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Past and recent contributions to indefinite sublinear elliptic problems

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Dedicated to Professor J. López-Gómez on the occasion of his 60th birthday

Abstract. *We review the indefinite sublinear elliptic equation $-\Delta u = a(x)u^q$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, with Dirichlet or Neumann homogeneous boundary conditions. Here $0 < q < 1$ and a is continuous and changes sign, in which case the strong maximum principle does not apply. As a consequence, the set of nonnegative solutions of these problems has a rich structure, featuring in particular both dead core and/or positive solutions. Overall, we are interested in sufficient and necessary conditions on a and q for the existence of positive solutions. We describe the main results from the past decades, and combine it with our recent contributions. The proofs are briefly sketched.*

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1. Introduction

Let $N \geq 1$, $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, and Δ the usual Laplace operator. This article is devoted to the semilinear equation

$$-\Delta u = a(x)u^q \quad \text{in} \quad \Omega, \tag{1}$$

under the condition

$$(\mathbf{AQ}) \qquad a \text{ changes sign and } 0 < q < 1.$$

This is a prototype of *indefinite* (due to the change of sign of a) and sublinear (with respect to u) elliptic pde, which is motivated by the porous medium type equation [22, 44]

$$w_t = \Delta(w^m) + a(x)w, \quad m > 1,$$

after the change of variables $u = w^m$ and $q = 1/m$. Indefinite elliptic problems have attracted considerable attention since the 70's, mostly in the linear ($q = 1$)

and superlinear ($q > 1$) cases [2, 4, 7, 13, 18, 24, 37, 39, 40, 43]. We intend here to give an overview of the main results known in the *sublinear* case. For the sign-definite case $a \geq 0$ we refer to [3, 11, 10, 35, 36, 42].

We shall consider (1) under Dirichlet and Neumann homogeneous boundary conditions, i.e. the problems

$$(P_{\mathcal{D}}) \qquad \begin{cases} -\Delta u = a(x)u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(P_{\mathcal{N}}) \qquad \begin{cases} -\Delta u = a(x)u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ∂_ν is the exterior normal derivative.

Throughout this article, we assume that $a \in C(\overline{\Omega})$. By a *solution* of $(P_{\mathcal{D}})$ we mean a *strong* solution $u \in W^{2,r}_{\mathcal{D}}(\Omega)$ for some $r > N$, where

$$W^{2,r}_{\mathcal{D}}(\Omega) := \{u \in W^{2,r}(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

Note that $u \in C^1(\overline{\Omega})$, and so the boundary condition is satisfied in the usual sense. A similar definition holds for $(P_{\mathcal{N}})$. We say that a solution u is *nontrivial* if $u \not\equiv 0$, and *positive* if $u > 0$ in Ω. Among positive solutions of $(P_{\mathcal{D}})$, we are interested in *strongly positive* solutions (denoted by $u \gg 0$), namely, solutions in

$$\mathcal{P}^\circ_{\mathcal{D}} := \left\{u \in C^1_0(\overline{\Omega}) : u(x) > 0 \text{ in } \Omega, \text{ and } \partial_\nu u(x) < 0 \text{ on } \partial\Omega\right\}.$$

For $(P_{\mathcal{N}})$, a solution is *strongly positive* if it belongs to

$$\mathcal{P}^\circ_{\mathcal{N}} := \left\{u \in C^1(\overline{\Omega}) : u(x) > 0 \text{ on } \overline{\Omega}\right\}.$$

In case that *every* nontrivial solution of $(P_{\mathcal{D}})$ (respect. $(P_{\mathcal{N}})$) is *strongly positive* we say that this problem has the *positivity property.*

The condition (AQ) gives rise to the main feature of this class of problems, namely, the fact that the strong maximum principle (shortly **SMP**) does *not* apply. Let us recall the following version of this result (for a proof, see e.g. [38, Theorem 7.10]):

Strong maximum principle: Let $u \in W^{2,r}(\Omega)$ for some $r > N$ be such that $u \geq 0$ and $(-\Delta + M)u \geq 0$ in Ω, for some constant $M \geq 0$. Then either $u \equiv 0$ or $u > 0$ in Ω and $\partial_\nu u(x) < 0$ for any $x \in \partial\Omega$ such that $u(x) = 0$.

Given u satisfying (1), we see that under (AQ) we can't find in general some $M > 0$ such that $(-\Delta + M)u = a(x)u^q + Mu \geq 0$ in Ω, which prevents us to apply the **SMP**, unlike when $a \geq 0$ (the *definite* case) or $q \geq 1$ (the *linear* and *superlinear* cases). This fact is reinforced by a simple example of a nontrivial solution u (of both $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$) violating the conclusion of the **SMP** (see Example C below), which shows that the positivity property may fail. Moreover, such example also provides us with nontrivial *dead core* solutions of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$, i.e. solutions vanishing in some open subset of Ω. The formation of dead cores has already been investigated by Diaz in [16, Proposition 1.11] for a more general class of problems.

To the best of our knowledge, the study of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ in the indefinite and sublinear case was launched in the late 80's by Bandle, Pozio and Tesei [5, 6, 41]. These works were then followed by the contributions of Hernández, Mancebo and Vega [23], Delgado and Suarez [15], and Godoy and Kaufmann [20, 21]. We shall review the main results of these papers in the next section and complement it with our main recent results from [27, 28, 30] in the subsequent sections. Since the proofs can be found in the aforementioned articles, in most cases we shall only sketch them here.

2. First results

Let us recall the first existence and uniqueness results for the problems above. For the Neumann problem, the following condition on a plays an important role:

$$(\mathbf{A.0}) \qquad \int_{\Omega} a < 0.$$

Indeed, we shall see that (A.0) is *necessary* for the existence of a positive solution of $(P_{\mathcal{N}})$, and *sufficient* for the existence of a nontrivial solution, for any $q \in (0, 1)$. As for the uniqueness results, some merely technical conditions (see also the beginning of Section 7) on the set

$$\Omega_+ := \{x \in \Omega : a(x) > 0\}$$

shall be used, namely:

$$(\mathbf{A.1}) \qquad \Omega_+ \text{ has } \textit{finitely} \text{ many connected components,}$$

$$(\mathbf{A.2}) \qquad \partial\Omega_+ \text{ satisfies an inner sphere condition with respect to } \Omega_+.$$

The following results were proved by Bandle, Pozio and Tesei [5, 6], and Delgado and Suárez [15]. Although [5, 6] require that $a \in C^\theta(\overline{\Omega})$ for some $0 < \theta < 1$, one can easily see from the proofs that these results still hold for strong solutions assuming that $a \in C(\overline{\Omega})$.

THEOREM A. *(i) The Dirichlet case:*

(a) $(P_{\mathcal{D}})$ *has at most one positive solution [15, Theorem 2.1]. Moreover, if (A.1) and (A.2) hold then* $(P_{\mathcal{D}})$ *has at most one solution positive in* Ω_+ *[5, Theorem 2.1].*

(b) $(P_{\mathcal{D}})$ *has at least one nontrivial solution [5, Theorem 2.2].*

(ii) The Neumann case:

(a) $(P_{\mathcal{N}})$ *has at most one solution in* $\mathcal{P}_{\mathcal{N}}^{\circ}$ *[6, Lemma 3.1]. Moreover, if (A.1) and (A.2) hold then* $(P_{\mathcal{N}})$ *has at most one solution positive in* Ω_+ *[6, Theorem 3.1].*

(b) If (A.0) holds then $(P_{\mathcal{N}})$ *has at least one nontrivial solution. Conversely, if* $(P_{\mathcal{N}})$ *has a positive solution then (A.0) holds [6, Theorem 2.1].*

Sketch of the proof. The uniqueness assertions rely on the following change of variables: if $u > 0$ and $-\Delta u = a(x)u^q$ in Ω then $v := (1-q)^{-1}u^{1-q}$ solves $-\Delta v = qu^{q-1}|\nabla v|^2 + a(x)$ in Ω. Let u_1, u_2 be positive solutions of (P_D) and assume that $\tilde{\Omega} := \{x \in \Omega : u_1(x) > u_2(x)\}$ is nonempty. We set $v_i := (1-q)^{-1}u_i^{1-q}$ for $i = 1, 2$, so that $\Phi := v_1 - v_2 > 0$ in $\tilde{\Omega}$. In addition,

$$-\Delta\Phi = q\left(u_1^{q-1}|\nabla v_1|^2 - u_2^{q-1}|\nabla v_2|^2\right) < qu_1^{q-1}\left(|\nabla v_1|^2 - |\nabla v_2|^2\right),$$

i.e.

$$-\Delta\Phi - qu_1^{q-1}\nabla(v_1 + v_2)\nabla\Phi < 0 \quad \text{in} \quad \tilde{\Omega}. \tag{2}$$

Since $\Phi = 0$ on $\partial\tilde{\Omega}$, we obtain a contradiction with the maximum principle. This shows that $(P_{\mathcal{D}})$ has at most one positive solution. Now, if $u_1, u_2 \in \mathcal{P}_{\mathcal{N}}^{\circ}$ solve $(P_{\mathcal{N}})$ then Φ satisfies (2) and for any $x \in \partial\tilde{\Omega}$ we have either $\Phi(x) = 0$ or $\partial_\nu\Phi(x) = 0$. By the maximum principle, we infer that Φ is constant in $\tilde{\Omega}$, which contradicts (2). The proof of the uniqueness of a solution of $(P_{\mathcal{D}})$ positive in Ω_+ (respect. a solution of $(P_{\mathcal{N}})$ in $\mathcal{P}_{\mathcal{N}}^{\circ}$) uses the same change of variables, but is more involved. We refer to [5, 6] for the details.

The existence results can be proved either by a variational argument or by the sub-supersolutions method. In the first case, it suffices to show that the functional

$$I_q(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{q+1}a(x)|u|^{q+1}\right)$$

has a negative global minimum in $H_0^1(\Omega)$ or $H^1(\Omega)$. In the latter case the condition (A.0) is crucial. The second approach consists in taking a ball $B \subset \Omega_+$ and a sufficiently small first positive eigenfunction of $-\Delta$ on $H_0^1(B)$ extended by zero to Ω, to find a (nontrivial) subsolution of both $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$. An arbitrary large supersolution of $(P_{\mathcal{D}})$ is given by kz, where z is the unique solution of $-\Delta z = a^+$ in Ω, $z = 0$ on $\partial\Omega$, and $k > 0$ is large enough (as usual, we write $a = a^+ - a^-$, with $a^\pm := \max(\pm a, 0)$). The construction of a suitable supersolution of $(P_{\mathcal{N}})$ under (A.0) is more delicate, and we refer to [6] for the details.

Finally, if $(P_{\mathcal{N}})$ has a positive solution u then, multiplying the equation by $(u+\varepsilon)^{-q}$ (with $0 < \varepsilon < 1$) and integrating by parts, we find that

$$\int_\Omega a \left(\frac{u}{u+\varepsilon}\right)^q = -q \int_\Omega (u+\varepsilon)^{-(q+1)} |\nabla u|^2 < -q \int_\Omega (u+1)^{-(q+1)} |\nabla u|^2 < 0.$$

Letting $\varepsilon \to 0$ we can check that $\int_\Omega a < 0$. $\square$

Although not stated explicitly in [5, 6], the next corollary follows almost directly from the existence and uniqueness results in these papers.

Corollary B. *Let Ω_+ be connected and satisfy (A.2). Then $(P_{\mathcal{D}})$ has a unique nontrivial solution. The same conclusion holds for $(P_{\mathcal{N}})$ assuming in addition (A.0).*

Sketch of the proof. It is based on the fact that a nontrivial solution u of $(P_{\mathcal{D}})$ or $(P_{\mathcal{N}})$ satisfies $u \not\equiv 0$ in Ω_+, which follows from the inequality $0 < \int_\Omega |\nabla u|^2 \leq \int_\Omega a^+(x) u^{q+1}$. Since Ω_+ is connected, by the maximum principle we find that $u > 0$ in Ω_+. And there is only one solution having this property, by Theorem A. $\square$

Remark 2.1: (i) Let us remark that the nontrivial solutions provided by Theorem A (i-b) and (ii-b) are *not* necessarily unique, see e.g. [5, 6].

(ii) Regarding Theorem A (ii-b), it is worth pointing out that (A.0) is *not* necessary for the existence of a nontrivial solution of $(P_{\mathcal{N}})$ for some $q \in (0, 1)$, cf. [6, Section 4] and [28, Remark 4.3].

Let us now give an example of a nontrivial solution $u \not\gg 0$ of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$. It is essentially due to [20], where the case $q = \frac{1}{2}$ was considered (see Figure 1).

Example C. Let $\Omega := (0, \pi)$ and $q \in (0, 1)$. We choose

$$r = r_q := \frac{2}{1-q} \in (2, \infty), \quad a(x) = a_q(x) := r^{1-\frac{2}{r}} \left(1 - r\cos^2 x\right) \quad \text{for } x \in \overline{\Omega}.$$

Then $u(x) := \frac{\sin^r x}{r} \in C^2(\overline{\Omega})$ satisfies

$$\begin{cases} -u'' = a(x)u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = u' = u'' = 0 & \text{on } \partial\Omega. \end{cases}$$

The above example also provides *dead core* solutions of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ both. Indeed, it suffices to consider any bounded open interval Ω' with $\Omega' \supset \overline{\Omega}$, and extend u by zero and a in any way to Ω'. Then u is a nontrivial dead core solution of both $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$, considered now in Ω'.

Since the **SMP** does not apply and *dead core* solutions may exist, obtaining a positive solution for these problems is a delicate issue which has been given little consideration. Let $\varphi \in W^{2,r}_{\mathcal{D}}(\Omega)$ be the unique solution of the Poisson equation

$$\begin{cases} -\Delta\varphi = a(x) & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\mathcal{S} : L^r(\Omega) \to W^{2,r}_{\mathcal{D}}(\Omega)$ be the corresponding *solution operator*, i.e. $\mathcal{S}(a) = \varphi$. In [23] Hernández, Mancebo and Vega showed that the condition

$$(\mathbf{A.3}) \qquad \mathcal{S}(a) \gg 0$$

implies the existence of a positive solution of $(P_{\mathcal{D}})$ for all $q \in (0,1)$. Later on Godoy and Kaufmann [20, 21] provided other sufficient conditions, namely, that a^- is small enough, or q is close enough to 1 (for some particular choices of N, Ω, and a). We shall state a simplified version of these results in the sequel, and refer to [23, Theorem 4.4], [21, Theorems 3.1 and 3.2], and [20, Theorems 2.1 (i) and 3.2] for the precise statements.

Theorem D. *(i) If a satisfies (A.3) then $(P_{\mathcal{D}})$ has a positive solution for every $q \in (0,1)$.*

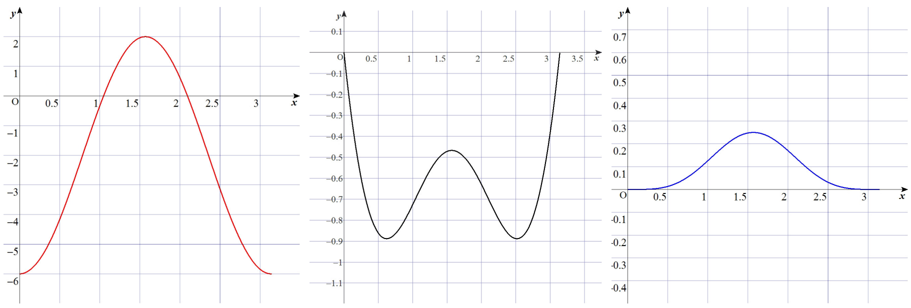
(ii) Let q and a^+ be fixed. Then there exists a constant $C > 0$ such that $(P_{\mathcal{D}})$ has a positive solution if $\|a^-\|_{C(\overline{\Omega})} < C$.

(iii) If either $N = 1$ or Ω is a ball, a is radial, and $0 \not\equiv a \geq 0$ in some smaller ball, then there exists $\overline{q} = \overline{q}(a)$ such that $(P_{\mathcal{D}})$ has a positive solution for $\overline{q} < q < 1$.

Remark 2.2: Let us mention that Theorem D (i) is still true for a linear second order elliptic operator with nonnegative zero order coefficient. On the other side, it may happen that $\mathcal{S}(a) < 0$ *everywhere* in Ω and yet $(P_{\mathcal{D}})$ admits a positive solution for some $q \in (0,1)$. Indeed, if we take $q = \frac{1}{2}$ in Example C then $\mathcal{S}(a_q) = x^2 - \pi x + 1 - \cos 2x < 0$ in $(0,\pi)$, see Figure 1 (ii). Note also that (A.3) is *not* compatible with the existence of a positive solution for $(P_{\mathcal{N}})$, since it implies $\int_\Omega a > 0$, contradicting (A.0), which is necessary by Theorem A (ii-b).

Sketch of the proof. All assertions follow by the well known sub-supersolutions method. Let us note that (*unlike* for $(P_{\mathcal{N}})$) it is easy to provide arbitrary big supersolutions for $(P_{\mathcal{D}})$. Indeed, a few computations show that $k\mathcal{S}(a^+)$ is a supersolution of $(P_{\mathcal{D}})$ for all $k > 0$ large enough. So the only task is to provide a *positive* subsolution. In (i), after some computations one can check that $[(1-q)\,\mathcal{S}\,(a)]^{1/(1-q)}$ is the desired subsolution.

In both (ii) and (iii), the subsolution is constructed by splitting the domain in two parts (a ball B in which $0 \not\equiv a \geq 0$, and $\Omega \setminus B$), constructing "subsolutions" in each of them, and checking that they can be glued appropriately to get a subsolution in the entire domain (see [8]). This fact depends on obtaining estimates for the normal derivatives of these subsolutions on the boundaries of the subdomains. In (iii) these bounds can be computed rather explicitly using the symmetry of a and the fact that Ω is a ball, while in (ii) the key tool is an estimate due to Brezis and Cabré [9, Lemma 3.2]. The proof of both (ii) and (iii) involve several computations, and we refer to [20, 21] for the details. $\square$



(i) (ii) (iii)

Figure 1: (i) The indefinite weight $a_{\frac{1}{2}}$; (ii) $\mathcal{S}(a_{\frac{1}{2}})$; (iii) The positive solution $u \ggg 0$ for $a_{\frac{1}{2}}$.

Godoy and Kaufmann [21] also proved that when a is too negative in a ball there are no positive solutions of $(P_{\mathcal{D}})$ (see also Remark 2.3 (i) below). This result, which has been proved in [16, Proposition 1.11] in a more general setting, can also be seen as a first step towards the construction of dead core solutions.

Theorem E. *Let q and a^+ be fixed. Given a ball $B = B_R(x_0) \subset \Omega \setminus \Omega_+$ there exists a constant $C = C(\Omega, N, q, R, a^+) > 0$ such that any solution of $(P_{\mathcal{D}})$ vanishes at x_0 if $\min_{\overline{B}} a^- > C$.*

Sketch of the proof. We use a comparison argument: let u be a nontrivial solution of $(P_{\mathcal{D}})$, and $\underline{a} := \min_{\overline{B}} a^-$. Set

$$C_{N,q} := \frac{(1-q)^2}{2\,(N\,(1-q)+2q)} \quad \text{and} \quad w(x) := \left(C_{N,q}\underline{a}\,|x-x_0|^2\right)^{\frac{1}{1-q}}.$$

One can check that $\Delta w \leq a^- w^q$ in B. On the other hand, note that $\Delta u = a^- u^q$ in B and $\|u\|_\infty \leq (\|\mathcal{S}\| \|a^+\|_\infty)^{\frac{1}{1-q}}$, and so $u \leq w$ on ∂B if

$$\underline{a} \geq \frac{\|\mathcal{S}\| \|a^+\|_\infty}{R^2 C_{N,q}}. \tag{3}$$

It follows then from the comparison principle that $u \leq w$ in B. In particular, $u(x_0) = 0$. □

REMARK 2.3: (i) The latter proof can be adapted for the Neumann problem, taking into account the following *a priori* bound: Under (A.0), there exists $C > 0$ (independent of a^-) such that $\|u\|_{C(\overline{\Omega})} \leq C$ for every subsolution of $(P_{\mathcal{N}})$.

(ii) Note that $C_{N,q} \to 0$ as $q \to 1^-$, i.e. the closer is q to 1, the larger is the right-hand side in (3), and the more negative a needs to be in $B_R(x_0)$ to satisfy (3). This fact is consistent with Theorem D (ii) and (iii).

3. Recent results

Let us now briefly describe our main contributions to the study of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$, which can be found in [27, 28, 30]:

(I) We determine the values of $q \in (0,1)$ for which $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ have the positivity property. In other words, we provide a characterization of the following *positivity sets*:

$$\mathcal{A}_{\mathcal{D}} = \mathcal{A}_{\mathcal{D}}(a) := \{q \in (0,1) : \ u \gg 0 \text{ for any nontrivial solution } u \text{ of } (P_{\mathcal{D}})\},$$
$$\mathcal{A}_{\mathcal{N}} = \mathcal{A}_{\mathcal{N}}(a) := \{q \in (0,1) : \ u \gg 0 \text{ for any nontrivial solution } u \text{ of } (P_{\mathcal{N}})\}.$$

Thanks to a continuity argument inspired by Jeanjean [25], and based on the fact that the **SMP** applies when $q = 1$, we shall see in Theorem 4.1 that under (A.1) we have $\mathcal{A}_{\mathcal{D}} = (q_{\mathcal{D}}, 1)$ and, assuming additionally (A.0), $\mathcal{A}_{\mathcal{N}} = (q_{\mathcal{N}}, 1)$, for some $q_{\mathcal{D}}, q_{\mathcal{N}} \in [0,1)$ (see also Corollary 4.3 and Theorem 4.2).

Note that in view of the existence and uniqueness results in Theorem A, the sets $\mathcal{A}_{\mathcal{D}}$ and $\mathcal{A}_{\mathcal{N}}$ can also be expressed as follows:

$$\mathcal{A}_{\mathcal{D}} = \{q \in (0,1) : (P_{\mathcal{D}}) \text{ has a unique nontrivial solution } u, \text{ and } u \gg 0\},$$
$$\mathcal{A}_{\mathcal{N}} = \{q \in (0,1) : (P_{\mathcal{N}}) \text{ has a unique nontrivial solution } u, \text{ and } u \gg 0\}.$$

We also obtain some positivity properties for the *ground state* solution of $(P_{\mathcal{D}})$.

(II) By the previous discussion we deduce that $(P_{\mathcal{D}})$ (respect. $(P_{\mathcal{N}})$, under (A.0)) has a solution $u \gg 0$ for $q \in \mathcal{A}_{\mathcal{D}}$ (respect. $q \in \mathcal{A}_{\mathcal{N}}$). Thus, setting

$$\mathcal{I}_{\mathcal{D}} = \mathcal{I}_{\mathcal{D}}(a) := \{q \in (0,1) : (P_{\mathcal{D}}) \text{ has a solution } u \gg 0\},$$
$$\mathcal{I}_{\mathcal{N}} = \mathcal{I}_{\mathcal{N}}(a) := \{q \in (0,1) : (P_{\mathcal{N}}) \text{ has a solution } u \gg 0\},$$

we observe that $\mathcal{A}_{\mathcal{D}} \subseteq \mathcal{I}_{\mathcal{D}}$ and $\mathcal{A}_{\mathcal{N}} \subseteq \mathcal{I}_{\mathcal{N}}$. We will further investigate $\mathcal{I}_{\mathcal{D}}$ (respect. $\mathcal{I}_{\mathcal{N}}$) and analyze how close $\mathcal{A}_{\mathcal{D}}$ and $\mathcal{I}_{\mathcal{D}}$ (respect. $\mathcal{A}_{\mathcal{N}}$ and $\mathcal{I}_{\mathcal{N}}$) can be to each other, see Theorems 5.3, 5.8, and also Proposition 4.4 and Remark 5.2 (i).

Note that Corollary B tells us that if Ω_+ is connected and satisfies (A.2), then $\mathcal{A}_{\mathcal{D}} = \mathcal{I}_{\mathcal{D}}$, and if additionally (A.0) holds, then $\mathcal{A}_{\mathcal{N}} = \mathcal{I}_{\mathcal{N}}$. Assuming moreover (A.3), we find by Theorem 5.1 (iv-c) that $\mathcal{A}_{\mathcal{D}} = (0,1)$.

(III) We consider $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ via a bifurcation approach, looking at q as a bifurcation parameter and taking advantage of the fact that $(P_{\mathcal{D}})$ has a trivial line of strongly positive solutions when $q = 1$, see Theorems 5.1 and 5.5 for $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$, respectively. We also analyze the structure of the nontrivial solutions set (with respect to q) of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ via variational methods and the construction of sub and supersolutions, see Theorem 5.1 and Remark 5.9 for $(P_{\mathcal{D}})$; Remarks 5.7 and 5.9 for $(P_{\mathcal{N}})$. In particular, we describe the asymptotic behaviors of nontrivial solutions as $q \to 0^+$ and $q \to 1^-$.

(IV) Finally, in Section 6 we present, without proofs, two further kind of results. On the one hand, we provide *explicit* sufficient conditions for the existence of positive solutions for $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$, see Theorems 6.1 and 6.2. And on the other hand, in Theorem 6.3 we state sufficient conditions for the existence of dead core solutions for $(P_{\mathcal{N}})$.

The above issues will be developed in the forthcoming sections. In the last section we include some final remarks and list some open questions.

4. The positivity property

The next theorem extends Theorem D (iii) under (A.1), showing that $(P_{\mathcal{D}})$, as well as $(P_{\mathcal{N}})$ under $(A.0)$, has a positive solution (and no other nontrivial solution) if q is close enough to 1. In other words, we show that under (A.1) the positivity property holds for such values of q [27, Theorems 1.3 and 1.7]:

THEOREM 4.1. *Assume (A.1). Then:*

(i) $\mathcal{A}_{\mathcal{D}} = (q_{\mathcal{D}}, 1)$ *for some* $q_{\mathcal{D}} \in [0,1)$.

(ii) If (A.0) holds then $\mathcal{A}_{\mathcal{N}} = (q_{\mathcal{N}}, 1)$ *for some* $q_{\mathcal{N}} \in [0, 1)$.

Sketch of the proof. First we show that $\mathcal{A}_{\mathcal{D}}$ is nonempty. We proceed by contradiction, assuming that $q_n \to 1^-$ and u_n are nontrivial solutions of $(P_{\mathcal{D}})$ with $q = q_n$ and $u_n \not\gg 0$. We know that $u_n \not\equiv 0$ in Ω_+, and thanks to (A.1) we can assume that, for every $n \in \mathbb{N}$, $u_n > 0$ in some fixed connected component of Ω_+. If $\{u_n\}$ is bounded in $H_0^1(\Omega)$ then, by standard compactness arguments, up to a subsequence, we have $u_n \to u_0$ in $H_0^1(\Omega)$ and u_0 solves $-\Delta u_0 = a(x)u_0$. Moreover, we can show that $\{u_n\}$ is away from zero, so that $u_0 \not\equiv 0$. By the **SMP** we get that $u_0 \gg 0$. Finally, by standard elliptic regularity, we find that $u_n \to u_0$ in $C^1(\overline{\Omega})$, up to a subsequence. Thus $u_n \gg 0$ for n large enough, and we have a contradiction. If $\{u_n\}$ is unbounded in $H_0^1(\Omega)$ then, normalizing it, we obtain a sequence v_n converging to some $v_0 \not\equiv 0$ that solves an eigenvalue problem. Once again, the **SMP** implies that $v_0 \gg 0$, a contradiction. A similar argument shows that $\mathcal{A}_{\mathcal{D}}$ is open. Indeed, assume to the contrary that there exist $q_0 \in \mathcal{A}_{\mathcal{D}}$ and $q_n \notin \mathcal{A}_{\mathcal{D}}$ such that $q_n \to q_0$. We take nontrivial solutions $u_n \not\gg 0$ of $(P_{\mathcal{D}})$ with $q = q_n$. It is easily seen that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Up to a subsequence, $u_n \to u_0$ in $C^1(\overline{\Omega})$, where u_0 is a nontrivial solution of $(P_{\mathcal{D}})$ with $q = q_0$. Since $q_0 \in \mathcal{A}_{\mathcal{D}}$, we have $u_0 \gg 0$, and so $u_n \gg 0$ for n large enough, which is a contradiction. Thus $\mathcal{A}_{\mathcal{D}}$ is open. The proof of the connectedness of $\mathcal{A}_{\mathcal{D}}$ is more technical, and we refer to [27] for the details. The proof of (ii) follows similarly, see also [27]. $\square$

Following a similar strategy, we show that the positivity property also holds in the Dirichlet case if a^- is small enough (assuming now that $q \in (0, 1)$ is fixed), which extends Theorem D (ii) under (A.1). Let us add that this theorem is also true for some non-powerlike nonlinearities [27, Theorem 1.1].

THEOREM 4.2. *Assume (A.1). Then there exists* $\delta > 0$ *(possibly depending on* q *and* a^+*) such that every nontrivial nonnegative solution* u *of* $(P_{\mathcal{D}})$ *satisfies that* $u \gg 0$ *if* $\|a^-\|_{C(\overline{\Omega})} < \delta$.

Note that since (A.0) is necessary for the existence of positive solutions of $(P_{\mathcal{N}})$, we can't expect an analogue of the above theorem for this problem.

As an immediate consequence of Theorem 4.1 and Corollary B, we infer:

COROLLARY 4.3. *Assume that* Ω_+ *is connected and satisfies (A.2), and let* u_q *be the unique nontrivial solution of* $(P_{\mathcal{D}})$. *Then* $u_q \not\gg 0$ *for all* $q \in (0, q_{\mathcal{D}}]$ *and* $u_q \gg 0$ *for all* $q \in (q_{\mathcal{D}}, 1)$. *A similar result holds for* $(P_{\mathcal{N}})$ *assuming, in addition, (A.0).*

Let us mention that, if in addition to the assumptions of Corollary 4.3, Ω_+ includes a *tubular neighborhood of* $\partial\Omega$ (i.e., a set of the form $\{x \in \Omega : d(x, \partial\Omega) < \rho\}$, for some $\rho > 0$) then the **SMP** shows that the solution u_q above satisfies either $u_q \gg 0$ or $u_q = 0$ somewhere in Ω, see Figure 4.

Although Theorem 4.1 claims that under (A.1) the sets $\mathcal{A}_{\mathcal{D}}$ and $\mathcal{A}_{\mathcal{N}}$ are *always* nonempty, by Example C we see that given *any* $q \in (0,1)$, we may find $a = a_q$ satisfying (A.1) and such that $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ have a nontrivial solution $u \not\gg 0$. In view of Theorem 4.1, this fact shows that $\mathcal{A}_{\mathcal{D}}$ and $\mathcal{A}_{\mathcal{N}}$ can be arbitrarily small for a suitable a.

The next result (cf. [28, Theorem 1.4 (i)], [30, Proposition 5.1 (i)]) shows that for any $q \in (0,1)$, we may find a such that $q \in \mathcal{I}(a) \setminus \mathcal{A}(a)$ (and so, in general, $\mathcal{A} \subsetneq \mathcal{I}$).

PROPOSITION 4.4. *(i) Given $\Omega \subset \mathbb{R}$ and $q \in (0,1)$, there exists $a \in C(\overline{\Omega})$ such that $q \in \mathcal{I}_{\mathcal{N}} \setminus \mathcal{A}_{\mathcal{N}}$.*

(ii) Given $\Omega \subset \mathbb{R}$ and $q \in (0,1)$, there exists $a \in C(\Omega) \cap L^r(\Omega)$, $r > 1$, such that $q \in \mathcal{I}_{\mathcal{D}} \setminus \mathcal{A}_{\mathcal{D}}$.

4.1. The ground state solution

Recall that the Dirichlet eigenvalue problem

$$(E_{\mathcal{D}}) \qquad \begin{cases} -\Delta\phi = \mu a(x)\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

has a first positive eigenvalue $\mu_{\mathcal{D}}(a)$, which is principal and simple, and a positive eigenfunction $\phi_{\mathcal{D}}(a) \gg 0$ associated with $\mu_{\mathcal{D}}(a)$ and normalized by $\int_\Omega \phi_{\mathcal{D}}^2 = 1$.

Let $I_q : H_0^1(\Omega) \to \mathbb{R}$ be given by

$$I_q(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{q+1}\int_\Omega a(x)|u|^{q+1}$$

for $q \in [0,1)$. It is well-known that nonnegative critical points (in particular minimizers) of I_q are solutions of $(P_{\mathcal{D}})$. By a *ground state* of I_q we mean a global minimizer of this functional.

PROPOSITION 4.5. *I_q has a unique nonnegative ground state U_q for every $q \in (0,1)$. In addition:*

(i) $U_q > 0$ in Ω_+ and $q \mapsto U_q$ is continuous from $(0,1)$ to $W_{\mathcal{D}}^{2,r}(\Omega)$.

(ii) There exists $q_0 \in (0,1)$ such that $U_q \gg 0$ for $q \in (q_0,1)$.

(iii) As $q \to 1^-$ we have $U_q \to 0$ in $C_0^1(\overline{\Omega})$ if $\mu_{\mathcal{D}}(a) > 1$, whereas $\|U_q\|_{C(\overline{\Omega})} \to \infty$ if $\mu_{\mathcal{D}}(a) < 1$.

(iv) If $q_n \to 0^+$ then, up to a subsequence, $U_{q_n} \to U_0$ in $C_0^1(\overline{\Omega})$, where U_0 is a nonnegative global minimizer of I_0. In particular, if $0 \not\equiv \mathcal{S}(a) \geq 0$ in Ω, then $U_q \to \mathcal{S}(a)$ in $C_0^1(\overline{\Omega})$ as $q \to 0^+$.

Sketch of the proof. By a standard minimization argument, one may easily prove the existence of a global minimizer of I_q. Moreover, there is a 1 to 1 correspondence between global minimizers of I_q and minimizers of $\int_\Omega |\nabla u|^2$ over the C^1 manifold $\left\{u \in H_0^1(\Omega) : \int_\Omega a(x)|u|^{q+1} = 1\right\}$. By [34, Theorem 1.1], we infer that if U_q and V_q are global minimizers of I_q then $U_q = tV_q$ for some $t > 0$. But since U_q and V_q solve $(P_{\mathcal{D}})$, we deduce that $t = 1$, i.e. U_q is the unique nonnegative global minimizer of I_q. If $U_q(x) = 0$ for some $x \in \Omega_+$ then, by the **SMP**, U_q vanishes is some ball $B \subset \Omega_+$. We choose a nontrivial and smooth $\psi \geq 0$ supported in B and extend it by zero to Ω. Then $I_q(U_q + t\psi) = I_q(U_q) + I_q(t\psi) < I_q(U_q)$ if t is small enough, which yields a contradiction. Using standard compactness arguments and the uniqueness of U_q, we can show that $U_q \to U_{q_0}$ in $W_{\mathcal{D}}^{2,r}(\Omega)$ as $q \to q_0$, for any $q_0 \in (0,1)$. Arguing as in the proof of Theorem 4.1 we prove that $U_q \gg 0$ for q close to 1, and $U_q \to 0$ in $C_0^1(\overline{\Omega})$ if $\mu_{\mathcal{D}}(a) > 1$. If $\mu_{\mathcal{D}}(a) < 1$ and $\{u_n\}$ is bounded in $H_0^1(\Omega)$, where $u_n := U_{q_n}$ and $q_n \to 1^-$, then again as in the proof of Theorem 4.1, we find that $u_n \to u_0$ and $u_0 \geq 0$ solves $-\Delta u_0 = a(x)u_0$ in Ω, $u_0 = 0$ on $\partial\Omega$. Using the fact that u_n are ground state solutions, we can show that $u_0 \not\equiv 0$, so that $\mu_{\mathcal{D}}(a) = 1$, a contradiction. Finally, we refer to [30] for the proof of (iv). □

Remark 4.6: (i) It is not hard to show that under (A.0) the functional I_q, considered now in $H^1(\Omega)$, has a ground state, which is positive in Ω_+, and strongly positive for q close enough to 1.

(ii) Proposition 4.5 (ii) extends Theorem D and Theorem 4.1(i) (as long as the existence of a positive solution is concerned) without assuming (A.1).

5. Structure of the positive solutions set

This section is devoted to a further investigation of the set $\mathcal{I}_{\mathcal{D}}$ (respect. $\mathcal{I}_{\mathcal{N}}$), which provides a rather complete description of the positive solutions set of $(P_{\mathcal{D}})$ (respect. $(P_{\mathcal{N}})$). From Theorem 4.1 we observe that $(q_{\mathcal{D}}, 1) \subseteq \mathcal{I}_{\mathcal{D}}$ and $(q_{\mathcal{N}}, 1) \subseteq \mathcal{I}_{\mathcal{N}}$. Taking advantage of the ground state solution, constructing suitable sub and supersolutions, and also using a bifurcation approach, we analyze the asymptotic behavior of the positive solutions as $q \to 1^-$ and $q \to 0^+$.

5.1. The Dirichlet problem

Let us consider $(P_{\mathcal{D}})$, with $q \in (0,1)$ as a bifurcation parameter. To this end, we introduce two further conditions on a. The first one slightly weakens (A.3) requiring that

$$(\mathbf{A.3'}) \qquad \mathcal{S}(a) > 0 \ \text{ in } \ \Omega,$$

whereas the second one is a technical decay condition near $\partial\Omega$:

$$(\mathbf{A.4}) \qquad |a(x)| \leq Cd(x,\partial\Omega)^{\eta} \text{ a.e. in } \Omega_{\rho_0}, \text{ for some } \rho_0 > 0 \text{ and } \eta > 1 - \frac{1}{N},$$

where

$$\Omega_\rho =: \{x \in \Omega : d(x,\partial\Omega) < \rho\} \tag{4}$$

is the tubular neighborhood of $\partial\Omega$. It turns out that (A.3′) is sufficient to deduce the conclusion of Theorem D (i), i.e. that $(P_\mathcal{D})$ has a positive solution for every $q \in (0,1)$. In addition, we shall use (A.3′) to show that this solution converges to $\mathcal{S}(a)$ as $q \to 0^+$. On the other hand, (A.4) is needed to obtain solutions of $(P_\mathcal{D})$ bifurcating from $t\phi_\mathcal{D}$, for some $t > 0$, when $\mu_\mathcal{D}(a) = 1$. Since $\phi_\mathcal{D} = 0$ on $\partial\Omega$, we assume (A.4) to ensure that $a\,\phi_\mathcal{D}^{q-2}$ has the appropriate integrability to carry out this bifurcation procedure, see Subsection 5.1.1.

Denoting by $u_\mathcal{D}(q)$ the unique positive solution of $(P_\mathcal{D})$ for $q \in (0,1)$ whenever it exists, we see from Proposition 4.5 (ii) that $U_q = u_\mathcal{D}(q)$ for q close to 1, so that Proposition 4.5 (iii) provides the asymptotics of $u_\mathcal{D}(q)$ when $\mu_\mathcal{D}(a) \neq 1$. We treat now the case $\mu_\mathcal{D}(a) = 1$ and also provide the asymptotic behavior of $u_\mathcal{D}(q)$ as $q \to 0^+$, as well as sufficient conditions to have $u_\mathcal{D}(q) \gg 0$ for *every* $q \in (0,1)$. Under these conditions, we obtain a rather complete description of the positive solutions set $\{(q, u_\mathcal{D}(q)) : q \in (0,1)\}$ of $(P_\mathcal{D})$, see Figure 2. We shall present here a simplified version of these results. For the precise assumptions required in each of following items we refer to [30, Theorems 1.2 and 1.4, Corollary 1.6]. Under (A.4), let us set

$$t_\mathcal{D}^* := \exp\left[-\frac{\int_\Omega a(x)\phi_\mathcal{D}^2 \log\phi_\mathcal{D}}{\int_\Omega a(x)\phi_\mathcal{D}^2}\right]. \tag{5}$$

Theorem 5.1. *Let $r > N$. Assume (A.1), (A.2), (A.3′) and (A.4). Then $u_\mathcal{D}(q) = U_q > 0$ in Ω for every $q \in (0,1)$. In addition, if we set $u_\mathcal{D}(0) := \mathcal{S}(a)$ then $q \mapsto u_\mathcal{D}(q)$ is continuous from $[0,1)$ to $W_\mathcal{D}^{2,r}(\Omega)$. The asymptotic behavior of $u_\mathcal{D}(q)$ as $q \to 1^-$ is characterized as follows:*

(i) If $\mu_\mathcal{D}(a) \geq 1$ and we set

$$u_\mathcal{D}(1) := \begin{cases} t_\mathcal{D}^*\,\phi_\mathcal{D}, & \text{if } \mu_\mathcal{D}(a) = 1, \\ 0, & \text{if } \mu_\mathcal{D}(a) > 1 \text{ (bifurcation from zero)}, \end{cases}$$

then $q \mapsto u_\mathcal{D}(q)$ is left continuous at $q = 1$ (see Figure 2 (i), (ii)).

(ii) If $\mu_\mathcal{D}(a) < 1$ then the curve $\{(q, u_\mathcal{D}(q)) : q \in [0,1)\}$ bifurcates from infinity at $q = 1$ (see Figure 2 (iii)).

Finally, as for the strong positivity of $u_\mathcal{D}(q)$, we have the following two assertions:

(iii) If (A.3) holds then $u_{\mathcal{D}}(q) \gg 0$ *for* q *close to* 0 *or* 1.

(iv) In the following cases, we have $u_{\mathcal{D}}(q) \gg 0$ *for all* $q \in (0,1)$ *(and so,* $\mathcal{I}_{\mathcal{D}} = (0,1)$*):*

(a) $a \geq 0$ *in* Ω_{ρ_0} *for some* $\rho_0 > 0$,

(b) Ω *is a ball and* a *is radial,*

(c) (A.3) holds and Ω_+ *is connected.*

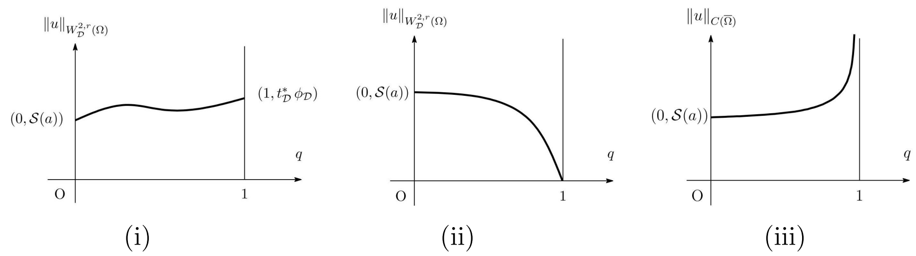


Figure 2: The curve of positive solutions emanating from $(0, \mathcal{S}(a))$: Cases (i) $\mu_{\mathcal{D}}(a) = 1$, (ii) $\mu_{\mathcal{D}}(a) > 1$, (iii) $\mu_{\mathcal{D}}(a) < 1$.

REMARK 5.2: (i) From Theorem A and Proposition 4.5 (ii), it suffices to assume (A.1) and (A.2) to have $U_q = u_{\mathcal{D}}(q)$ whenever $u_{\mathcal{D}}(q)$ exists. Moreover, under these conditions,

$$\mathcal{I}_{\mathcal{D}} = \{q \in (0,1) : U_q \gg 0\}, \tag{6}$$

and $\mathcal{I}_{\mathcal{D}}$ is open.

(ii) The assertion in Theorem 5.1 (i) when $\mu_{\mathcal{D}}(a) = 1$ also gives a better asymptotic estimate for U_q as $q \to 1^-$ if (A.4) holds and $\mu_{\mathcal{D}}(a) \neq 1$. Indeed, a rescaling argument yields that

$$U_q \sim \mu_{\mathcal{D}}(a)^{-\frac{1}{1-q}}\, t^*_{\mathcal{D}}\, \phi_{\mathcal{D}} \quad \text{as} \quad q \to 1^-,$$

i.e.

$$\mu_{\mathcal{D}}(a)^{\frac{1}{1-q}} U_q \to t^*_{\mathcal{D}}\, \phi_{\mathcal{D}} \ \text{ in } \ W^{2,r}_{\mathcal{D}}(\Omega) \ \text{ as } \ q \to 1^-.$$

(iii) As already stated, under (A.3′) we have a positive solution for every $q \in (0,1)$. Assuming additionally (A.3), we can deduce the conclusion of Theorem 5.1 (iii), which extends Theorem D (i). Let us add that in some cases, by Theorem 6.1 below, the condition $\int_\Omega a \geq 0$ (which is weaker than (A.3′)) is also sufficient to have a positive solution of $(P_{\mathcal{D}})$ for all $q \in (0,1)$.

(iv) Under the assumptions of Theorem 5.1 (iv-c), we infer from Corollary B that $\mathcal{A}_{\mathcal{D}} = (0, 1)$.

Next we consider the linearized stability of a solution in $\mathcal{P}^{\circ}_{\mathcal{D}}$ of $(P_{\mathcal{D}})$ for $q \in \mathcal{I}_{\mathcal{D}}$. Let us recall that a solution $u \gg 0$ of $(P_{\mathcal{D}})$ is said to be *asymptotically stable* if $\gamma_1(q, u) > 0$, where $\gamma_1(q, u)$ is the first eigenvalue of the linearized eigenvalue problem at u, namely,

$$\begin{cases} -\Delta\varphi = qa(x)u^{q-1}\varphi + \gamma\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Observe that under the decay condition (A.4), given $q \in [0, 1)$ and $u \gg 0$, we have $au^{q-1} \in L^t(\Omega)$ for some $t > N$, so that $\gamma_1(q, u)$ is well defined.

The implicit function theorem (**IFT** for short) provides us with the following result [30, Theorem 1.5]:

THEOREM 5.3. *If (A.4) holds then $\mathcal{I}_{\mathcal{D}}$ is open, and $u_{\mathcal{D}}(q)$ is asymptotically stable for $q \in \mathcal{I}_{\mathcal{D}}$.*

5.1.1. Local bifurcation analysis in the case $\mu_{\mathcal{D}}(a) = 1$

Let us give a sketch of the proof of Theorem 5.1 (i) when $\mu_{\mathcal{D}}(a) = 1$. In this case, $(P_{\mathcal{D}})$ has the trivial line of strongly positive solutions:

$$\Gamma_1 := \{(q, u) = (1, t\phi_{\mathcal{D}}) : t > 0\}.$$

For $q \simeq 1$, where q is a bifurcation parameter, we shall construct solutions of $(P_{\mathcal{D}})$ bifurcating at certain $(1, t\phi_{\mathcal{D}}) \in \Gamma_1$ in $\mathbb{R} \times W^{2,\xi}_{\mathcal{D}}(\Omega)$, for some fixed $\xi > N$. This bifurcation result (Proposition 5.4 below) complements Proposition 4.5 (iii).

Under (A.4), choose $\sigma_0 > 0$ such that $\eta > 1 + \sigma_0 - \frac{1}{N}$ and set $J_0 := (1 - \frac{\sigma_0}{2}, 1 + \frac{\sigma_0}{2})$. We fix then $\xi \in (N, r)$, depending only on N and σ_0, in such a way that $\xi(\eta + q - 2) > -1 + \frac{\sigma_0 N}{4}$ for $q \in J_0$. Following the Lyapunov-Schmidt procedure, we reduce $(P_{\mathcal{D}})$ to a bifurcation equation. Set $A := -\Delta - a(x)$ with domain $D(A) := W^{2,\xi}_{\mathcal{D}}(\Omega)$. Then $\mathrm{Ker}A = \{t\phi_{\mathcal{D}} : t \in \mathbb{R}\}$ and $\mathrm{Im}A = \left\{f \in L^{\xi}(\Omega) : \int_{\Omega} f\phi_{\mathcal{D}} = 0\right\}$. Let Q be the projection of $L^{\xi}(\Omega)$ to $\mathrm{Im}A$, given by $Q[f] := f - \left(\int_{\Omega} f\phi_{\mathcal{D}}\right)\phi_{\mathcal{D}}$. As long as we consider solutions $u \gg 0$, $(P_{\mathcal{D}})$ is equivalent to the following coupled equations: for $u = t\phi_{\mathcal{D}} + w \in D(A) = \mathrm{Ker}A + X_2$ with $t = \int_{\Omega} u\phi_{\mathcal{D}}$ and $X_2 = \{u \in D(A) : \int_{\Omega} u\phi_{\mathcal{D}} = 0\}$,

$$Q\left[A(t\phi_{\mathcal{D}} + w)\right] = Q\left[a(x)\left((t\phi_{\mathcal{D}} + w)^q - (t\phi_{\mathcal{D}} + w)\right)\right], \tag{8}$$

$$(1 - Q)\left[A(t\phi_{\mathcal{D}} + w)\right] = (1 - Q)\left[a(x)\left((t\phi_{\mathcal{D}} + w)^q - (t\phi_{\mathcal{D}} + w)\right)\right]. \tag{9}$$

Given $t_0 > 0$, first we solve (8) with respect to w at $(q, t, w) = (1, t_0, 0)$, where $(1, t_0, 0)$ is a solution of (8). Note that (A.4) gives that (8) is C^2 for

$(q,t,w) \simeq (1,t_0,0)$, since the choice of ξ ensures that $a(t\phi_{\mathcal{D}}+w)^{q-2} \in L^{\xi}(\Omega)$ for such (q,t,w). An **IFT** argument shows the existence of a unique $w = w(q,t)$ for every $(q,t) \simeq (1,t_0)$ such that (q,t,w) solves (8). We plug $w(q,t)$ into (9), and thus, deduce the desired bifurcation equation

$$\Phi(q,t) := \int_{\Omega} a(x)\{(t\phi_{\mathcal{D}} + w(q,t))^q - (t\phi_{\mathcal{D}} + w(q,t)\}\phi_{\mathcal{D}} = 0, \quad (q,t) \simeq (1,t_0),$$

where we note that Φ is C^2 for $(q,t) \simeq (1,t_0)$.

As an application of the **IFT**, we find that if $(1, t_0\phi_{\mathcal{D}})$ is a bifurcation point on Γ_1 then

$$\frac{\partial \Phi}{\partial q}(1,t_0) = t_0 \left\{ (\log t_0) \int_{\Omega} a(x)\phi_{\mathcal{D}}^2 + \int_{\Omega} a(x)\phi_{\mathcal{D}}^2 \log \phi_{\mathcal{D}} \right\} = 0,$$

so that $t_0 = t_{\mathcal{D}}^*$, given by (5). Conversely, since direct computations [30, Lemma 4.3] provide

$$\frac{\partial \Phi}{\partial t}(1,t_{\mathcal{D}}^*) = \frac{\partial^2 \Phi}{\partial t^2}(1,t_{\mathcal{D}}^*) = 0, \quad \frac{\partial^2 \Phi}{\partial t \partial q}(1,t_{\mathcal{D}}^*) = \int_{\Omega} a\,(x)\,\phi_{\mathcal{D}}^2 > 0,$$

the Morse Lemma [17, Theorem 4.3.19] yields the following existence result [30, Proposition 4.4]:

PROPOSITION 5.4. *Suppose (A.4) with $\mu_{\mathcal{D}}(a) = 1$. Then the set of solutions of $(P_{\mathcal{D}})$ near $(1, t_{\mathcal{D}}^*\phi_{\mathcal{D}})$ consists of two continuous curves in $\mathbb{R} \times W_{\mathcal{D}}^{2,\xi}(\Omega)$ intersecting only at $(1, t_{\mathcal{D}}^*\phi_{\mathcal{D}})$ transversally, given by $\Gamma_1 \cup \Gamma_2$, where Γ_2 for $q < 1$ represents the ground state solution U_q.*

Let us mention that Proposition 5.4 remains true in $\mathbb{R} \times W_{\mathcal{D}}^{2,r}(\Omega)$ by elliptic regularity.

5.2. The Neumann problem

Under (A.0), the Neumann eigenvalue problem

$$(E_{\mathcal{N}}) \qquad \begin{cases} -\Delta\phi = \mu a(x)\phi & \text{in } \Omega, \\ \partial_\nu \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

has a first positive eigenvalue $\mu_{\mathcal{N}}(a)$, which is principal and simple, and an eigenfunction $\phi_{\mathcal{N}}(a) \gg 0$ associated to $\mu_{\mathcal{N}}(a)$ and satisfying $\int_{\Omega} \phi_{\mathcal{N}}^2 = 1$.

The bifurcation scheme from the previous subsection also applies to $(P_{\mathcal{N}})$, with the advantage of *not* requiring the decay condition (A.4), since $\phi_{\mathcal{N}} > 0$ on $\overline{\Omega}$. We look at q as a bifurcation parameter in $(P_{\mathcal{N}})$. Similarly as in the

Dirichlet case, if $\mu_{\mathcal{N}}(a) = 1$ then $u = t\phi_{\mathcal{N}}$ solves $(P_{\mathcal{N}})$ with $q = 1$, i.e., $(P_{\mathcal{N}})$ has the trivial line

$$\Gamma_1 := \{(q, u) = (1, t\phi_{\mathcal{N}}) : t > 0\}.$$

We shall obtain, for q close to 1, a curve of solutions $u \gg 0$ bifurcating from Γ_1 (see Figure 3).

The definition of *asymptotically stable* for solutions $u \gg 0$ of $(P_{\mathcal{N}})$ is similar to the one for $(P_{\mathcal{D}})$, see (7). Setting

$$t^*_{\mathcal{N}} := \exp\left[-\frac{\int_\Omega a(x)\phi^2_{\mathcal{N}} \log \phi_{\mathcal{N}}}{\int_\Omega a(x)\phi^2_{\mathcal{N}}}\right], \tag{10}$$

we have the following result [28, Theorem 1.2].

THEOREM 5.5. *Assume (A.0) and $r > N$. Then there exists $q_0 = q_0(a) \in (0, 1)$ such that $(P_{\mathcal{N}})$ has a solution $u_q \gg 0$ for $q_0 < q < 1$. Moreover, u_q is asymptotically stable and satisfies*

$$u_q \sim \mu_{\mathcal{N}}(a)^{-\frac{1}{1-q}}\, t^*_{\mathcal{N}}\, \phi_{\mathcal{N}} \quad \textit{as} \quad q \to 1^-,$$

i.e. $\mu_{\mathcal{N}}(a)^{\frac{1}{1-q}} u_q \to t^*_{\mathcal{N}}\, \phi_{\mathcal{N}}$ *in* $W^{2,r}(\Omega)$ *as* $q \to 1^-$. *More specifically (see Figure 3):*

(i) If $\mu_{\mathcal{N}}(a) = 1$, *then* $u_q \to t^*_{\mathcal{N}}\, \phi_{\mathcal{N}}$ *in* $W^{2,r}(\Omega)$ *as* $q \to 1^-$.

(ii) If $\mu_{\mathcal{N}}(a) > 1$, *then* $u_q \to 0$ *in* $W^{2,r}(\Omega)$ *as* $q \to 1^-$.

(iii) If $\mu_{\mathcal{N}}(a) < 1$, *then* $\min_{\overline{\Omega}} u_q \to \infty$ *as* $q \to 1^-$.

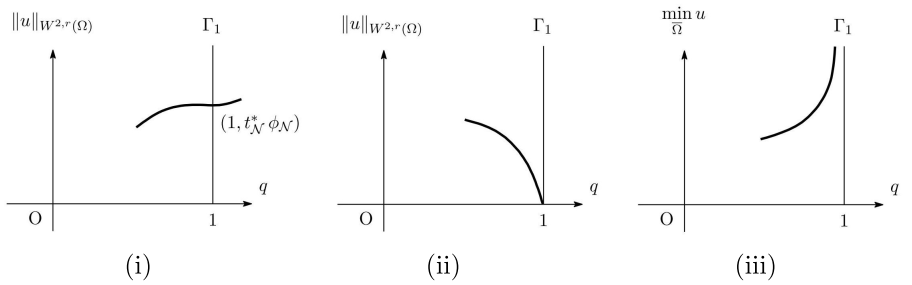


Figure 3: Bifurcating solutions $u \gg 0$ (i) from Γ_1 at $(1, t^*_{\mathcal{N}}\, \phi_{\mathcal{N}})$ in case $\mu_{\mathcal{N}}(a) = 1$; (ii) from zero in case $\mu_{\mathcal{N}}(a) > 1$; (iii) from infinity in case $\mu_{\mathcal{N}}(a) < 1$.

Let us point out that, in general, it is hard to give a lower estimate for $q_0(a)$, as one can see from Example C. As a direct consequence of Theorem 5.5, we complement Theorem A (ii-b) showing that (A.0) is also sufficient for the existence of a positive solution of $(P_{\mathcal{N}})$, for some $q \in (0,1)$:

COROLLARY 5.6. *$(P_{\mathcal{N}})$ has a positive solution (or a solution $u \gg 0$) for some $q \in (0,1)$ if and only if (A.0) holds.*

REMARK 5.7: Differently from the Dirichlet case, under (A.0) and (A.1) one may deduce the existence of a *dead core* limit function for nontrivial solutions of $(P_{\mathcal{N}})$ as $q \to 0^+$. Indeed, thanks to an *a priori* bound [28, Proposition 2.1], we may assume that a nontrivial solution u_n of $(P_{\mathcal{N}})$ with $q = q_n \to 0^+$ converges to $u_0 \geq 0$ in $C^1(\overline{\Omega})$. We claim that u_0 *vanishes somewhere* in Ω. Indeed, if $u_0 > 0$ in Ω then Lebesgue's dominated convergence theorem shows that $\int_\Omega \nabla u_0 \nabla v = \int_\Omega a(x)v$ for all $v \in C^1(\overline{\Omega})$, i.e. u_0 is a nontrivial solution of $(P_{\mathcal{N}})$ with $q = 0$, implying $\int_\Omega a = 0$, a contradiction. This situation does not occur in $(P_{\mathcal{D}})$ under (A.3$'$) (see Theorem 5.1).

The final result of this section is a characterization of the set $\mathcal{I}_{\mathcal{N}}$, which is proved by combining the **IFT** and the sub-supersolutions method [28, Theorem 1.4 (i)]. Also, using the **IFT** approach developed by Brown and Hess [12, Theorem 1], we have a stability result analogous to the one in Theorem 5.3.

THEOREM 5.8. *Assume (A.0). Then $\mathcal{I}_{\mathcal{N}} = (\hat{q}_{\mathcal{N}}, 1)$ for some $\hat{q}_{\mathcal{N}} \in [0,1)$. Moreover, for $q \in \mathcal{I}_{\mathcal{N}}$, the unique solution in $\mathcal{P}^\circ_{\mathcal{N}}$ is asymptotically stable.*

In addition to the local result given by Theorem 5.5, we can give a *global* description (i.e. for all $q \in (0,1)$) of the nontrivial solutions set of $(P_{\mathcal{N}})$ when Ω_+ is connected and satisfies (A.2):

REMARK 5.9: If Ω_+ is connected and satisfies (A.2), then Corollary 4.3 yields that $u_{\mathcal{D}}(q) \gg 0$ for $q \in (q_{\mathcal{D}}, 1)$, and the unique nontrivial solution of $(P_{\mathcal{D}})$ does not belong to $\mathcal{P}^\circ_{\mathcal{D}}$ for $q \in (0, q_{\mathcal{D}}]$. Moreover, if additionally Ω_+ includes a tubular neighborhood of $\partial\Omega$, then this solution vanishes somewhere in Ω. Note that the asymptotic behavior of $u_{\mathcal{D}}(q)$ as $q \to 1^-$, i.e. assertions (i) and (ii) of Theorem 5.1, remain valid, assuming additionally (A.4), see Figure 4. A similar result holds for $(P_{\mathcal{N}})$ if we assume, in addition, (A.0). In this case, the asymptotic behavior of the solution $u_q \gg 0$ as $q \to 1^-$, i.e. assertions (i)-(iii) of Theorem 5.5, also remain valid without assuming (A.4).

6. Some further results

In this section we present some results (without proofs) on the two following issues:

- *Explicit* sufficient conditions for the existence of positive solutions for $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$.
- Sufficient conditions for the existence of dead core solutions for $(P_{\mathcal{N}})$.

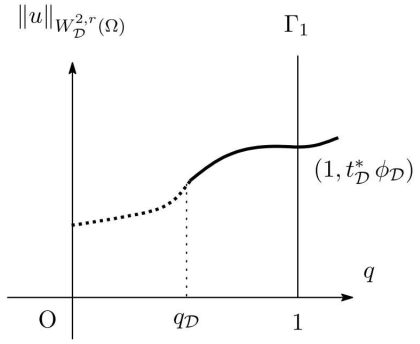


Figure 4: The bifurcation curve of the unique nontrivial solution in the case $\mu_{\mathcal{D}}(a) = 1$, assuming that Ω_+ is connected, satisfies (A.2), and includes a tubular neighborhood of $\partial\Omega$. Here the full curve represents $u_{\mathcal{D}}(q) \gg 0$, whereas the dotted curve represents solutions vanishing somewhere in Ω.

Given $0 < R_0 < R$, we write $B_{R_0} := \{x \in \mathbb{R}^N : |x| < R_0\}$. When $\Omega = B_R$ and a is radial, we shall exhibit some explicit conditions on q and a so that $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ admit a positive solution. In Theorem 6.1 below we consider the case that supp a^+ is contained in B_{R_0} and give a condition that guarantees the existence of a positive solution u (not necessarily $\gg 0$), while in Theorem 6.2 we consider the case that supp a^- is contained in B_{R_0} and provide a solution $u \gg 0$. These theorems are based on a sub-supersolutions approach and are inspired in the proofs of [20, Section 3] (see also the proof of Theorem D (iii)). If f is a radial function, we shall write $f(x) := f(|x|) := f(r)$, and we also set $A_{R_0,R} := \{x \in \mathbb{R}^N : R_0 < |x| < R\}$.

Theorem 6.1. *Let $a \in C(\overline{B_R})$ be a radial function such that*

- *$a \geq 0$ in B_{R_0} and $a \leq 0$ in $A_{R_0,R}$;*
- *$r \to a(r)$ is differentiable and nonincreasing in (R_0, R), and*

$$\frac{1-q}{1+q}\int_{A_{R_0,R}} a^- \leq \int_{B_{R_0}} a^+. \tag{11}$$

Then, $(P_{\mathcal{D}})$ has a positive solution. If, in addition, (A.0) holds, then $(P_{\mathcal{N}})$ has a positive solution.

Note that (11) holds for all $q \in (0,1)$ if $\int_{B_R} a \geq 0$, and this condition can also be formulated as

$$\frac{-\int_\Omega a}{\int_\Omega |a|} \leq q < 1. \tag{12}$$

In particular, we see that (11) is satisfied if q is close enough to 1. Note that if we replace a by

$$a_\delta = a^+ - \delta a^-, \quad \text{with} \quad \delta > \delta_0 := \frac{\int_\Omega a^+}{\int_\Omega a^-},$$

then the left-hand side in (12) approaches 1 as $\delta \to \infty$, so that this condition becomes very restrictive for a_δ as $\delta \to \infty$. On the other side, $\int_\Omega a_\delta \to 0^-$ as $\delta \to \delta_0^+$, so that (12) becomes much less constraining for a_δ as $\delta \to \delta_0^+$.

We denote by ω_{N-1} the surface area of the unit sphere ∂B_1 in $\mathbb{R}^N$.

THEOREM 6.2. *Let $a \in C(\overline{B_R})$ be a radial function satisfying (A.0). Assume that $a \geq 0$ in $A_{R_0,R}$ and*

$$\frac{1-q}{2q + N(1-q)} \omega_{N-1} R_0^N \left\| a^- \right\|_{C(\overline{B_{R_0}})} < \int_{A_{R_0,R}} a^+. \tag{13}$$

Then $(P_\mathcal{N})$ has a solution $u \gg 0$.

Unlike in Theorem 6.1, we observe that no differentiability nor monotonicity condition is imposed on a^- in Theorem 6.2. Note again that (13) is also clearly satisfied if q is close enough to 1.

Finally, we consider the existence of nontrivial dead core solutions of $(P_\mathcal{N})$. From [5, 6] we recall that the set $\{x \in \Omega : u(x) = 0\}$ is called the *dead core* of a nontrivial solution u of $(P_\mathcal{N})$ if it contains an interior point. Recall that in Theorem E we have already given sufficient conditions for the existence of a nontrivial solution of $(P_\mathcal{N})$ vanishing somewhere in Ω. We proceed now with the construction of dead cores for solutions of $(P_\mathcal{N})$. To this end, let us first introduce the following assumption:

$$0 \leq b_1, b_2 \in C(\overline{\Omega}) \quad \text{and} \quad \operatorname{supp} b_1 \cap \{x \in \Omega : b_2(x) > 0\} = \emptyset. \tag{14}$$

Given a nonempty open subset $G \subseteq \Omega$ and $\rho > 0$, we set

$$G^\rho := \{x \in G : \operatorname{dist}(x, \partial G) > \rho\}. \tag{15}$$

The following result is based on a comparison argument from [19]:

THEOREM 6.3. *Let $a_\delta := b_1 - \delta b_2$, with $b_1, b_2 \not\equiv 0$ satisfying (14), and $\delta > 0$. If we set $G := \{x \in \Omega : b_2(x) > 0\}$ then, given $0 < \overline{q} < 1$ and $\rho > 0$, there exists $\delta_0 = \delta_0(\rho, \overline{q}) > 0$ such that any nontrivial solution of $(P_\mathcal{N})$ with $a = a_\delta$ and $q \in (0, \overline{q}]$ vanishes in G^ρ if $\delta \geq \delta_0$.*

Theorem 6.3 holds also for the Dirichlet problem $(P_\mathcal{D})$. In particular, it complements Theorem 4.2 as follows: given $q \in (0,1)$ there exist $0 < \delta_1 < \delta_0$ such that every nontrivial solution u of $(P_\mathcal{D})$ with $a = a_\delta$ satisfies $u \gg 0$ for $\delta < \delta_1$, whereas u has a nonempty dead core for $\delta > \delta_0$.

7. Final remarks

Several conditions in this paper are assumed for the sake of presentation or technical reasons. As a matter of fact, the results in Sections 4 and 5 remain true more generally for $a \in L^r(\Omega)$ with $r > N$. In this situation, we assume, instead of (A.1), that

$$\begin{cases} \Omega_+ \text{ is the largest open subset of } \Omega \text{ where } a > 0 \text{ a.e.,} \\ \text{satisfies } |(\operatorname{supp} a^+) \setminus \Omega_+| = 0 \text{ and has a finite number} \\ \text{of connected components,} \end{cases}$$

where supp is the support in the measurable sense.

It is also important to highlight that the uniqueness results in Theorem A hold without assuming (A.1) and (A.2). Indeed, one may prove that the ground state solution U_q is the only solution of $(P_{\mathcal{D}})$ being positive in Ω_+, and a similar result applies to $(P_{\mathcal{N}})$ under (A.0), see [33]. A similar situation occurs in Theorem 5.1: without (A.1) and (A.2) the solution $u_{\mathcal{D}}(q)$ still exists for every $q \in (0,1)$, and satisfies assertions (i)-(iv) in Theorem 5.1 (cf. Remark 5.2 (i)).

Also, let us mention that some of the results in this paper can be extended to the Robin problem

$$\begin{cases} -\Delta u = a(x)u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \partial_\nu u = \alpha u & \text{on } \partial\Omega, \end{cases} \tag{16}$$

with $\alpha \in \mathbb{R}$. Some work in this direction has already been done in [31, 32]. Let us note that there are striking differences between (16) and the problems considered here. For instance, under (A.0)–(A.2) and some additional assumptions, for any $q \in \mathcal{I}_{\mathcal{N}}$ fixed, there exists some $\overline{\alpha} > 0$ such that (16) has *exactly two* strongly positive solutions for $\alpha \in (0, \overline{\alpha})$, *one* strongly positive solution for $\alpha = \overline{\alpha}$, and *no* strongly positive solutions for $\alpha > \overline{\alpha}$ [32, Theorem 1.3].

It is also worth pointing out that the positivity results in Section 4 can be applied to the study of positive solutions for indefinite concave-convex equations of the form $-\Delta u = a(x)u^q + b(x)u^p$, where $0 < 1 < q < p$, see [27, 29]. Finally, let us mention that several results presented here can be extended to problems involving a class of fully nonlinear homogeneous operators [14].

We conclude now with some interesting questions that remain open in the context of this paper:

(i) Is the set $\mathcal{I}_{\mathcal{D}}$ connected?

(ii) Is there some a such that $\mathcal{I}_{\mathcal{N}} = (0,1)$? Let us note that we can construct a sequence $a_n \in L^\infty(\Omega)$ such that $\mathcal{I}_{\mathcal{N}}(a_n) = (q_n, 1)$ with $q_n \searrow 0$ [28, Remark 4.5].

(iii) Assume $\mathcal{I}_{\mathcal{N}}(a) = (0,1)$. Can we characterize the limiting behavior of the solution $u_q \gg 0$ of $(P_{\mathcal{N}})$ as $q \to 0^+$?

(iv) By Theorem E, we see that we may have $q_{\mathcal{D}} > 0$ or $q_{\mathcal{N}} > 0$. On the other side, Theorem 5.1 (iv-c) shows a situation in which $q_{\mathcal{D}} = 0$. Can we have $q_{\mathcal{N}} = 0$ (i.e., $\mathcal{A}_{\mathcal{N}} = (0,1)$) ?

(v) Can we obtain explicit sufficient conditions for the existence of positive solutions of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ (as e.g. the ones in Theorems 6.1 and 6.2 for $(P_{\mathcal{N}})$; or the ones in Theorem 6.1 and [20, Theorem 3.2 (i)] for $(P_{\mathcal{D}})$) without assuming that Ω is a ball and a is radial?

(vi) Is it possible to extend the results in this paper to a general operator of the form
$$Lu = -\operatorname{div}(A(x)\nabla u) + \langle b(x), \nabla u\rangle + c\,(x)\,u,$$
under suitable assumptions on the coefficients? Let us note that if $b \not\equiv 0$ variational techniques do not apply. Furthermore, the size of the coefficient c plays an important role: in the one-dimensional Dirichlet case no positive solutions exist if $c > 0$ is large enough, cf. [26, Theorem 3.11]. Let us add that the Neumann case with $A \equiv 1$, $b \equiv 0$, and c constant has been treated in [1, 32].

(vii) We believe that many of the resuts and techniques reviewed here also apply to the corresponding p-Laplacian equation
$$-\Delta_p u = a(x)u^q,$$
with $p > 1$ and $0 < q < p - 1$. Some progress in this direction has been achieved in [33].

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