R_{i\,i} = r_i^{-1}, i = 1, \ldots, N+1$, see [33] and [27].

Finally, one should extract the N_{θ} - first rows of L_{Δ} because they correspond to the discretization of the Laplacian on the boundary points (r_1, θ_j) for j = $1, \ldots, N_{\theta}$. So, the discretization matrix of the Laplace operator on the inner collocations points is given by \tilde{L} where \tilde{L} is obtained by stripping L_{Δ} of its N_{θ} first rows, so,

$$L_{\Delta} = \left(\begin{array}{c} \dots \\ \tilde{L} \end{array}\right)$$

Throughout the rest of this section, we will set:

$$u = (u(r_1, \theta_1), \dots, u(r_1, \theta_{N_{\theta}}), u(r_2, \theta_1), \dots, u(r_2, \theta_{N_{\theta}}), \dots \dots \dots, u(r_{N+1}, \theta_1), \dots, u(r_{N+1}, \theta_{N_{\theta}}))^T,$$

$$u_0 = (u(r_1, \theta_1), \dots, u(r_1, \theta_{N_{\theta}}))^T,$$

$$\tilde{u} = (u(r_2, \theta_1), \dots, u(r_2, \theta_{N_{\theta}}), \dots, u(r_{N+1}, \theta_1), \dots, u(r_{N+1}, \theta_{N_{\theta}}))^T$$

Note that $u = (u_0, \tilde{u})^T$ and $N + 1 = \frac{N_r + 1}{2}$. Finally,

$$(\tilde{L}\,\tilde{u})_{(i-2)N_{\theta}+j} = \left. \left(\frac{\partial^2 u_{N+1,N_{\theta}}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{N+1,N_{\theta}}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_{N+1,N_{\theta}}}{\partial \theta^2} \right) \right|_{(r_i,\theta_j)}$$

for every i = 2, ..., N + 1 and $j = 1, ..., N_{\theta}$. It should be noted that the subsequent analysis depends on the nature of the boundary conditions of the problem we want to solve.

A.1. Inhomogeneous Dirichlet condition $(u = f \neq 0 \text{ on } \partial B_1((0,0)))$

To impose the boundary condition we fix $(u_0)_j = u(r_1, \theta_j) = f(\theta_j)$ for $j = 1, \ldots, N_{\theta}$. Then, we divide \tilde{L} as:

$$\tilde{L} = \left(\begin{array}{c|c} L_1 & L_2 \end{array} \right) \tag{31}$$

where L_1 and L_2 are the matrices obtained by stripping \tilde{L} of its NN_{θ} -last and N_{θ} -first columns, respectively. Note that

$$L u = L_1 u_0 + L_2 \tilde{u}.$$

Thus, L_2 provides us with the discretization matrix of the Laplace operator on the inner collocation points.

A.2. Homogeneous Neumann conditions $\left(\frac{\partial u}{\partial \eta} = 0 \text{ on } \partial B_1((0,0))\right)$

Let *E* be the differentiation matrix of $\frac{\partial}{\partial r}$ on the colocation points (r_i, θ_j) for $i = 1, \ldots, N$ and $j = 1, \ldots, N_{\theta}$:

$$E := E_1 \otimes \left(\begin{array}{cc} I & 0 \\ 0 & I \end{array}\right) + E_2 \otimes \left(\begin{array}{cc} 0 & I \\ I & 0 \end{array}\right)$$

where E_1 and E_2 are the matrices defined in (30), and I stands for the identity of order $\frac{N_{\theta}}{2} \times \frac{N_{\theta}}{2}$. In order to impose the Neumann boundary conditions on the collocation points on $\partial B_1((0,0))$, we are interested in the portion of E that discretizes the first derivative on these points. Thus, we introduce the matrix

$$A = F_1^{E_1} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + F_1^{E_2} \otimes \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

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where $F_1^{E_1}$ and $F_1^{E_2}$ denote the first row of the matrices E_1 and E_2 , respectively. Note that, for any $j = 1, \ldots, N_{\theta}$ fixed, the discrete partial derivative with respect to r in (r_1, θ_j) corresponds to the j-th row of the matrix A. That is,

$$\left. \frac{\partial}{\partial r} \right|_{(r_1,\theta_j)} = F_j^A \quad \text{for } j = 1, \dots, N_{\theta}.$$

Then, we obtain the following partial of E:

$$E = \left(\begin{array}{c} A \\ \hline \\ \hline \\ \hline \\ \end{array}\right).$$

Next, we break up A as follows:

$$A = \left(\begin{array}{c|c} A_1 & A_2 \end{array} \right)$$

where A_1 and A_2 stand for the matrices obtained by stripping of A the NN_{θ} last and the N_{θ} -first columns, respectively. Finally, the homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta}\Big|_{\partial B_1((0,0))} = \left.\frac{\partial u}{\partial r}\right|_{\partial B_1((0,0))} = 0$$

implies that

$$0 = A \, u = A_1 \, u_0 + A_2 \, \tilde{u}.$$

Thus, u_0 satisfies

$$u_0 = -A_1^{-1} A_2 \tilde{u}.$$

Considering \tilde{L} as in (31), we have:

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$$\dot{L}u = L_1 u_0 + L_2 \tilde{u}$$

= $(-L_1 A_1^{-1} A_2 + L_2) \tilde{u}$

Therefore, the discretization matrix of the Laplacian on the inner collocation points with homogeneous Neumann boundary conditions becomes

$$\tilde{L} = -L_1 A_1^{-1} A_2 + L_2.$$

We claim that A_1 is non-singular. Indeed, since

$$A_1 = (E_1)_{11} \otimes \left(\begin{array}{cc} I & 0\\ 0 & I \end{array}\right) + (E_2)_{11} \otimes \left(\begin{array}{cc} 0 & I\\ I & 0 \end{array}\right),$$

using (30) and the well known Chebyshev differentiation matrix (see, e.g., [26], [5] or [33]), it follows that

$$(E_1)_{11} = \frac{2N_r^2 + 1}{6}, \quad (E_2)_{11} = \frac{1}{2}(-1)^{N_r}.$$

Thus,

$$\det(A_1) = \left[((E_1)_{11})^2 - ((E_2)_{11})^2 \right]^{\frac{N_{\theta}}{2}} \neq 0.$$

B. The differentiation matrices in a circular annulus

The rotational symmetry of $A(R_0, R_1)$ enables us to use polar coordinates. In such case, there is an isomorphism between $A(R_0, R_1)$ and the rectangle $[R_0, R_1] \times [0, 2\pi]$. Hence, we need to take a linear transformation of the Chebyshev grid in the *r*-direction and a periodic Fourier grid in θ . The grid in the ρ -direction is obtained from the usual Chebyshev grid $r \in [-1, 1]$. So, the collocation points in the annulus are

$$(\rho_i, \theta_j) = \left(\frac{(R_1 - R_0)r_i + (R_1 + R_0)}{2}, \frac{2\pi j}{N_\theta}\right) \quad 1 \le i \le N_r + 1, \ 1 \le j \le N_\theta.$$

It should be remembered that $\rho_1 = R_1$ and $\rho_{N_r+1} = R_0$ correspond to the boundary points of the annulus. As a by-product, the discretization of the Laplace operator in polar coordinates in the annulus is the matrix of order $((N_r + 1)N_\theta) \times ((N_r + 1)N_\theta)$ defined by

$$L_{\Delta} = (p^2 D_r^2 + p R D_r) \otimes I + R^2 \otimes D_{\theta}^{(2)}$$

where $p = \frac{2}{R_1 - R_0}$, *I* stands for the $N_{\theta} \times N_{\theta}$ identity matrix, *R* is the diagonal matrix with entries $R_{i\,i} = \frac{1}{\rho_i}$ for $i = 1, \ldots, N_r + 1$, and D_r is the full Chebyshev differentiation matrix

$$(D_r)_{i,j} = L'_i(r_i); \quad 1 \le i, j \le N_r + 1.$$
 (32)

Note that in this case we are not discarding any blocks of D_r because we need to consider exactly r in the closed interval [-1, 1]. Before imposing the boundary conditions on L_{Δ} , we set

$$u_0 = (u(\rho_1, \theta_1), \dots, u(\rho_1, \theta_{N_\theta}))^T,$$

$$\tilde{u} = (u(\rho_2, \theta_1), \dots, u(\rho_2, \theta_{N_\theta}), \dots, u(\rho_{N_r}, \theta_1), \dots, u(\rho_{N_r}, \theta_{N_\theta}))^T,$$

$$u_1 = (u(\rho_{N_r+1}, \theta_1), \dots, u(\rho_{N_r+1}, \theta_{N_\theta}))^T.$$

So, u is factorized as $(u_0, \tilde{u}, u_1)^T$.

B.1. Homogeneous Dirichlet conditions $(u = 0 \text{ on } \partial A(R_0, R_1))$

First, set $w := L_{\Delta} u$. Next, we factorize w in the same way as u, so that $w = (w_0, \tilde{w}, w_1)^T$ with $w_0, w_1 \in \mathbb{R}^{N_{\theta}}$ and $\tilde{w} \in \mathbb{R}^{N_{\theta}(N_r-1)}$. Then, the procedure scheme adopted here to impose the homogeneous Dirichlet conditions on L_{Δ} consists in fixing the vectors u_0 and u_1 at zero, and ignoring w_0 and w_1 because, as already mentioned above, the Laplacian is computed in the interior of domain where the differential equation holds. This implies that the N_{θ} -first and N_{θ} -last columns of L_{Δ} have no computational effects, because they correspond to the discretization of the Laplacian at points along the boundary. Accordingly, the discretization matrix for the Laplacian is the matrix \tilde{L} obtained by stripping L_{Δ} of its N_{θ} -first and N_{θ} -last rows and columns.

$$L_{\Delta} = \left(\begin{array}{c} \tilde{L} \end{array} \right).$$

B.2. Inhomogeneous Dirichlet conditions $(u = f \neq 0 \text{ on } \partial A(R_0, R_1))$

We consider w as in the previous subsection. In the present situation, to impose the inhomogeneous Dirichlet condition on L_{Δ} we first fix u_0 and u_1 at the vectors f_{N_r+1} and f_1 , respectively, where $(f_i)_j = f(\rho_i, \theta_j)$ with $i \in \{1, N_r + 1\}$ fixed and $j = 1, \ldots, N_{\theta}$, and we ignore w_0 and w_1 . So, the N_{θ} -first and N_{θ} -last rows have no effects and they can be ignored. Accordingly, the matrix L_{Δ} is split into the three blocks

$$L_{\Delta} = \left(\begin{array}{c} \hline \cdots \\ \hline \overline{L} \\ \hline \\ \hline \end{array} \right)$$

where \overline{L} is the matrix obtained by stripping L_{Δ} of its N_{θ} -first and N_{θ} -last rows. Consequently, we can discard the top and bottom blocks of L_{Δ} . Next, we split \overline{L} into another three blocks, as follows

$$\overline{L} = \left(\begin{array}{c|c|c} L_1 & L_2 & L_3 \end{array} \right), \tag{33}$$

where

• L_1 is the matrix formed by the first N_{θ} columns of \overline{L} .

- L_2 is obtained by stripping \overline{L} of its N_{θ} -first and N_{θ} -last columns.
- L_3 is the matrix formed by the last N_{θ} columns of \overline{L} .

Note that, owing to (33), we have

$$\overline{L}u = L_1 u_0 + L_2 \tilde{u} + L_3 u_1.$$
(34)

Naturally, L_2 is the discrete matrix of the Laplacian in polar coordinates on the inner collocation points of the annulus.

B.3. Homogeneous Neumann conditions $\left(\frac{\partial u}{\partial \eta} = 0 \text{ on } \partial A(R_0, R_1)\right)$

In this case, we denote by E the corresponding discratization matrix of the first partial derivative with respect to r on the collocation points (ρ_i, θ_j) for $i = 1, \ldots, N_r + 1$ and $j = 1, \ldots, N_{\theta}$. That is,

$$E = p D_r \otimes I$$

where I is the identity matrix of dimension $N_{\theta} \times N_{\theta}$.

Now, to impose the Neumann boundary conditions on the collocation points contained in $\partial A(R_0, R_1)$, we are just interested into the portion of E that discretizes the first derivative on the inner and outer components of the boundary of the annulus. Accordingly, we introduce the matrices A and B as follows:

$$A = p F_1^{D_r} \otimes I$$

$$B = p F_{N_r+1}^{D_r} \otimes I$$
(35)

where $F_1^{D_r}$ and $F_{N_r+1}^{D_r}$ denote the first and last rows, respectively, of the matrix D_r . Note that, for any fixed $j = 1, \ldots, N_{\theta}$, the discrete partial derivative with respect to ρ at (ρ_1, θ_j) and (ρ_{N_r+1}, θ_j) corresponds to the j-th row of the matrices A and B, respectively. That is,

$$p \left. \frac{\partial}{\partial r} \right|_{(\rho_1, \theta_j)} = F_j^A \quad \text{for } j = 1, \dots, N_{\theta},$$
$$p \left. \frac{\partial}{\partial r} \right|_{(\rho_{N_r+1}, \theta_j)} = F_j^B \quad \text{for } j = 1, \dots, N_{\theta}.$$

Now, we divide both, A and B, in three blocks

$$A = \left(\begin{array}{c|c} A_1 & A_2 & A_3 \end{array} \right), \tag{36}$$

$$B = \left(\begin{array}{c|c} B_1 & B_2 & B_3 \end{array} \right), \tag{37}$$

where:

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- A_1 (resp. B_1) is obtained by stripping A (resp. B) of its $N_r N_{\theta}$ -last (resp. -first) columns.
- A_2 (resp. B_2) is obtained by stripping A (resp. B) of its N_{θ} -first (resp. -last) and N_{θ} -last (resp. -first) columns.
- A_3 (resp. B_3) is obtained by stripping A (resp. B) of its $N_r N_{\theta}$ -first (resp. -last) columns.

Imposing $0 = \frac{\partial u}{\partial r}(R_0, \theta) = \frac{\partial u}{\partial r}(R_1, \theta)$, yields

$$0 = A u = A_1 u_0 + A_2 \tilde{u} + A_3 u_1 \implies A_1 u_0 + A_3 u_1 = -A_2 \tilde{u}$$
$$0 = B u = B_1 u_0 + B_2 \tilde{u} + B_3 u_1 \implies B_1 u_0 + B_3 u_1 = -B_2 \tilde{u}$$

an solving the matricial system

$$\begin{cases} A_1 u_0 + A_3 u_1 = -A_2 \tilde{u} \\ B_1 u_0 + B_3 u_1 = -B_2 \tilde{u}, \end{cases}$$

we obtain

$$\begin{cases} u_0 = -A_1^{-1}(A_2 + A_3(B_3 - B_1A_1^{-1}A_3)^{-1})(B_1A_1^{-1}A_2 - B_2)\tilde{u} \\ u_1 = (B_3 - B_1A_1^{-1}A_3)^{-1}(B_1A_1^{-1}A_2 - B_2)\tilde{u}. \end{cases}$$

We claim that $(B_3 - B_1 A_1^{-1} A_3)$ is non-singular. Indeed, from (35), (36) and (37) it becomes apparent that

$$A_{1} = (D_{r})_{1 I}I,$$

$$A_{3} = (D_{r})_{1 N_{r}+1}I,$$

$$B_{1} = (D_{r})_{N_{r}+1 I}I,$$

$$B_{3} = (D_{r})_{N_{r}+1 N_{r}+1}I.$$

Using (32) and the coefficients of the Chebyshev differentiation matrix, it follows that

$$(D_r)_{1\,1} = -(D_r)_{N_r+1\,N_r+1} = \frac{2N_r^2 + 1}{6}$$

and that

$$(D_r)_{1 N_r+1} = -(D_r)_{N_r+1 1} = \frac{1}{2}(-1)^{N_r}$$

Therefore,

$$\det(B_3 - B_1 A_1^{-1} A_3) = \left[(D_r)_{N_r+1 N_r+1} - (D_r)_{N_r+1 1} ((D_r)_{1 1})^{-1} (D_r)_{1 N_r+1} \right]^{N_{\theta}} \neq 0.$$

Consequently, by substituting u_0 and u_1 in (34), we find the discretization matrix of the Laplacian. Namely,

$$\tilde{L} = -L_1 A_1^{-1} (A_2 + A_3 (B_3 - B_1 A_1^{-1} A_3)^{-1}) (B_1 A_1^{-1} A_2 - B_2) + L_2 + L_3 (B_3 - B_1 A_1^{-1} A_3)^{-1} (B_1 A_1^{-1} A_2 - B_2).$$

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