Rend. Istit. Mat. Univ. Trieste
Volume 52 (2020), 131–149
DOI: 10.13137/2464-8728/30765

Boundedness of solutions to the Cauchy problem for an attraction-repulsion chemotaxis system in two-dimensional space

Toshitaka Nagai and Tetsuya Yamada

Dedicated to Professor Julián López-Gómez on his 60th birthday

Abstract. *We consider the Cauchy problem for an attraction-repulsion chemotaxis system in two-dimensional space. The system consists of three partial differential equations; a drift-diffusion equation incorporating terms for both chemoattraction and chemorepulsion, and two elliptic equations. We denote by β_1 the coefficient of the attractant and by β_2 that of the repellent. The boundedness of nonnegative solutions to the Cauchy problem was shown in the repulsive dominant case $\beta_1 < \beta_2$ and the balance case $\beta_1 = \beta_2$. In this paper, we study the boundedness problem to the Cauchy problem in the attractive dominant case $\beta_1 > \beta_2$.*

Keywords: attraction-repulsion chemotaxis system, attractive dominant case, boundedness of solutions.
MS Classification 2010: 35B45, 35K15, 35K55.

1. Introduction

We consider the Cauchy problem for the following attraction-repulsion chemotaxis system in $\mathbb{R}^2$:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (\beta_1 u \nabla v_1) + \nabla \cdot (\beta_2 u \nabla v_2), & t > 0,\ x \in \mathbb{R}^2, \\ 0 = \Delta v_j - \lambda_j v_j + u, & t > 0,\ x \in \mathbb{R}^2 \quad (j = 1, 2), \\ u(0, x) = u_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{P}$$

where β_j and λ_j $(j = 1, 2)$ are positive constants. We assume that

$$u_0 \geq 0 \text{ on } \mathbb{R}^2,\ u_0 \not\equiv 0,\ u_0 \in L^1 \cap L^\infty, \tag{1}$$

and consider nonnegative solutions to the Cauchy problem (P). Here, $L^p := L^p(\mathbb{R}^2)$ $(1 \le p \le \infty)$ stand for the usual Lebesgue spaces on $\mathbb{R}^2$ with norm $\|\cdot\|_{L^p}$, and in what follows, we denote $\|\cdot\|_{L^p}$ by $\|\cdot\|_p$ for simplicity.

The system (P) is a simplified mathematical model introduced in [19] to describe the aggregation of *Microglia* in the central nervous system. In the system (P), the functions u, v_1 and v_2 denote the density of *Microglia*, the concentration of attractive and repulsive chemical substances, respectively.

In the case $\beta_2 = 0$, the system (P) becomes a minimal version of the classical Keller-Segel model (e.g., [10, 13]):

$$\begin{cases} \partial_t u = \Delta u - \beta_1 \nabla \cdot (u \nabla v_1), & t > 0, x \in \mathbb{R}^2, \\ 0 = \Delta v_1 - \lambda_1 v_1 + u, & t > 0, x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{KS}$$

where λ_1 is a nonnegative constant. The mass conservation for u holds and plays an important role in the existence of nonnegative global solutions to the Cauchy problem (KS). Indeed, in the case $\beta_1 \int_{\mathbb{R}^2} u_0\,dx \le 8\pi$, the nonnegative solutions exist globally in time (e.g., [4, 5, 6, 21, 22, 29]), meanwhile, in the case $\beta_1 \int_{\mathbb{R}^2} u_0\,dx > 8\pi$, a nonnegative solution may blow up in finite time (e.g., [2, 5, 15, 29]). The boundedness of nonnegative solutions to the Cauchy problem (KS) was shown under the assumption $\beta_1 \int_{\mathbb{R}^2} u_0\,dx < 8\pi$ by using rearrangement techniques ([6, 20]). In the critical mass case $\int_{\mathbb{R}^2} u_0\,dx = 8\pi$ to the Cauchy problem (KS) with $\beta_1 = 1$ and $\lambda_1 = 0$, the boundedness of nonnegative solutions has been studied in [3, 18, 23], and it was shown in [23] that $\sup_{t>0} \|u(t)\|_\infty < \infty$ for the nonnegative *radial* solutions under the assumption $\liminf_{R\to\infty}(R^2 \int_{|x|>R} u_0\,dx) > 0$. We also remark that $\lim_{t\to\infty} \|u(t)\|_\infty = \infty$ if $\int_{\mathbb{R}^2} |x|^2 u_0(x)\,dx < \infty$ ([4]).

The Cauchy problem (P) has a unique nonnegative smooth solution locally in time for initial data u_0 satisfying (1) ([26]). The nonnegative solutions exist globally in time and are bounded in the repulsive dominant case $\beta_1 < \beta_2$ ([26]) and the balance case $\beta_1 = \beta_2$ ([12, 24]). In the attractive dominant case $\beta_1 > \beta_2$, the nonnegative solutions exist globally in time under the assumption $(\beta_1 - \beta_2)\int_{\mathbb{R}^2} u_0\,dx \le 8\pi$ ([24, 25]), whereas there exists a blowing-up solution in finite time under the assumption $(\beta_1 - \beta_2)\int_{\mathbb{R}^2} u_0\,dx > 8\pi$ ([26]). We remark that if $\sup_{t>0} \|(u(t), v_1(t), v_2(t))\|_\infty < \infty$, then for all $1 < p \le \infty$,

$$\|(u(t), v_1(t), v_2(t))\|_p \le C(1+t)^{-1+1/p} \quad (t > 0)$$

(see the proof of [26, Theorem 1.3]), and

$$\lim_{t\to\infty} t^{1-1/p} \Big\| u(t) - \int_{\mathbb{R}^2} u_0\,dx G(t) \Big\|_p = 0,$$

where $G(t,x) = (4\pi t)^{-1} e^{-|x|^2/(4t)}$ is the heat kernel (see the proof of Theorem 1.2 and Remark 1.1 in [12]). Concerning the boundedness problem to the

Cauchy problem for the parabolic system of an attraction-repulsion chemotaxis model, see, e.g., [12] for the balance case.

The boundedness problem to attraction-repulsion chemotaxis systems has been studied on a smooth bounded domain under Neumann boundary conditions (e.g., [7, 11, 16, 17, 28]). When the system (P) is considered on a smooth bounded domain Ω in $\mathbb{R}^2$ under Neumann boundary conditions for u and v_j $(j = 1, 2)$, the boundedness of nonnegative solutions in the attractive dominant case $\beta_1 > \beta_2$ was obtained in [7] under the assumption $(\beta_1 - \beta_2) \int_\Omega u_0 \, dx < 4\pi$ by showing the boundedness of the entropy $\int_\Omega u(t) \log u(t) \, dx$ with respect to $t \in [0, \infty)$. However, the entropy $\int_{\mathbb{R}^2} u(t) \log u(t) \, dx$ on $\mathbb{R}^2$ is not appropriate to get the boundedness of nonnegative solutions to the Cauchy problem (P). The reason is that if $\lim_{t\to\infty} \|u(t)\|_2 = 0$, we observe that

$$\begin{aligned}\int_{\mathbb{R}^2} u(t) \log u(t) \, dx &\le \|u(t)\|_1 \log \Big(\frac{1}{\|u(t)\|_1} \int_{\mathbb{R}^2} u^2(t) \, dx \Big) \\ &= \|u_0\|_1 \big(\log \|u(t)\|_2^2 - \log \|u_0\|_1 \big) \to -\infty \quad (t \to \infty).\end{aligned}$$

Here we used Jensen's inequality for the concave function $\log u$ and $\|u(t)\|_1 = \|u_0\|_1$ $(t > 0)$. For this reason, we introduce the modified entropy $\int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t)) \, dx$ in place of $\int_{\mathbb{R}^2} u(t) \log u(t) \, dx$.

For the nonnegative solutions (u, v_1, v_2) to the Cauchy problem (P), the following relation is satisfied ([26, Lemma 3.1]): For $p > 1$,

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + \frac{4(p-1)}{p^2}\|\nabla u^{p/2}(t)\|_2^2 + (\beta_2 - \beta_1)\Big(1 - \frac{1}{p}\Big)\|u(t)\|_{p+1}^{p+1} \\ &= -\beta_1\lambda_1\Big(1 - \frac{1}{p}\Big)\int_{\mathbb{R}^2} u^p(t) v_1(t) \, dx + \beta_2\lambda_2\Big(1 - \frac{1}{p}\Big)\int_{\mathbb{R}^2} u^p(t) v_2(t) \, dx.\end{aligned} \tag{2}$$

In the repulsive dominant case $\beta_1 < \beta_2$, we get the boundedness of $\|u(t)\|_p$ in $t > 0$ from (2) thanks to $\beta_2 - \beta_1 > 0$ in the third term on the left-hand side of (2). In the attractive dominant case $\beta_1 > \beta_2$, we need a smallness condition on initial data to get the boundedness of $\|u(t)\|_p$ in $t > 0$. Hence, we first study the boundedness of the modified entropy $\int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t)) \, dx$ in $t > 0$ under the assumption $(\beta_1 - \beta_2) \int_{\mathbb{R}^2} u_0 \, dx < 4\pi$, and then apply (2) to get the boundedness of $\|u(t)\|_p$ in $t > 0$.

The a priori estimate of $\int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t)) \, dx$ has been studied for the Keller-Segel model (KS) in [21] and for the Cauchy problem (P) in [24] by applying the Brezis-Merle type inequality established in [21]. However, the a priori bound of $\int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t)) \, dx$ for $0 < t < T$ obtained in [21, 24] depends on T, which does not give the uniform boundedness of the solutions on $[0, \infty)$. Another approach from the application of radially symmetric decreasing rearrangement does not seem to work for the Cauchy problem (P) due to the term for chemorepulsion, although it is useful for getting the uniform boundedness of the solutions to the Keller-Segel model (KS) (e.g., [6, 18, 20]). We prove the boundedness of $\int_{\mathbb{R}^2}(1+u(t))\log(1+u(t))\,dx$ on $[0,\infty)$ by applying the sharp form of the Gagliardo-Nirenberg inequality under the assumption $(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx<4\pi$, but the uniform boundedness of the solutions is expected under the assumption $(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx<8\pi$.

THEOREM 1.1. *Let $\beta_1>\beta_2$ and assume that*

$$(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx<4\pi. \tag{3}$$

Then, $\sup_{t>0}\|(u(t),v_1(t),v_2(t))\|_p<\infty$ *for all* $1\le p\le\infty$.

We next study the boundedness of nonnegative radial solutions to the Cauchy problem (P). For the nonnegative radial initial data u_0 satisfying (1), the uniqueness of solutions to the Cauchy problem (P) ensures that the solution (u,v_1,v_2) for the initial data u_0 is radial in x. Considering the mass function $U(t,s)=\int_0^s\tilde{u}(t,\sigma)\,d\sigma$ of u, where $u(t,x)=\tilde{u}(t,s)$ $(s=\pi|x|^2)$, we reduce the boundedness of u to the following (see Lemma 4.2): There exist $s_0>0$ and $C>0$ such that

$$U(t,s)\le C\sqrt{s}\quad(t\ge0,\ 0\le s\le s_0). \tag{4}$$

Constructing a comparison function and applying the comparison principle for parabolic equations, we show (4) to have the following.

THEOREM 1.2. *Let $\beta_1>\beta_2$ and assume that the nonnegative initial data u_0 is radial and*

$$(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx<8\pi. \tag{5}$$

Then, $\sup_{t>0}\|(u(t),v_1(t),v_2(t))\|_p<\infty$ *for all* $1\le p\le\infty$.

We lastly study the boundedness problem to the Cauchy problem (P) in the critical mass case $(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx=8\pi$. Using the idea of getting the boundedness of radial solutions to the Cauchy problem (KS) in [23], we have the following theorem under a restricted condition on β_j and λ_j $(j=1,2)$.

THEOREM 1.3. *Let $\beta_1>\beta_2$, $\lambda_1\le\lambda_2$ and $\beta_1\lambda_1\ge\beta_2\lambda_2$. Assume that the nonnegative initial data u_0 is radial and*

$$(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx=8\pi, \tag{6}$$

$$\liminf_{R\to\infty}\left(R^2\int_{|x|>R}u_0\,dx\right)>0. \tag{7}$$

Then, $\sup_{t>0}\|(u(t),v_1(t),v_2(t))\|_p<\infty$ *for all* $1\le p\le\infty$.

The rest of the paper is organized as follows. In Section 2, we mention some properties of nonnegative solutions to the Cauchy problem (P) and give function inequalities on $\mathbb{R}^2$ used in the proof of Theorem 1.1. In Section 3, we give the proof of Theorem 1.1, and in Section 4, the proofs of Theorems 1.2 and 1.3.

Throughout the paper, we use a universal constant C to describe a various constant, and $C(*, \cdots, *)$ when C depends on the quantities appearing in parentheses.

2. Preliminaries

For the nonnegative solutions to the Cauchy problem (P), the conservation of mass is one of important properties, which is obtained by integrating the equations for u and v_j $(j = 1, 2)$ over $\mathbb{R}^2$.

Lemma 2.1. *Let* (u, v_1, v_2) *be the nonnegative solution to the Cauchy problem* (P) *with nonnegative initial data* u_0 *satisfying* (1). *Then,*

$$\int_{\mathbb{R}^2} u(t)\,dx = \lambda_1 \int_{\mathbb{R}^2} v_1(t)\,dx = \lambda_2 \int_{\mathbb{R}^2} v_2(t)\,dx = \int_{\mathbb{R}^2} u_0\,dx \quad (t > 0).$$

For $\lambda > 0$ and $f \in L^p$ $(1 \le p \le \infty)$, we denote by $(\lambda - \Delta)^{-1} f$ the convolution of the Bessel kernel B_λ and f, namely,

$$(\lambda - \Delta)^{-1} f = B_\lambda * f,$$

where

$$B_\lambda(x) = \int_0^\infty e^{-\lambda\sigma} G(\sigma, x)\,d\sigma, \quad x \in \mathbb{R}^2$$

and $G(t, x)$ is the heat kernel given by $G(t, x) = (4\pi t)^{-1} e^{-|x|^2/(4t)}$. For $f \in L^p$ $(1 < p < \infty)$, the function $v := (\lambda - \Delta)^{-1} f$ on $\mathbb{R}^2$ belongs to $W^{2,p}$ and a solution of

$$(\lambda - \Delta)v = f \quad \text{in} \ \ \mathbb{R}^2.$$

By the following estimates

$$\|\partial_x^\alpha B_\lambda\|_p < \infty \text{ for } 1 \le p < \infty \text{ if } |\alpha| = 0 \text{ and } 1 \le p < 2 \text{ if } |\alpha| = 1,$$

applying Young's inequality for convolution gives L^p estimates on $(\lambda - \Delta)^{-1} f$ in Lemma 2.2 below, which are often used in the course of the proof of Theorem 1.1. For the Bessel kernel, see, e.g., [9, 27].

Lemma 2.2. *For $\lambda > 0$, it holds that*

$$\|(\lambda - \Delta)^{-1} f\|_p \leq C(\lambda, p, q)\|f\|_q, \ 1 \leq q \leq p < \infty,$$
$$\|(\lambda - \Delta)^{-1} f\|_\infty \leq C(\lambda, q)\|f\|_q, \ 1 < q \leq \infty,$$
$$\|\nabla(\lambda - \Delta)^{-1} f\|_\infty \leq C(\lambda, q)\|f\|_q, \ 2 < q \leq \infty.$$

For later uses, we give some function inequalities on $\mathbb{R}^2$. We begin with the Gagliardo-Nirenberg inequality on $\mathbb{R}^2$ (e.g., [8]): For $1 < p < \infty$, there is a positive constant C depending on p such that for any $f \in L^1$ with $|\nabla f| \in L^2$,

$$\|f\|_p \leq C\|\nabla f\|_2^{1-1/p}\|f\|_1^{1/p}. \tag{8}$$

The next inequality is a version of the Gagliardo-Nirenberg inequality on $\mathbb{R}^2$: For any $f \in L^2$ with $|\nabla f| \in L^1$,

$$\|f\|_2 \leq \frac{1}{\sqrt{4\pi}}\|\nabla f\|_1. \tag{9}$$

Here, $1/\sqrt{4\pi}$ is the best constant (e.g., [30, Theorem 2.7.4]).

We give two lemmas below, which are proven by applying (9).

Lemma 2.3. *For $0 < \varepsilon < 1$ and nonnegative functions $g \in L^1 \cap W^{1,2}$,*

$$\int_{\mathbb{R}^2} g^2\,dx \leq \frac{1+\varepsilon}{4\pi}\Big(\int_{\mathbb{R}^2} g\,dx\Big)\Big(\int_{\mathbb{R}^2} \frac{|\nabla g|^2}{1+g}\,dx\Big) + \frac{2}{\varepsilon}\int_{\mathbb{R}^2} g\,dx. \tag{10}$$

Proof. Let $\alpha > 1$. We have that

$$\begin{aligned}
\int_{\mathbb{R}^2} g^2\,dx &= \int_{g>\alpha} g^2\,dx + \int_{g\leq\alpha} g^2\,dx = \int_{g>\alpha} \{(g-\alpha)+\alpha\}^2\,dx + \int_{g\leq\alpha} g^2\,dx \\
&= \int_{g>\alpha} (g-\alpha)^2\,dx + 2\alpha\int_{g>\alpha} (g-\alpha)\,dx + \int_{g>\alpha} \alpha^2\,dx + \int_{g\leq\alpha} g^2\,dx \\
&\leq \int_{\mathbb{R}^2} (g-\alpha)_+^2\,dx + 2\alpha\int_{g>\alpha} g\,dx + \alpha\int_{g\leq\alpha} g\,dx \\
&\leq \int_{\mathbb{R}^2} (g-\alpha)_+^2\,dx + 2\alpha\int_{\mathbb{R}^2} g\,dx,
\end{aligned}$$

where $(g-\alpha)_+ = \max\{g-\alpha, 0\}$. We estimate $\int_{\mathbb{R}^2} (g-\alpha)_+^2\,dx$ as follows. By the Gagliardo-Nirenberg inequality (9),

$$\begin{aligned}
\int_{\mathbb{R}^2} (g-\alpha)_+^2\,dx &\leq \frac{1}{4\pi}\Big(\int_{\mathbb{R}^2} |\nabla(g-\alpha)_+|\,dx\Big)^2 = \frac{1}{4\pi}\Big(\int_{g>\alpha} |\nabla g|\,dx\Big)^2 \\
&= \frac{1}{4\pi}\Big(\int_{g>\alpha} \sqrt{1+g}\frac{|\nabla g|}{\sqrt{1+g}}\,dx\Big)^2 \\
&\leq \frac{1}{4\pi}\Big(\int_{g>\alpha} (1+g)\,dx\Big)\Big(\int_{g>\alpha} \frac{|\nabla g|^2}{1+g}\,dx\Big),
\end{aligned}$$

and then,

$$\int_{g>\alpha} (1+g)\,dx = \int_{g>\alpha} dx + \int_{g>\alpha} g\,dx \le \frac{1}{\alpha}\int_{g>\alpha} g\,dx + \int_{g>\alpha} g\,dx = \Big(1+\frac{1}{\alpha}\Big)\int_{g>\alpha} g\,dx.$$

Hence,

$$\int_{\mathbb{R}^2} (g-\alpha)_+^2\,dx \le \frac{1}{4\pi}\Big(1+\frac{1}{\alpha}\Big)\Big(\int_{\mathbb{R}^2} g\,dx\Big)\Big(\int_{\mathbb{R}^2} \frac{|\nabla g|^2}{1+g}\,dx\Big).$$

Therefore,

$$\int_{\mathbb{R}^2} g^2\,dx \le \frac{1}{4\pi}\Big(1+\frac{1}{\alpha}\Big)\Big(\int_{\mathbb{R}^2} g\,dx\Big)\Big(\int_{\mathbb{R}^2} \frac{|\nabla g|^2}{1+g}\,dx\Big) + 2\alpha\int_{\mathbb{R}^2} g\,dx.$$

By putting $\varepsilon = 1/\alpha$, (10) is derived. □

LEMMA 2.4. *It holds that for any nonnegative function $g \in L^1 \cap W^{1,2}$,*

$$\int_{\mathbb{R}^2} g^3\,dx \le \varepsilon\Big(\int_{\mathbb{R}^2} (1+g)\log(1+g)\,dx\Big)\Big(\int_{\mathbb{R}^2} |\nabla g|^2\,dx\Big) + C(\varepsilon)\int_{\mathbb{R}^2} g\,dx,$$

where ε is any positive number and $C(\varepsilon) \to \infty$ $(\varepsilon \to 0)$.

For the proof of Lemma 2.4, see, e.g., [21, Lemma 2.1].

We lastly mention the following interpolation inequality, which is obtained by applying Hölder's inequality: Let $1 \le p_1 < p_2 \le \infty$ and $f \in L^{p_1} \cap L^{p_2}$. Then $f \in L^p$ for all p with $p_1 \le p \le p_2$ and

$$\|f\|_p \le \|f\|_{p_1}^{\lambda}\|f\|_{p_2}^{1-\lambda} \quad \text{where} \quad \frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2},\ 0 \le \lambda \le 1. \tag{11}$$

3. Boundedness of solutions by entropy estimates

Let (u, v_1, v_2) be the nonnegative solution to the Cauchy problem (P) corresponding to the initial data u_0 satisfying (1). For the proof of Theorem 1.1, we need the following proposition, which is proven in Subsection 3.1.

PROPOSITION 3.1. *Let $0 < T \le \infty$ and assume that*

$$E := \sup_{0<t<T} \|(1+u(t))\log(1+u(t))\|_1 < \infty. \tag{12}$$

Then,

$$\|u(t)\|_\infty \le C(\|u_0\|_1, \|u_0\|_\infty, E), \quad 0 < t < T.$$

REMARK 3.2: The assumption $\beta_1 > \beta_2$ is not required for proving Proposition 3.1.

We put $\psi = \beta_1 v_1 - \beta_2 v_2$ in the equations of u and v_j $(j = 1, 2)$ in (P). Then,

$$\partial_t u = \Delta u - \nabla \cdot (u \nabla \psi), \quad -\Delta \psi = (\beta_1 - \beta_2) u + h \quad (t > 0,\ x \in \mathbb{R}^2), \tag{13}$$

where $h = \lambda_2 \beta_2 v_2 - \lambda_1 \beta_1 v_1$. As $v_j = (\lambda_j - \Delta)^{-1} u$ $(j = 1, 2)$, applying Lemma 2.2 as $f = u(t)$, we observe that for $j = 1, 2$ and $t > 0$,

$$\|v_j(t)\|_p \le C(p, q) \|u(t)\|_q,\ 1 \le q \le p < \infty, \tag{14}$$

$$\|v_j(t)\|_\infty \le C(q) \|u(t)\|_q,\ 1 < q \le \infty, \tag{15}$$

$$\|\nabla v_j(t)\|_\infty \le C(q) \|u(t)\|_q,\ 2 < q \le \infty. \tag{16}$$

Here and in what follows, we drop λ_j from $C(\lambda_j, p, q)$ and $C(\lambda_j, q)$ for simplicity. In particular, thanks to (14) for $q = 1$ and $\|u(t)\|_1 = \|u_0\|_1$ by Lemma 2.1, we have

$$\|v_j(t)\|_p \le C(p) \|u_0\|_1,\ 1 \le p < \infty. \tag{17}$$

We give the following lemma for the modified entropy.

LEMMA 3.3. *It holds that*

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^2} (1 + u) \log(1 + u)\, dx + \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\, dx \\ &\quad = -\int_{\mathbb{R}^2} u \Delta \psi\, dx + \int_{\mathbb{R}^2} \log(1 + u)\, \Delta \psi\, dx, \end{aligned} \tag{18}$$

where $\psi = \beta_1 v_1 - \beta_2 v_2$.

Proof. Using $\partial_t u = \Delta u - \nabla \cdot (u \nabla \psi)$ in (13) and noting $\int_{\mathbb{R}^2} \partial_t u\, dx = 0$, by integration by parts, we have that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} (1 + u) \log(1 + u)\, dx &= \int_{\mathbb{R}^2} \partial_t u\, \log(1 + u)\, dx + \int_{\mathbb{R}^2} \partial_t u\, dx \\ &= \int_{\mathbb{R}^2} \Delta u\, \log(1 + u)\, dx - \int_{\mathbb{R}^2} \nabla \cdot (u \nabla \psi)\, \log(1 + u)\, dx \\ &= -\int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\, dx + \int_{\mathbb{R}^2} \frac{u}{1 + u} \nabla u \cdot \nabla \psi\, dx \\ &= -\int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\, dx + \int_{\mathbb{R}^2} \nabla u \cdot \nabla \psi\, dx - \int_{\mathbb{R}^2} \frac{1}{1 + u} \nabla u \cdot \nabla \psi\, dx \\ &= -\int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\, dx - \int_{\mathbb{R}^2} u \Delta \psi\, dx - \int_{\mathbb{R}^2} \nabla \log(1 + u) \cdot \nabla \psi\, dx \\ &= -\int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\, dx - \int_{\mathbb{R}^2} u \Delta \psi\, dx + \int_{\mathbb{R}^2} \log(1 + u) \Delta \psi\, dx. \end{aligned}$$

Thus, we derive (18). □

Proof of Theorem 1.1. Since the nonnegative solution exists globally in time under the assumption $(\beta_1-\beta_2)\int_{\mathbb{R}^2} u_0\,dx < 8\pi$ by [24, Theorem 1.1], all we have to do is to show boundedness under the assumption $(\beta_1-\beta_2)\int_{\mathbb{R}^2} u_0\,dx < 4\pi$ by applying Proposition 3.1 as $T=\infty$.

Since $-\Delta\psi = (\beta_1-\beta_2)u + h$ ($h = \lambda_2\beta_2 v_2 - \lambda_1\beta_1 v_1$) by (13), plugging this relation into the right-hand side of (18) yields that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx + \int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}\,dx \\
&\quad = (\beta_1-\beta_2)\int_{\mathbb{R}^2} u^2\,dx + \int_{\mathbb{R}^2} uh\,dx - (\beta_1-\beta_2)\int_{\mathbb{R}^2} u\log(1+u)\,dx \\
&\qquad - \int_{\mathbb{R}^2}\log(1+u)\,h\,dx \\
&\quad \le (\beta_1-\beta_2)\int_{\mathbb{R}^2} u^2\,dx + \frac{\varepsilon}{2}\int_{\mathbb{R}^2}\left\{u^2 + (\log(1+u))^2\right\}dx \\
&\qquad - (\beta_1-\beta_2)\int_{\mathbb{R}^2} u\log(1+u)\,dx + C(\varepsilon)\int_{\mathbb{R}^2} h^2\,dx,
\end{aligned} \tag{19}$$

where $0<\varepsilon<1$. By $\log(1+u)\le u$, we have that

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_{\mathbb{R}^2}\left\{u^2 + (\log(1+u))^2\right\}dx - (\beta_1-\beta_2)\int_{\mathbb{R}^2} u\log(1+u)\,dx \\
&\le \varepsilon\int_{\mathbb{R}^2} u^2\,dx - (\beta_1-\beta_2)\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx + (\beta_1-\beta_2)\int_{\mathbb{R}^2} u\,dx.
\end{aligned} \tag{20}$$

Substituting (20) into the right-hand side of (19) and using $\|h\|_2^2 \le C\|u_0\|_1^2$ obtained by (17), we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx + \int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}\,dx \\
&\quad \le (\beta_1-\beta_2+\varepsilon)\int_{\mathbb{R}^2} u^2\,dx - (\beta_1-\beta_2)\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx \\
&\qquad + C(\|u_0\|_1,\varepsilon).
\end{aligned} \tag{21}$$

Applying Lemma 2.3 as $g=u(t)$ yields that

$$\int_{\mathbb{R}^2} u^2(t)\,dx \le \frac{1+\varepsilon}{4\pi}\|u_0\|_1\int_{\mathbb{R}^2}\frac{|\nabla u(t)|^2}{1+u(t)}\,dx + \frac{2}{\varepsilon}\|u_0\|_1.$$

Here we used $\|u(t)\|_1 = \|u_0\|_1$. Plugging this inequality into (21), we have that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx \\
&\qquad + \left\{1-(\beta_1-\beta_2+\varepsilon)\frac{1+\varepsilon}{4\pi}\|u_0\|_1\right\}\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}\,dx \qquad (22)\\
&\le -(\beta_1-\beta_2)\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx + C(\|u_0\|_1,\varepsilon).
\end{aligned}$$

Thanks to $(\beta_1-\beta_2)\|u_0\|_1 < 4\pi$ by assumption (3), we can take $0<\varepsilon<1$ such that

$$1-(\beta_1-\beta_2+\varepsilon)\frac{1+\varepsilon}{4\pi}\|u_0\|_1 \ge 0.$$

Hence, it follows from (22) that

$$\|(1+u(t))\log(1+u(t))\|_1 \le e^{-(\beta_1-\beta_2)t}\|(1+u_0)\log(1+u_0)\|_1 + C(\|u_0\|_1), \quad t>0.$$

Therefore, we conclude the boundedness of $\|u(t)\|_\infty$ on $[0,\infty)$ by Proposition 3.1. □

3.1. Proof of Proposition 3.1

The proof of Proposition 3.1 relies on the following lemma, which is proven by Moser's iteration technique (e.g., [1, 14, 26]).

LEMMA 3.4. *Let $0<T\le\infty$ and assume*

$$A := \sup_{0<t<T}\|\nabla(\beta_1 v_1(t)-\beta_2 v_2(t))\|_\infty < \infty.$$

Then, $\|u(t)\|_\infty \le C(\|u_0\|_1,\|u_0\|_\infty,A), \quad 0<t<T.$

To prove Proposition 3.1, we begin with showing

$$\|u(t)\|_2 \le C(\|u_0\|_1,\|u_0\|_2,E), \quad 0<t<T, \tag{23}$$

where $E=\sup_{0<t<T}\|(1+u(t))\log(1+u(t))\|_1$. By (2) for $p=2$, we have that

$$\begin{aligned}
&\frac{d}{dt}\|u(t)\|_2^2 + 2\|\nabla u(t)\|_2^2 - (\beta_1-\beta_2)\|u(t)\|_3^3 \\
&\quad \le \beta_2\lambda_2\int_{\mathbb{R}^2}u^2 v_2\,dx \le \beta_2\lambda_2\|u(t)\|_3^2\|v_2(t)\|_3 \le \beta_2\|u(t)\|_3^3 + C\|v_2(t)\|_3^3,
\end{aligned}$$

from which it follows that

$$\frac{d}{dt}\|u(t)\|_2^2 + 2\|\nabla u(t)\|_2^2 - \beta_1\|u(t)\|_3^3 \le C\|v_2(t)\|_3^3 \le C\|u_0\|_1^3.$$

Here we used $\|v_2(t)\|_3 \le C\|u_0\|_1$ by (17). To control $\|u(t)\|_3$, we recall the following inequality on $\mathbb{R}^2$ (see Lemma 2.4): For any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for any nonnegative function $g \in L^1 \cap W^{1,2}$,

$$\|g\|_3^3 \le \varepsilon \|(1+g)\log(1+g)\|_1 \|\nabla g\|_2^2 + C(\varepsilon)\|g\|_1. \tag{24}$$

Thanks to $E = \sup_{0<t<T} \|(1+u(t))\log(1+u(t))\|_1 < \infty$ by assumption (12), applying (24) as $g = u(t)$ and using $\|u(t)\|_1 = \|u_0\|_1$, we have

$$\|u(t)\|_3^3 \le \varepsilon E \|\nabla u(t)\|_2^2 + C(\varepsilon)\|u_0\|_1, \quad 0 < t < T,$$

and hence,

$$\frac{d}{dt}\|u(t)\|_2^2 + (2 - \varepsilon\beta_1 E)\|\nabla u(t)\|_2^2 \le C(\|u_0\|_1, \varepsilon), \quad 0 < t < T,$$

where $0 < \varepsilon < 1$. Take ε such as $2 - \varepsilon\beta_1 E \ge 1$, that is, $0 < \varepsilon \le 1/(\beta_1 E)$. Then,

$$\frac{d}{dt}\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \le C(\|u_0\|_1, E), \quad 0 < t < T. \tag{25}$$

Applying (8) as $f = u(t)$ and using $\|u(t)\|_1 = \|u_0\|_1$ yield that

$$\|u(t)\|_2^2 \le C\|\nabla u(t)\|_2 \|u_0\|_1 \le \|\nabla u(t)\|_2^2 + C\|u_0\|_1^2.$$

Substituting this inequality into (25), we have that

$$\frac{d}{dt}\|u(t)\|_2^2 + \|u(t)\|_2^2 \le C(\|u_0\|_1, E), \quad 0 < t < T,$$

from which (23) follows.

We next show that

$$\|u(t)\|_4 \le C(\|u_0\|_1, \|u_0\|_4, E), \quad 0 < t < T. \tag{26}$$

By (2) for $p = 4$,

$$\begin{aligned}&\frac{d}{dt}\|u(t)\|_4^4 + 3\|\nabla u^2(t)\|_2^2 - 3(\beta_1 - \beta_2)\|u(t)\|_5^5 \\ &\le 3\beta_2\lambda_2 \int_{\mathbb{R}^2} u^4(t)v_2(t)\,dx \le 3\beta_2\lambda_2\|u(t)\|_5^4\|v_2(t)\|_5 \le 3\beta_2\|u(t)\|_5^5 + C\|v_2(t)\|_5^5.\end{aligned}$$

Putting $w = u^2$ yields that

$$\frac{d}{dt}\|w(t)\|_2^2 + 3\|\nabla w(t)\|_2^2 - 3\beta_1\|w(t)\|_{5/2}^{5/2} \le C\|u_0\|_1^5.$$

Here we used $\|v_2(t)\|_5 \le C\|u_0\|_1$ by (17). Applying the Gagliardo-Nirenberg inequality (8) for $p = 5/2$ and using Young's inequality, we have that

$$\|w(t)\|_{5/2}^{5/2} \le C\|\nabla w(t)\|_2^{3/2}\|w(t)\|_1 \le \eta\|\nabla w(t)\|_2^2 + C(\eta)\|w(t)\|_1^4,$$

where η is a positive number determined later. Hence, for $0 < t < T$,

$$\frac{d}{dt}\|w(t)\|_2^2 + 3(1-\beta_1\eta)\|\nabla w(t)\|_2^2 \le 3\beta_1 C(\eta)\|w(t)\|_1^4 + C\|u_0\|_1^5. \tag{27}$$

Take $\eta > 0$ such that $3(1-\beta_1\eta) \ge 1$ and note that by (23),

$$\|w(t)\|_1 = \|u(t)\|_2^2 \le C(\|u_0\|_1, \|u_0\|_2, E), \quad 0 < t < T.$$

Then, as in the proof of the boundedness of $\|u(t)\|_2$, we derive from (27) that

$$\|u(t)\|_4^4 = \|w(t)\|_2^2 \le C(\|u_0\|_1, \|u_0\|_4, E), \quad 0 < t < T.$$

Here we used the fact that $\|u_0\|_2$ is estimated by $\|u_0\|_1$ and $\|u_0\|_4$ by virtue of interpolation inequality (11). Thus, (26) is derived.

By (15) for $q = 2$ and (23),

$$\|v_j(t)\|_\infty \le C\|u(t)\|_2 \le C(\|u_0\|_1, \|u_0\|_2, E), \quad 0 < t < T,$$

and by (16) for $q = 4$ and (26),

$$\|\nabla v_j(t)\|_\infty \le C\|u(t)\|_4 \le C(\|u_0\|_1, \|u_0\|_4, E), \quad 0 < t < T.$$

Hence, since $\|\nabla(\beta_1 v_1(t) - \beta_2 v_2(t))\|_\infty \le C(\|u_0\|_1, \|u_0\|_4, E)$ $(0 < t < T)$, Lemma 3.4 ensures that

$$\|u(t)\|_\infty \le C(\|u_0\|_1, \|u_0\|_\infty, E), \quad 0 < t < T.$$

Thus, we establish the assertion of Proposition 3.1.

4. Boundedness of radial solutions

In this section, we assume that the nonnegative initial data u_0 satisfying (1) is radial in x. Then, by the uniqueness of solutions to the Cauchy problem (P), the nonnegative solution (u, v_1, v_2) corresponding to the initial data u_0 is radial in x.

Define the functions $\tilde{u}(t,s)$ and $\tilde{v}_j(t,s)$ $(j = 1, 2)$ by

$$u(t,x) = \tilde{u}(t,s), \quad v_j(t,x) = \tilde{v}_j(t,s), \quad s = \pi|x|^2$$

and $\tilde{u}_0(s)$ by $u_0(x) = \tilde{u}_0(s)$. We next define U and V_j $(j = 1, 2)$ by

$$U(t,s) = \int_0^s \tilde{u}(t,\sigma)\,d\sigma, \quad V_j(t,s) = \int_0^s \tilde{v}_j(t,\sigma)\,d\sigma \tag{28}$$

and $U_0(s) = \int_0^s \tilde{u}_0(\sigma)\,d\sigma$. By Lemma 2.1, we observe that

$$U(t,\infty) = \int_0^\infty \tilde{u}(t,s)\,ds = 2\pi \int_0^\infty \tilde{u}(t,\pi r^2)\,rdr = \int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx,$$

and

$$V_j(t,\infty) = \int_0^\infty \tilde{v}_j(t,s)\,ds = \int_{\mathbb{R}^2} v_j(t,x)\,dx = \frac{1}{\lambda_j}\int_{\mathbb{R}^2} u_0(x)\,dx.$$

Lemma 4.1. *It holds that*

$$\partial_t U = 4\pi s\partial_s^2 U + (\beta_1-\beta_2)U\partial_s U - (\beta_1\lambda_1 V_1 - \beta_2\lambda_2 V_2)\partial_s U \quad (t>0,\ s>0). \tag{29}$$

Proof. Calculating that

$$\begin{aligned}
&\partial_{x_j} u = \partial_s \tilde{u}\partial_{x_j} s = 2\pi x_j \partial_s \tilde{u}, \quad \partial_{x_j}^2 u = 4\pi^2 x_j^2 \partial_s^2 \tilde{u} + 2\pi\partial_s \tilde{u},\\
&\Delta u = 4\pi s\partial_s^2 \tilde{u} + 4\pi\partial_s \tilde{u} = 4\pi\partial_s(s\partial_s \tilde{u}),\\
&\nabla\cdot(u\nabla v_j) = 4\pi s\partial_s(\tilde{u}\partial_s \tilde{v}_j) + 4\pi\tilde{u}\partial_s\tilde{v}_j = 4\pi\partial_s(s\tilde{u}\partial_s\tilde{v}_j),
\end{aligned}$$

we have

$$\partial_t \tilde{u} = 4\pi\partial_s(s\partial_s\tilde{u}) - 4\pi\partial_s(s\tilde{u}\partial_s(\beta_1\tilde{v}_1 - \beta_2\tilde{v}_2)), \tag{30}$$

$$0 = 4\pi\partial_s(s\partial_s\tilde{v}_j) - \lambda_j\tilde{v}_j + \tilde{u} \quad (j=1,2). \tag{31}$$

Integrating (30) and (31) with respect to s, we have that

$$\begin{aligned}
\partial_t U &= 4\pi s\partial_s\tilde{u} - 4\pi s\tilde{u}\partial_s(\beta_1\tilde{v}_1 - \beta_2\tilde{v}_2)\\
&= 4\pi s\partial_s^2 U - \partial_s U\{4\pi s\partial_s(\beta_1\tilde{v}_1 - \beta_2\tilde{v}_2)\},
\end{aligned}$$

$$4\pi s\partial_s\tilde{v}_j = -U + \lambda_j V_j \quad (j=1,2).$$

Hence,

$$\partial_t U = 4\pi s\partial_s^2 U + (\beta_1-\beta_2)U\partial_s U - (\beta_1\lambda_1 V_1 - \beta_2\lambda_2 V_2)\partial_s U.$$

□

To obtain the boundedness of the solution (u, v_1, v_2), by Lemma 3.4 it suffices to show that

$$\sup_{t>0}\|\nabla v_j(t)\|_\infty < \infty \quad (j=1,2). \tag{32}$$

Thanks to $4\pi s\partial_s\tilde{v}_j = \lambda_j V_j - U$ and $s = \pi|x|^2$, we have that

$$|\nabla v_j(t,x)| = 2\pi|x||\partial_s\tilde{v}_j(t,s)| = \frac{1}{2\sqrt{\pi s}}|U(t,s) - \lambda_j V_j(t,s)|. \tag{33}$$

By Hölder's inequality we observe that for $0 < \lambda < 1$,

$$\begin{aligned} 0 \le V_j(t,s) &= \int_0^s \tilde{v}_j(t,\sigma)\,d\sigma \le s^\lambda\Big(\int_0^\infty |\tilde{v}_j(t,\sigma)|^{1/(1-\lambda)}\,d\sigma\Big)^{1-\lambda} \\ &= s^\lambda\|v_j(t)\|_{1/(1-\lambda)} \le C(\lambda)\|u_0\|_1 s^\lambda. \end{aligned} \tag{34}$$

Here we used $\|v_j(t)\|_{1/(1-\lambda)} \le C(\lambda)\|u_0\|_1$ $(t > 0)$ by (17). Since $0 \le V_j(t,s) \le C\|u_0\|_1\sqrt{s}$ by (34) for $\lambda = 1/2$, we have the following lemma by virtue of (33).

LEMMA 4.2. *If there exist $s_0 > 0$ and $C > 0$ such that*

$$U(t,s) \le C\sqrt{s} \quad (t \ge 0,\ 0 \le s \le s_0),$$

then (32) *is satisfied. Hence,* $\sup_{t>0}\|(u(t), v_1(t), v_2(t))\|_\infty < \infty$.

Proof of Theorem 1.2. By (29) and $\partial_s U \ge 0$,

$$\partial_t U \le 4\pi s\partial_s^2 U + \beta U\partial_s U + \beta_2\lambda_2 V_2\partial_s U,$$

where $\beta = \beta_1 - \beta_2 > 0$. As $V_2(t,s) \le C(\lambda)\|u_0\|_1 s^\lambda$ for $0 < \lambda < 1$ by (34), putting $B(\lambda) = \beta_2\lambda_2 C(\lambda)\|u_0\|_1$, we have that

$$\partial_t U \le 4\pi s\partial_s^2 U + \beta U\partial_s U + B(\lambda)s^\lambda\partial_s U, \quad t > 0,\ s > 0.$$

In what follows, for simplicity we put

$$\mathcal{N}g = 4\pi s\partial_s^2 g + \beta g\partial_s g + B(\lambda)s^\lambda\partial_s g, \tag{35}$$

where $0 < \lambda < 1$. We then get the following:

$$\begin{cases} \partial_t U \le \mathcal{N}U, \quad t > 0,\ s > 0, \\ U(t,0) = 0,\ U(t,\infty) = \|u_0\|_1, \quad t > 0, \\ U(0,s) = U_0(s), \quad s \ge 0. \end{cases}$$

For $b > 0$ and $\gamma > 0$, we define $W_{b,\gamma}(s)$ by

$$W_{b,\gamma}(s) = \frac{8\pi}{\gamma}\frac{s}{s+b} \quad (s \ge 0).$$

The function satisfies

$$4\pi s \frac{d^2 W_{b,\gamma}}{ds^2} + \gamma W_{b,\gamma} \frac{dW_{b,\gamma}}{ds} = 0 \quad (s > 0). \tag{36}$$

As $\beta\|u_0\|_1 < 8\pi$ by assumption (5), we can choose γ and λ in (35) such that

$$1 < \frac{\gamma}{\beta} < \min\Big\{\frac{8\pi}{\beta\|u_0\|_1}, 2\Big\}, \quad \Big(\frac{1}{2} <\Big)\frac{\beta}{\gamma} < \lambda < 1,$$

and as a comparison function we take

$$W(s) = W_{b,\gamma}(s^\lambda) \quad (s \geq 0).$$

Take $b > 0$ so small that

$$s_0 := \Big(\frac{\gamma\|u_0\|_1 b}{8\pi - \gamma\|u_0\|_1}\Big)^{1/\lambda} < 1, \quad \frac{\lambda\gamma - \beta}{\gamma} \cdot \frac{8\pi - \gamma\|u_0\|_1}{b} \geq B(\lambda). \tag{37}$$

Here we used $\|u_0\|_1 < 8\pi/\gamma = W_{b,\gamma}(\infty)$ and $\lambda\gamma > \beta$. Since $W_{b,\gamma}(s)$ is decreasing in b and converges to $8\pi/\gamma$ as $b \to +0$ for each $s > 0$ and $W'_{b,\gamma}(0) = 8\pi/(\gamma b)$ where $' = d/ds$, we can shorten b such that

$$U_0(s) \leq \|u_0\|_\infty s < W_{b,\gamma}(s) \quad \text{for} \quad 0 < s \leq s_0.$$

By $W(s_0) = W_{b,\gamma}(s_0^\lambda) = \|u_0\|_1$, it is apparent that

$$U(t, s_0) \leq \|u_0\|_1 = W(s_0) \quad \text{for} \quad t \geq 0.$$

As $W_{b,\gamma}(s)$ is increasing in s and $0 < s < s^\lambda$ for $0 < s \leq s_0 (< 1)$, we observe that for $0 < s \leq s_0$,

$$U_0(s) < W_{b,\gamma}(s) < W_{b,\gamma}(s^\lambda) = W(s).$$

Since

$$\frac{dW}{ds} = \lambda s^{\lambda-1}\frac{dW_{b,\gamma}}{ds}(s^\lambda), \quad s\frac{d^2W_{b,\gamma}}{ds^2} = -\frac{\gamma}{4\pi}W_{b,\gamma}\frac{dW_{b,\gamma}}{ds},$$

we have

$$\begin{aligned}
\frac{d^2W}{ds^2} &= \lambda^2 s^{\lambda-2} \cdot s^\lambda \frac{d^2W_{b,\gamma}}{ds^2}(s^\lambda) - \lambda(1-\lambda)s^{\lambda-2}\frac{dW_{b,\gamma}}{ds}(s^\lambda) \\
&= -\frac{\lambda^2\gamma}{4\pi}s^{\lambda-2}W_{b,\gamma}(s^\lambda)\frac{dW_{b,\gamma}}{ds}(s^\lambda) - \lambda(1-\lambda)s^{\lambda-2}\frac{dW_{b,\gamma}}{ds}(s^\lambda) \\
&= -\frac{\lambda\gamma}{4\pi}s^{-1}W\frac{dW}{ds} - (1-\lambda)s^{-1}\frac{dW}{ds},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}W &= 4\pi s\frac{d^2W}{ds^2} + \beta W\frac{dW}{ds} + B(\lambda)s^\lambda\frac{dW}{ds}\\
&= -(\lambda\gamma-\beta)W\frac{dW}{ds} - 4\pi(1-\lambda)\frac{dW}{ds} + B(\lambda)s^\lambda\frac{dW}{ds}\\
&= -4\pi(1-\lambda)\frac{dW}{ds} - s^\lambda\left\{(\lambda\gamma-\beta)\frac{8\pi}{\gamma}\frac{1}{s^\lambda+b} - B(\lambda)\right\}\frac{dW}{ds}.
\end{aligned}$$

Let $0 < s < s_0$, where s_0 is given by (37). As $\lambda\gamma > \beta$, we observe that

$$\begin{aligned}
(\lambda\gamma-\beta)\frac{8\pi}{\gamma}\frac{1}{s^\lambda+b} - B(\lambda) &\geq (\lambda\gamma-\beta)\frac{8\pi}{\gamma}\frac{1}{s_0^\lambda+b} - B(\lambda)\\
&= \frac{\lambda\gamma-\beta}{\gamma}\cdot\frac{8\pi-\gamma\|u_0\|_1}{b} - B(\lambda)\\
&\geq 0.
\end{aligned}$$

Hence, $\mathcal{N}W < 0$ $(0 < s < s_0)$ because of $dW/ds > 0$. Therefore,

$$\begin{cases}
\partial_t U \leq \mathcal{N}U, \quad \mathcal{N}W < 0\ (t > 0,\ 0 < s < s_0),\\
U(t,0) = W(0) = 0, \quad U(t,s_0) \leq W(s_0)\ (t \geq 0),\\
U(0,s) = U_0(s) \leq W(s)\ (0 \leq s \leq s_0).
\end{cases}$$

Then, the comparison principle ensures that

$$U(t,s) \leq W(s) \leq \frac{8\pi}{\gamma}\frac{s^\lambda}{b} \quad (t \geq 0,\ 0 \leq s \leq s_0).$$

Therefore, as $1/2 < \lambda < 1$, we establish Theorem 1.2 thanks to Lemma 4.2. □

Proof of Theorem 1.3. Let $U(t,s)$ and $V_j(t,s)$ $(j = 1,2)$ be the functions defined by (28). We claim that

$$V_1(t,s) \geq V_2(t,s) \quad (t > 0,\ s \geq 0).$$

In fact, as $v_j \geq 0$ $(j = 1,2)$ and $\lambda_1 \leq \lambda_2$, by the equations for v_j $(j = 1,2)$,

$$-\Delta(v_1 - v_2) + \lambda_1(v_1 - v_2) = (\lambda_2 - \lambda_1)v_2 \geq 0 \quad \text{in } \mathbb{R}^2.$$

By the maximum principle, we have $v_1 \geq v_2$ on $\mathbb{R}^2$. Thus $V_1 \geq V_2$.

By Lemma 4.1,

$$\partial_t U - 4\pi s\partial_s^2 U - \beta U\partial_s U = (\beta_2\lambda_2 V_2 - \beta_1\lambda_1 V_1)\partial_s U,$$

where $\beta = \beta_1 - \beta_2$. It follows from $V_1 \geq V_2$ and $\beta_1\lambda_1 \geq \beta_2\lambda_2$ that

$$\beta_2\lambda_2 V_2 - \beta_1\lambda_1 V_1 = \beta_2\lambda_2(V_2 - V_1) + (\beta_2\lambda_2 - \beta_1\lambda_1)V_1 \leq 0.$$

Hence, as $\partial_s U \geq 0$, we have

$$\partial_t U - 4\pi s \partial_s^2 U - \beta U \partial_s U \leq 0 \quad (t > 0,\ s > 0).$$

We note that assumption (7) is equivalent to

$$\liminf_{s\to\infty} \left(s \int_s^\infty \tilde{u}_0 \, d\sigma \right) > 0, \tag{38}$$

where $\tilde{u}_0$ is defined by $u_0(x) = \tilde{u}_0(s)$, $s = \pi|x|^2$. As in the proof of [23, Lemma 3.1.(ii)], by assumption (6) and (38) we can choose $b > 0$ such that

$$U(0,s) = \int_0^s \tilde{u}_0 \, d\sigma \leq W_{b,\beta}(s) = \frac{8\pi}{\beta} \frac{s}{s+b} \quad (s \geq 0).$$

Since $U(t,0) = W_{b,\beta}(0) = 0$, $U(t,\infty) = W_{b,\beta}(\infty) = 8\pi/\beta$ $(t > 0)$ and

$$4\pi s \frac{d^2 W_{b,\beta}}{ds^2} + \beta W_{b,\beta} \frac{dW_{b,\beta}}{ds} = 0 \quad (s > 0)$$

by (36), the comparison principle ensures that

$$U(t,s) \leq W_{b,\beta}(s) = \frac{8\pi}{\beta} \frac{s}{s+b} \quad (t > 0,\ s \geq 0).$$

Therefore, Theorem 1.3 is established by Lemma 4.2. □

References

[1] N. D. Alikakos, *L^p bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations **4** (1979), 827–868.

[2] P. Biler and T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interaction of particles, I*, Colloq. Math. **66** (1994), 319–334.

[3] A. Blanchet, E. Carlen, and J. A. Carrillo, *Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model*, J. Funct. Anal. **262** (2012), 2142–2230.

[4] A. Blanchet, J. A. Carrillo, and N. Masmoudi, *Infinite time aggregation for the critical Paltak-Keller-Segel model in $\mathbb{R}^2$*, Comm. Pure Appl. Math. **61** (2008), 1449–1481.

[5] A. Blanchet, J. Dolbeault, and B. Perthame, *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations **2006** (2006), 1–33.

[6] J. I. Diaz, T. Nagai, and J. M. Rakotoson, *Symmetrization techniques on unbounded domains: application to a chemotaxis system on* $\mathbb{R}^n$, J. Differential Equations **145** (1998), 156–183.
[7] E. Espejo and T. Suzuki, *Global existence and blow-up for a system describing the aggregation of microglia*, Appl. Math. Lett. **35** (2014), 29–34.
[8] M. H. Giga, Y. Giga, and J. Saal, *Nonlinear Partial Differential Equations: Asymptotic Behavior of Solutions and Self-Similar Solutions*, vol. 79, Birkhäuser Boston, Ltd., Boston, MA, 2010.
[9] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, vol. 840, Springer, Berlin-New York, 1981.
[10] T. Hillen and K. J. Painter, *A user's guide to pde models for chemotaxis*, J. Math. Biol. **58** (2009), 183–217.
[11] H. Y. Jin, *Boundedness of the attraction-repulsion Keller-Segel system*, J. Math. Anal. Appl. **422** (2015), 1463–1478.
[12] H. Y. Jin and Z. Liu, *Large time behavior of the full attraction-repulsion Keller-Segel system in the whole space*, Appl. Math. Lett. **47** (2015), 13–20.
[13] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), 399–415.
[14] M. Kurokiba, T. Nagai, and T. Ogawa, *The uniform boundedness and threshold for the global existence of the radial solution to a drift-diffusion system*, Commun. Pure Appl. Anal. **5** (2006), 97–106.
[15] M. Kurokiba and T. Ogawa, *Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type*, Differential Integral Equations **16** (2003), 427–452.
[16] K. Lin, C. Mu, and L. Wang, *Large-time behavior of an attraction-repulsion chemotaxis system*, J. Math. Anal. Appl. **426** (2015), 105–124.
[17] D. Liu and Y. Tao, *Global boundedness in a fully parabolic attraction-repulsion chemotaxis model*, Math. Methods Appl. Sci. **38** (2015), 2537–2546.
[18] J. López-Gómez, T. Nagai, and T. Yamada, *The basin of attraction of the steady-states for a chemotaxis model in* $\mathbb{R}^2$ *with critical mass*, Arch. Ration. Mech. Anal. **207** (2013), 159–184.
[19] M. Luca, A. Chavez-Ross, L. Edelstein-Keshet, and A. Mogilner, *Chemotactic signaling, microglia, and alzheimer's disease senile plaques: Is there a connection?*, Bull. Math. Biol. **65** (2003), 693–730.
[20] T. Nagai, *Global existence and decay estimates of solutions to a parabolic-elliptic system of drift-diffusion type in* $\mathbb{R}^2$, Differential Integral Equations **24** (2011), 29–68.
[21] T. Nagai and T. Ogawa, *Brezis-Merle inequalities and application to the global existence of the Cauchy problem of the Keller-Segel system*, Commun. Contemp. Math. **13** (2011), 795–812.
[22] T. Nagai and T. Ogawa, *Global existence of solutions to a parabolic-elliptic system of drift-diffusion type in* $\mathbb{R}^2$, Funkcial. Ekvac. **59** (2016), 67–112.
[23] T. Nagai and T. Yamada, *Boundedness of solutions to a parabolic-elliptic Keller-Segel equation in* $\mathbb{R}^2$ *with critical mass*, Adv. Nonlinear Stud. **18** (2018), 337–360.
[24] T. Nagai and T. Yamada, *Global existence of solutions to the Cauchy problem*

for an attraction-repulsion chemotaxis system in $\mathbb{R}^2$ *in the attractive dominant case*, J. Math. Anal. Appl. **462** (2018), 1519–1535.

[25] T. Nagai and T. Yamada, *Global existence of solutions to a two-dimensional attraction-repulsion chemotaxis system in the attractive dominant case with critical mass*, Nonlinear Anal. **190** (2020), 111615, 25pp.

[26] R. Shi and W. Wang, *Well-posedness for a model derived from an attraction-repulsion chemotaxis system*, J. Math. Anal. Appl. **423** (2015), 497–520.

[27] E. M. Stein, *Singular Integrals and Differentiability of Functions*, Princeton Mathematical Series, vol. 30, Princeton University Press, Princeton N. J, 1970.

[28] Y. Tao and Z. A. Wang, *Competing effects of attraction vs. repulsion in chemotaxis*, Math. Models Methods Appl. Sci. **23** (2013), 1–36.

[29] D. Wei, *Global well-posedness and blow-up for the* 2*-D Patlak-Keller-Segel equation*, J. Funct. Anal. **274** (2018), 388–401.

[30] W. P. Ziemer, *Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation*, Graduate Texts in Mathematics, vol. 120, Springer, Berlin-New York, 1989.

Authors' addresses:

Toshitaka Nagai
Department of Mathematics
Hiroshima University
Higashi-Hiroshima, 739-8526, Japan
E-mail: `tnagai@hiroshima-u.ac.jp`

Tetsuya Yamada
Course of General Education (Science)
National Institute of Technology
Fukui College
Sabae, Fukui 916-8507, Japan
E-mail: `yamada@fukui-nct.ac.jp`

Received November 20, 2019
Revised June 3, 2020
Accepted June 5, 2020