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Stability and periodicity of solutions to the Oldroyd-B model on exterior domains

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Dedicated to our good friend Julian Lopez-Gomez on the occasion of his 60^{th}-Birthday

Abstract. *Consider the Oldroyd-B system on exterior domains with nonzero external forces f. It is shown that this system admits under smallness assumptions on f a bounded, global solution (u, τ), which is stable in the sense that any other global solution to this system starting in a sufficiently small neighborhood of $(u(0), \tau(0))$ is tending to (u, τ). In addition, if the outer force is T-periodic and small enough, the Oldroyd-B system admits a T-periodic solution. Note that no smallness condition on the coupling coefficient is assumed.*

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1. Introduction

In this note we consider stability and periodicity questions related to viscoelastic fluids of Oldroyd-B type with non vanishing external forces on exterior domains. This type of fluids are described by the following set of equations

$$\begin{cases} \mathrm{Re}(u_t+(u\cdot\nabla)u)-(1-\alpha)\Delta u+\nabla p &= \operatorname{div}\tau+f \quad \text{in } \Omega\times(0,\infty),\\ \nabla\cdot u &= 0 \quad \text{in } \Omega\times(0,\infty),\\ \mathrm{We}(\tau_t+(u\cdot\nabla)\tau+g_a(\tau,\nabla u))+\tau &= 2\alpha D(u) \quad \text{in } \Omega\times(0,\infty),\\ u &= 0 \quad \text{on } \partial\Omega\times(0,\infty),\\ u(0) &= u_0 \quad \text{in } \Omega,\\ \tau(0) &= \tau_0 \quad \text{in } \Omega. \end{cases} \tag{1}$$

Here $\Omega \subset \mathbb{R}^3$ denotes a domain with smooth boundary $\partial\Omega$, u the velocity of the fluid, and the tensor τ represents the elastic part of the stress tensor. Furthermore, Re and We denote the Reynolds and Weissenberg number of the fluid, respectively. The term g_a is given by

$$g_a(\tau, \nabla u) := \tau W(u) - W(u)\tau - a(D(u)\tau + \tau D(u)) \tag{2}$$

for some $a \in [-1,1]$ and $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and $W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$ denote the deformation and vorticity tensors, respectively. The constant $\alpha \in (0,1)$ is the coupling coefficient between the two equations and represents in particular the strengthness of the coupling between the parabolic fluid type equation for u and the hyperbolic transport type equation for τ.

This set of equations has been introduced first by J.G. Oldroyd [24] and the analysis of this set of equations for viscoelastic fluids gained a lot of attention since then.

If $\Omega \subset \mathbb{R}^3$ is a *bounded domain* with smooth boundary, Guillopé and Saut [13] proved the existence and uniqueness and exponential stability of small solutions to (1) in the case of small coupling parameters α. They further proved the existence of periodic and stationary solutions to (1) by adapting Serrin's method to this situation. For extensions of this results to the L^p-setting we refer to the work of Fernandez-Cara et al [9]. Molinet and Talhouk [23] extended the result of Guillopé and Saut [13] to the case of non small coupling parameters $\alpha \in (0,1)$. For results concerning the critical L^p-framework, we refer to the work of Zi, Fang, and Zhang [25].

For the case $\Omega = \mathbb{R}^3$, Lions and Masmoudi [21] proved the existence of global weak solutions provided $a = 0$. For further results in this direction we refer to the works [4] and [19]. Blow-up criteria for Oldroyd-B type fluids were developed by Kupfermann, Mangoubi and Titi [18] in the case where the Navier-Stokes equation is replaced by the stationary Stokes system and in the general case by Lei, Masmoudi and Zhou [20] as well as by Feng, Zhu and Zi [8]. For global regularity results in the two dimensional setting, we refer to the work of Constantin and Kriegl [5].

If $\Omega \subset \mathbb{R}^3$ is an *exterior domain*, existence and uniqueness of solutions to (1) for small data were proved by Hieber, Naito and Shibata in [14] for small coupling parameter α and by Fang, Hieber and Zi in [7] for any $\alpha \in (0,1)$. For optimal decay rates for the case $\Omega = \mathbb{R}^3$, see [16].

For recent results on ill-posedness of these equations within the L^∞-setting we refer to the work of Elgindi and Masmoudi [6].

In this article we are interested in the global existence, stability and periodicity of solutions to the Oldroyd-B equations in *exterior domains* in the presence of external forces f of the form $f = \operatorname{div} F$ for certain F. One might think of applying the method developed in [11] to the given situation, however, it is unclear whether the Oldroyd semigroup constructed in [10] satisfies suitable decay estimates.

Note that the methods for obtaining results on stability, bifurcation and periodicity of solutions for viscoelastic fluids are quite different from the ones often used in the theory of second order parabolic equations, where comparison principles allow to develop a very rich and powerful theory. For beautiful results in this direction, we refer to the work of Julian Lopez-Gomez and mention here only his book [22] as well as the recent articles [2] and [1].

2. Existence of Bounded Solutions

We consider the Oldroyd-B equation with an external force f of the form $f = \operatorname{div} F$

$$\begin{cases} u_t + (u\cdot\nabla)u - (1-\alpha)\Delta u + \nabla p &= \operatorname{div}\tau + \operatorname{div} F \quad \text{in } \Omega\times(0,\infty), \\ \nabla\cdot u &= 0 \quad \text{in } \Omega\times(0,\infty), \\ \tau_t + (u\cdot\nabla)\tau + g_a(\tau,\nabla u) + \tau &= 2\alpha D(u) \quad \text{in } \Omega\times(0,\infty), \\ u &= 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ u(0) &= u_0 \quad \text{in } \Omega, \\ \tau(0) &= \tau_0 \quad \text{in } \Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^3$ is an exterior domain with boundary of class C^3. Let $A := -\mathbb{P}\Delta$ be the Stokes operator in the solenoidal space $L^2_\sigma(\Omega)$ with domain $D(A) = H^2(\Omega)\cap H^1_0(\Omega)\cap L^2_\sigma(\Omega)$ and set $V := H^1_0(\Omega)\cap L^2_\sigma(\Omega)$. For fixed $T > 0$ we put

$$\begin{aligned} E_1(T) &:= L^2(0,T;H^3(\Omega)) \cap L^\infty(0,T;D(A)), \\ E_2(T) &:= L^2(0,T;V)\cap L^\infty(0,T;L^2_\sigma(\Omega)), \\ G_1(T) &:= L^\infty(0,T;H^2(\Omega)), \\ G_2(T) &:= L^\infty(0,T;H^1(\Omega)). \end{aligned}$$

Our first result concerns the local existence of a unique, strong solution to (3) under certain conditions on F.

PROPOSITION 2.1 (Local Existence). *Let Ω be an exterior domain with C^3-boundary and let $u_0 \in D(A)$ and $\tau_0 \in H^2(\Omega)$. Then there exist $T_* > 0$ and $M > 0$ such that for $F \in G_1(T_*)$ and $F' \in G_2(T_*)$ with $\|F\|_{G_1(T_*)} + \|F'\|_{G_2(T_*)} < M$, equation (3) has a unique solutions (u,p,τ) on $(0,T_*)$ with*

$$\begin{aligned} u &\in E_1(T_*)\cap C([0,T_*],D(A)), \\ u' &\in E_2 \cap C([0,T_*],D(A))), \\ p &\in L^2(0,T_*;H^2_{loc}(\Omega)) \text{ with } \nabla p \in L^2(0,T_*;H^1(\Omega)), \\ \tau &\in C([0,T_*];H^2(\Omega)) \text{ with } \tau' \in C([0,T_*];H^1(\Omega)). \end{aligned}$$

In order to prove Proposition 2.1 we make use of the following version of Banach's fixed point theorem, see [17].

LEMMA 2.2 ([17]). *Let X be either reflexive Banach space or have a separable pre-dual. Let K be a convex, closed and bounded subset of X and assume that X is continuously embedded into a Banach space Y. Let $\Phi : X \to X$ maps K into K and assume there is $0 < q < 1$ such that*

$$\|\Phi(x) - \Phi(y)\|_Y \leqslant q\|x - y\|_Y \quad \textit{for all } x, y \in K.$$

Then there exists a unique fixed point of Φ in K.

Proof of Proposition 2.1. The proof follows the strategy described in [7, Prop. 3.1], however with a forcing term of the form $f = \operatorname{div} F$. For the reader's convenience we give here a short outline of the proof. For real numbers $B_1, B_2 > 0$ we set

$$\begin{aligned} K(T) := \{&(v,\theta) \in E_1(T) \times G_1(T) : v' \in E_2(T), \theta' \in G_2(T), v(0) = u_0, \theta(0) = \tau_0 \\ &\text{and } \|v\|^2_{E_1(T)} + \|v'\|^2_{E_2(T)} \leqslant B_1, \|\theta\|_{G_1(T)} \leqslant B_1, \|\theta'\|_{G_2(T)} \leqslant B_2\} \end{aligned}$$

Then, for $(v, \theta) \in K(T)$ we define the mapping

$$\Phi(v, \theta) := (u, \tau),$$

where (u, τ) is the unique solution of the linearized problem of (3), i.e.,

$$\left\{ \begin{aligned} u_t + (1-\alpha)Au &= -\mathbb{P}\mathrm{div}\,(v \otimes v) + \mathbb{P}\mathrm{div}\,\theta + \mathbb{P}\mathrm{div}\,F && \text{in } \Omega \times (0, \infty), \\ \tau_t + (u \cdot \nabla)\tau + \tau &= 2\alpha D(v) - g_a(\tau, \nabla v)) && \text{in } \Omega \times (0, \infty), \\ u &= 0 && \text{on } \partial\Omega \times (0, \infty), \\ u(0) &= u_0 && \text{in } \Omega, \\ \tau(0) &= \tau_0 && \text{in } \Omega. \end{aligned} \right. \tag{4}$$

Regularity results for the Stokes and the transport equation imply the existence of a constant $C > 0$ such that

$$\begin{aligned} &\|u\|^2_{L^2(H^3)\cap L^\infty(D(A))} + \|u'\|^2_{L^2(V)\cap L^\infty(L^\infty_\sigma)} \\ &\quad\leqslant C\big[\|u_0\|^2_{H^2} + \|v(0)\|^2_{H^2} + \|v\|^2_{L^2(H^3)} + \|v'\|_{L^\infty(H^1)}\|v\|_{L^2(H^3)} \\ &\qquad + \|\theta + F\|_{L^\infty(H^2)} + \|\theta' + F'\|_{L^\infty(H^1)}\big] \end{aligned}$$

and

$$\|\tau\|_{L^\infty(H^2)} + \|\tau'\|_{L^\infty(H^1)} \leqslant [2 + C\|v\|_{L^\infty(H^2)}]\Big(\|\tau_{H^2} + \frac{2\alpha}{C}\Big) \exp C\|v\|_{L^1(H^3)}.$$

Hence, choosing B_1, B_2 and T_1 appropriately, we see that Φ maps $K(T_1)$ into $K(T_1)$.

Next, similarly as in [7], for two solutions (u_i, τ_i) corresponding to given (v_i, θ_i) for $i = 1, 2$ we verify that

$$\|u_1 - u_2\|^2_{L^\infty(L^2)} + \|\tau_1 - \tau_2\|^2_{L^\infty(L^2)} + \int_0^T (\|\nabla u_1 - \nabla u_2\|^2_{L^2} + \|\tau_1 - \tau_2\|^2_{L^2}) dt$$
$$\leqslant \frac{1}{4}\Big(\|v_1 - v_2\|^2_{L^\infty(L^2)} + \|\theta_1 - \theta_2\|^2_{L^\infty(L^2)} + \int_0^T (\|\nabla v_1 - \nabla v_2\|^2_{L^2} + \|\theta_1 - \theta_2\|^2_{L^2}) dt \Big)$$

provided $T \leqslant T_* := \min\left\{T_1, \frac{\delta}{1+2B_1^2}, \frac{1}{B_1}, \frac{1-\alpha}{4C(1+2B_1)(1+2C\exp(2C))}\right\}$ with $\delta := \frac{1-\alpha}{4+8C\exp(2C)}$. Therefore, the mapping Φ is a contraction from

$$Y(T_*) := \left\{(v, \theta) \in L^\infty(0, T; L^2(\Omega))^2, \nabla v \in L^\infty(0, T; L^2(\Omega))\right\}$$

into itself and the assertion of Proposition 2.1 follows from Lemma 2.2. □

Our global existence result to (3) in the presence of outer forces f of the form $f = \operatorname{div} F$ reads as follows.

THEOREM 2.3 (Global Existence). *Let $F \in L^\infty(0, \infty; H^2(\Omega))$ such that $F' \in L^\infty(0, \infty; H^1(\Omega))$. Then there exists $\varepsilon_0 > 0$ such that if*

$$\|u_0\|_{D(A)} + \|\tau_0\|_{H^2} < \varepsilon_0 \text{ and}$$
$$\max\{\|F\|_{L^\infty(H^2)}, \|F'\|_{L^\infty(H^1)}\} < \min\{\varepsilon_0, 1 - \alpha\},$$

then equation (3) admits a unique, global strong solution (u, p, τ) on $(0, \infty)$ satisfying

$$u \in C_b([0, \infty); D(A)) \text{ with } \nabla u \in L^2([0, \infty); H^2(\Omega)) \text{ and}$$
$$u' \in L^2([0, \infty); H^1_0(\Omega) \cap L^2_\sigma(\Omega)),$$
$$\nabla p \in L^2([0, \infty), H^1(\Omega)) \cap L^\infty([0, \infty), H^1(\Omega)),$$
$$\tau \in C_b([0, \infty); H^2(\Omega)) \cap L^2([0, \infty); H^2(\Omega)) \text{ and } \tau' \in L^2([0, \infty); L^2(\Omega)).$$

Proof. The proof of Theorem 2.3 follows essentially the lines of the proof of Theorem 1.1 in [7], but we need to take into account the contributins due to the external force $\operatorname{div} F$. For the convenience of the reader, we sketch the main ideas of the proof here. Let (u, τ) be the local solution of (3) constructed in Proposition 2.1. Our aim is to to derive a priori estimates for u, τ, u' and τ'. Since the norms of F are assumed to be small, our strategy is to absorb these terms into the left-hand sides of the equations thanks to energy-type estimates.

Since the equation $(3)_2$ for τ does not contain external forces, estimates (4.1) and (4.2) of [7] yield

$$\frac{d}{dt}\|\tau\|_{H^2}^2 + \|\tau\|_{H^2}^2 \leqslant C\alpha^2\|\nabla u\|_{H^2}^2 + \frac{C}{\alpha^2}\|\tau\|_{H^2}^4.$$

Applying the Helmholtz projection $\mathbb{P}$ to the second line of (3) gives

$$u_t + \mathbb{P}(u\cdot\nabla)u + (1-\alpha)Au = \mathbb{P}\mathrm{div}\,\tau + \mathbb{P}\mathrm{div}\,F. \tag{5}$$

Similarly as in [7] we obtain

$$\begin{aligned}\|\nabla u\|_{H^2}^2 \leqslant C\Big(&\|Au\|_{H^2}^2 + \|\nabla u\|_{L^2}^2 + \frac{1}{(1-\alpha)^2}\|\nabla u_t\|_{L^2}^2 + \frac{1}{(1-\alpha)^2}\|\nabla\mathbb{P}\mathrm{div}\,\tau\|_{L^2}^2\\ &+ \frac{1}{(1-\alpha)^2}\|\nabla\mathbb{P}\mathrm{div}\,F\|_{L^2}^2 + \frac{1}{(1-\alpha)^2}\|Au\|_{L^2}^4 + \frac{1}{(1-\alpha)^2}\|\nabla u\|_{L^2}^4\Big).\end{aligned}$$

Next, taking the inner product of (5) with u yields

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + (1-\alpha)\|\nabla u\|_{L^2}^2 = (\mathrm{div}\,\tau \mid u) + (\mathrm{div}\,F \mid u).$$

Similarly as in [7] we arrive at

$$\begin{aligned}\frac{d}{dt}\big(\|u\|_{L^2}^2 + \frac{1}{2}\|\tau\|_{L^2}^2\big) + (1-\alpha-\|F\|_{L^2})\|\nabla u\|_{L^2}^2 + \frac{1}{2\alpha}\|\tau\|_{L^2}^2&\\ \leqslant \frac{C}{(1-\alpha)\alpha^2}\|\tau\|_{H^2}^4 + \frac{1}{2\alpha}\|F\|_{L^2}&\end{aligned}$$

and obtain the differential inequality

$$\frac{d}{dt}U(t) + V(t) \leqslant CH(t)V(t),$$

where

$$\begin{aligned}U(t) :=&(1-\alpha)(\kappa_4 C_0+1)(\|\mathbb{P}\mathrm{div}\,\tau\|_{L^2}^2 + \|\,\mathrm{curl}\,\mathrm{div}\,\tau\|_{L^2}^2) + \frac{\kappa_6+1}{1-\alpha}\|u\|_{L^2}^2\\ &+ \frac{\kappa_6+1}{2\alpha(1-\alpha)}\|\tau\|_{L^2}^2 + \|\tau\|_{H^2}^2 + \frac{1}{2}\|F\|_{L^2}\\ &+ \frac{(\kappa_1+1)(3-\alpha-\|F\|_{L^2}^2)}{1-\alpha}\|\nabla u\|_{L^2}^2\\ &+ \frac{\kappa_5+1}{1-\alpha}\|u_t\|_{L^2}^2 + \frac{\kappa_5+1}{2\alpha(1-\alpha)}\|\tau_t\|_{L^2}^2,\end{aligned}$$

$$
\begin{aligned}
V(t) :=& \frac{\kappa_1+1}{1-\alpha}\|u_t\|_{L^2}^2 + \|Au\|_{L^2}^2 + \|\tau\|_{H^2}^2 + \|\nabla u\|_{H^2}^2 + \|\nabla u_t\|_{L^2}^2 \\
&+ \frac{\kappa_5+1}{\alpha(1-\alpha)}\|\tau_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\tau\|_{L^2}^2 + \|\mathbb{P}\mathrm{div}\,\tau\|_{L^2}^2 \\
&+ \|\,\mathrm{curl}\,\mathrm{div}\,\tau\|_{L^2}^2 + \|F\|_{L^2}^2, \\
H(t) :=& \|u_t\|_{L^2}^2 + \|Au\|_{L^2}^2 + \|\tau\|_{H^2}^2 + \|\tau_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4.
\end{aligned}
$$

Following (4.28) in [7], there is a constant $M_1 = M_1(\alpha) > 0$ such that

$$
H(t) \leqslant M_1\big(U(t) + U(t)^2 + U(t)^3\big), \quad t \geq 0. \tag{6}
$$

Arguing as in (4.28) in [7] we see that for $\delta_0 > 0$ with $\delta + \delta^2 + \delta^3 < \frac{1}{2CM_1}$ and for $\epsilon_0 >$ such that $C(\epsilon_0^4 + \epsilon_0^4) < \delta_0$ we have

$$
\sup_{0\leqslant t\leqslant T_*} U(t) + \frac{1}{2}\int_0^{T_*} V(s)ds \leqslant \delta_0.
$$

Hence,

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T_*}\big(\|u(t)\|_{D(A)}^2 + \|u'(t)\|_{L^2}^2 + \|\tau(t)\|_{H^2}^2 + \|\tau'(t)\|_{L^2}^2\big) \\
&\quad + \frac{1}{2}\int_0^{T_*}\big(\|\nabla u(t)\|_{H^2}^2 + \|\nabla u'(t)\|_{L^2}^2 + \|\tau(t)\|_{H^2}^2 + \|\tau'(t)\|_{L^2}^2\big)dt \leqslant C,
\end{aligned}
$$

and the local solution (u, p, τ) can be extended to all $t > 0$. □

3. Stability of the Oldroyd-B Equations with Small External Forces

In this section we consider the stability of bounded solutions to the system (3). Applying the Helmholtz projection to (3) we obtain

$$
\begin{cases}
u_t + (u\cdot\nabla)u + (1-\alpha)Au &= \mathbb{P}\mathrm{div}\,\tau + \mathbb{P}\mathrm{div}\,F, \\
\tau_t + (u\cdot\nabla)\tau + g_a(\tau,\nabla u) + \tau &= 2\alpha D(u), \\
u(0) &= u_0, \\
\tau(0) &= \tau_0,
\end{cases} \tag{7}
$$

In the following we will prove that the bounded global solution (u, τ) to (7) obtained in Theorem 2.3 is stable in the sense that any other global solution to (3) starting in a sufficiently small neighborhood of $(u(0), \tau(0))$ is tending to (u, τ). To this end, we introduce the spaces

$$
W_1 := H^3(\Omega) \cap H_0^1(\Omega) \cap L_\sigma^2(\Omega), \qquad W_2 := H^2(\Omega)
$$

and set $W := W_1 \times W_2$. Moreover, for $r > 0$ and $(x_1, x_2) \in W$ we set

$$\mathcal{B}(x_1, x_2, r) := \{(y_1, y_2) \in W : \|(y_1, y_2) - (x_1, x_2)\|_W \leqslant r\}.$$

The following stability result is the first main result of this article.

Theorem 3.1. *There exist constants $\delta_0, A, R > 0$ such that for a solution (u, τ) to equation (7) with $\|(u(0), \tau(0))\|_W \leqslant \delta_0$ and any solution (v, μ) to equation (7) with $\alpha \leqslant A$ and initial data $(v(0), \mu(0)) \in \mathcal{B}(u(0), \tau(0), r)$ for $r \leqslant R$, the equality*

$$\lim_{t\to\infty} \|v(t) - u(t)\|_{L^2} = \lim_{t\to\infty} \|\mu(t) - \tau(t)\|_{L^2} = 0$$

holds.

In order to prove Theorem 3.1 we make use of Hölder's and Young's inequality in weak L^p-spaces. For proofs, see e.g., Section 1 of [12]. More specifically, for $1 < p < \infty$ we denote by $L^p_w := L^p_w(\mathbb{R})$ the space of all measurable functions f on $\mathbb{R}$ with norm

$$\|f\|_{p,w} = \sup_{0<|E|<\infty} |E|^{-1+\frac{1}{p}} \int_E |f| ds < \infty, \tag{8}$$

where $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}$.

Lemma 3.2 ([12], Section 1). *Let $p \in [1, \infty)$, $q, r \in (1, \infty)$. Then the following assertins hold.*

a) If $f \in L^p_w, g \in L^q_w$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then $fg \in L^r_w$ and

$$\|fg\|_{r,w} \leqslant C\|f\|_{p,w}\|g\|_{q,w}$$

for some constant C depending only on p and q.

b) If $f \in L^p_w, g \in L^q_w$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then $f * g \in L^r_w$ and there is a constant C, depending only on p and q, such that

$$\|f * g\|_{r,w} \leqslant C\|f\|_{p,w}\|g\|_{q,w}.$$

c) If $f \in L^p_w, g \in L^1$, then $f * g \in L^p_w$ and there is a constant C, depending only on p, such that

$$\|f * g\|_{p,w} \leqslant C\|f\|_{p,w}\|g\|_{L_1}.$$

Proof of Theorem 3.1. The strategy of our proof follows to a certain extent the one of Theorem 3.2 in [10]. In the present case, we need to deal, however, with two non trivial solutions to (7).

Let (u,τ) and (v,μ) be two solutions to (7) as in Theorem 3.1. Setting $\tilde u := v-u$ and $\tilde\tau := \mu-\tau$, we obtain from (7)

$$\begin{cases} \tilde u_t + (\tilde u\cdot\nabla)\tilde u + (u\cdot\nabla)\tilde u + (\tilde u\cdot\nabla)u + (1-\alpha)A\tilde u &= \mathbb{P}\mathrm{div}\tilde\tau,\\ \tilde\tau_t + (\tilde u\cdot\nabla)\tilde\tau + (\tilde u\cdot\nabla)\tau + (u\cdot\nabla)\tilde\tau + g_a(\tilde\tau,\nabla\tilde u) & \\ \qquad + g_a(\tilde\tau,\nabla u) + g_a(\tau,\nabla\tilde u) + \tilde\tau &= 2\alpha D(\tilde u),\\ \tilde u(0) &= v(0)-u(0),\\ \tilde\tau(0) &= \mu(0)-\tau(0). \end{cases} \tag{9}$$

We first estimate $\tilde\tau$ by the second equation of system (9). Denote by $\|\cdot\|$ the norm of $L^2(\Omega)$. Taking the scalar product in the second equation of (9) with $\tilde\tau$ we obtain

$$\frac{d}{dt}\|\tilde\tau\|^2 + 2\left\langle(\tilde u\cdot\nabla)\tau,\tilde\tau\right\rangle + 2\left\langle g_a(\tilde\tau,\nabla\tilde u),\tilde\tau\right\rangle + 2\left\langle g_a(\tilde\tau,\nabla u),\tilde\tau\right\rangle$$
$$+2\left\langle g_a(\tau,\nabla\tilde u),\tilde\tau\right\rangle + 2\|\tilde\tau\|^2 = 4\alpha\left\langle D(\tilde u),\tilde\tau\right\rangle,\ t\ge 0.$$

Integrating we obtain by Gronwall's lemma

$$\begin{aligned}\|\tilde\tau(t)\|^2 \leqslant e^{-2t}\|\tilde\tau(0)\|^2 + 2\int_0^t e^{-2(t-s)}\Big(&|\langle g_a(\tilde\tau(s),\nabla\tilde u),\tilde\tau(s)\rangle| \\ &+ |\langle g_a(\tilde\tau(s),\nabla u(s)),\tilde\tau(s)\rangle| + |\langle g_a(\tau(s),\nabla\tilde u(s)),\tilde\tau(s)\rangle| \\ &+ |\langle(\tilde u\cdot\nabla)\tau,\tilde\tau(s)\rangle| + 2\alpha\,|\langle D(\tilde u(s)),\tilde\tau(s)\rangle|\Big)ds, \quad t\ge 0.\end{aligned}$$

For $\|u\|_{W_1}\leqslant r$ we thus obtain

$$\begin{aligned}\|\tilde\tau(t)\|^2 &\leqslant e^{-2t}\|\tilde\tau(0)\|^2 + 8rC(|a|+1)\int_0^t e^{-2(t-s)}\|\tilde\tau(s)\|^2 ds \\ &\quad + Cr(4|a|+5)\int_0^t e^{-2(t-s)}\|\tilde\tau(s)\|\|\tau(s)\|ds \\ &\quad + 4\alpha\int_0^t e^{-2(t-s)}\|D(u(s))\|\|\tilde\tau(s)\|ds \\ &\leqslant e^{-2t}\|\tilde\tau(0)\|^2 + 8rC(|a|+1)\int_0^t e^{-2(t-s)}\|\tilde\tau(s)\|^2 ds \\ &\quad + Cr(4|a|+5)\int_0^t e^{-2(t-s)}\big(\tfrac12\|\tilde\tau(s)\|^2 + \tfrac12\|\tau(s)\|^2\big)ds \\ &\quad + 2\alpha\int_0^t e^{-2(t-s)}(\|D(u(s))\|^2 + \|\tilde\tau(s)\|^2)ds, \qquad t\ge 0,\end{aligned}$$

where C denotes the constant in Sobolev's embedding. Therefore,

$$\|\tilde{\tau}(t)\|^2 \leqslant e^{-2t}\|\tilde{\tau}(0)\|^2 + \frac{4\alpha + 8rC(6|a|+7)}{2}\int_0^t e^{-2(t-s)}\|\tilde{\tau}(s)\|^2 ds + \int_0^t e^{-2(t-s)}\Big(2\alpha\|D(u(s))\|^2 + \frac{Cr(4|a|+5)}{2}\|\tau(s)\|^2\Big)ds, \quad t \geq 0.$$

Choosing r so small that $K := \frac{4-4\alpha-8rC(6|a|+7)}{2} > 0$, Gronwall's inequality yields for $t \geq 0$

$$\|\tilde{\tau}(t)\|^2 \leqslant e^{-Kt}\|\tilde{\tau}(0)\|^2 + \int_0^t e^{-K(t-\xi)}\Big(2\alpha\|D(u(\xi))\|^2 + \frac{Cr(4|a|+5)}{2}\|\tau(s)\|^2\Big)d\xi. \quad (10)$$

In a second step we take the inner product of the first equation in (9) with $\tilde{u}$ and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{u}(t)\|^2 + (1-\alpha)\|\nabla\tilde{u}(t)\|^2 &= \langle \mathbb{P}\mathrm{div}\,\tau(t), u(t)\rangle - \langle(\tilde{u}\cdot\nabla)u, \tilde{u}\rangle \\ &= \langle \mathbb{P}\mathrm{div}\,\tau(t), u(t)\rangle + \langle(\tilde{u}\cdot\nabla)\tilde{u}, u\rangle\,.\end{aligned}$$

Integrating from s to t yields

$$\begin{aligned}&\|\tilde{u}(t)\|^2 + 2(1-\alpha)\int_s^t \|\nabla\tilde{u}(t)\|^2 dt \\ &\quad\leqslant \|\tilde{u}(s)\|^2 + 2\int_s^t \|\tilde{\tau}(t)\|\|\nabla\tilde{u}(\xi)\|d\xi + \int_s^t \|(\tilde{u}(\xi)\cdot\nabla)\tilde{u}(\xi)\|\|u(\xi)\|d\xi \\ &\quad\leqslant \|\tilde{u}(s)\|^2 + 2\int_s^t \|\tilde{\tau}(t)\|\|\nabla\tilde{u}(\xi)\|d\xi \\ &\qquad + 2\tilde{C}\int_s^t \|\tilde{u}(\xi)\|_{L^6}\|\nabla\tilde{u}(\xi)\|_{L^3}\|u(\xi)\|d\xi \\ &\quad\leqslant \|\tilde{u}(s)\|^2 + 2\int_s^t \|\tilde{\tau}(t)\|\|\nabla\tilde{u}(\xi)\|d\xi \\ &\qquad + 2C\int_s^t \|\nabla\tilde{u}(\xi)\|\|\nabla\tilde{u}(\xi)\|^{1/2}\|\nabla^2\tilde{u}\|^{1/2}\|u(\xi)\|d\xi \\ &\quad\leqslant \|\tilde{u}(s)\|^2 + 2\int_s^t \|\tilde{\tau}(t)\|\|\nabla\tilde{u}(\xi)\|d\xi + 2C\|u\|_{C_b}\int_s^t \|\nabla\tilde{u}(\xi)\|^2_{H^1}d\xi \\ &\quad\leqslant \|\tilde{u}(s)\|^2 + \int_s^t \|\tilde{\tau}(t)\|^2 d\tau + (1 + 2C\|u\|_{C_b})\int_s^t \|\nabla\tilde{u}(\xi)\|^2_{H^1}d\xi\end{aligned}$$

where $\tilde{C}$ and C are the constants arising in Gagliardo-Nirenberg and Sobolev inequalities and $\|u\|_{C_b} := \|u\|_{C_b([0,\infty),L^2)}$. Summing up, we obtain

$$\|\tilde{u}(t)\| \leqslant \|\tilde{u}(s)\| + \Big(\int_s^t \|\tilde{\tau}(\xi)\|^2 d\xi\Big)^{1/2} + (1+2C\|u\|_{C_b})^{1/2}\Big(\int_s^t \nabla\tilde{u}(\xi)\|_{H^1}^2 d\xi\Big)^{1/2},$$

and integrating with respect to $s \in (0,t)$ yields

$$\begin{aligned}\|\tilde{u}(t)\| \leqslant \frac{1}{t}\int_0^t \|\tilde{u}(s)\|ds + \Big(\frac{2}{t}\Big)^{1/2}\|\tilde{\tau}\|_{L^2(0,\infty;H^2)} \\ + \Big(\frac{1+2C\|u\|_{C_b}}{t}\Big)^{1/2}\|\nabla\tilde{u}\|_{L^2(0,\infty;H^2)}.\end{aligned} \tag{11}$$

Theorem 2.3 yields $\|\tilde{\tau}\|_{L^2(0,\infty;H^2)} < \infty$ as well as $\|\nabla\tilde{u}\|_{L^2(0,\infty;H^2)} < \infty$. Hence, the second and third term on the right-hand side of (11) tend to 0 as $t \to \infty$.

We now turn our attention to the first term on the right hand side of (11) and aim to show that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \|\tilde{u}(s)\|ds = 0. \tag{12}$$

To this end, we multiply the first line of equation (9) with $\phi \in C(\mathbb{R}_+, H_0^1(\Omega) \cap L^2_\sigma(\Omega)) \cap C^1(\mathbb{R}_+, L^2_\sigma(\Omega))$ and integrate from s to t to obtain

$$\begin{aligned}\langle\tilde{u}(t),\phi(t)\rangle + \int_s^t [(1-\alpha)\langle\nabla\tilde{u},\nabla\phi\rangle + \langle(\tilde{u}\cdot\nabla)\tilde{u},\phi\rangle + \langle(\tilde{u}\cdot\nabla)u,\phi\rangle \\ + \langle(u\cdot\nabla)\tilde{u},\phi\rangle]d\xi \\ = \langle\tilde{u}(s),\phi(s)\rangle + \int_s^t [\langle\tilde{u},\phi'\rangle + \langle\mathbb{P}\mathrm{div}\,\tilde{\tau},\phi\rangle]d\xi.\end{aligned} \tag{13}$$

Substituting $\phi(\xi) = e^{-(t-\xi)A}\psi$ with $\psi \in C^\infty_{0,\sigma}(\Omega)$ into (13) and setting $s = 0$ we arrive at

$$\begin{aligned}\langle\tilde{u}(t),\psi\rangle = \langle e^{-tA}\tilde{u}(0),\psi\rangle - \int_0^t \big[< (\tilde{u}\cdot\nabla)\tilde{u}(\xi), e^{-(t-\xi)A}\psi > \\ + < (\tilde{u}\cdot\nabla)u(\xi), e^{-(t-\xi)A}\psi > \big]d\xi \\ + \int_0^t < (u\cdot\nabla)\tilde{u}(\xi), e^{-(t-\xi)A}\psi > d\xi \\ + \alpha\int_0^t < \nabla\tilde{u}(\xi), \nabla e^{-(t-\xi)A}\psi > d\xi \\ + \int_0^t < \tilde{\tau}(\xi), \nabla e^{-(t-\xi)A}\psi > d\xi.\end{aligned}$$

We next note the following estimates for the Stokes semigroup on exterior domains (see e.g. [3], [15])

$$
\begin{aligned}
&\|e^{-tA}(w\cdot\nabla)v\| \leqslant Ct^{-1/2}(\|w\|\|v\|)^{1/4}(\|\nabla w\|\|\nabla v\|)^{3/4},\\
&\qquad\qquad\qquad\qquad\qquad t>0, w, v\in H^1(\Omega)\cap L^2_\sigma(\Omega),\\
&\|\nabla e^{-tA}\psi\| \leqslant Ct^{-1/2}\|\psi\| \quad\text{and}\quad \|\nabla e^{-tA}\psi\|_{L^3}\leqslant Ct^{-3/4}\|\psi\|,\\
&\qquad\qquad\qquad\qquad\qquad t>0, \psi\in C^\infty_{0,\sigma},
\end{aligned}
\tag{14}
$$

as well as the Gagliardo-Nirenberg inequality

$$
\|\nabla\tilde u(s)\|_{L^{\frac32}} \leqslant C\|\nabla\tilde u(s)\|^{\frac12}\|\nabla^2\tilde u(s)\|^{\frac12}\leqslant C\|\nabla\tilde u(s)\|_{H^1}.
$$

Taking the supremum over all $\psi\in C^\infty_{0,\sigma}$ with $\|\psi\|\leqslant 1$ yields

$$
\begin{aligned}
\|\tilde u(t)\| \leqslant{}& \|e^{-tA}\tilde u(0)\| + C\int_0^t (t-s)^{-\frac12}\Big(\|\tilde u(s)\|^{\frac12}\|\nabla\tilde u(s)\|^{\frac32}\\
&\qquad + 2(\|\tilde u(s)\|\|u(s)\|)^{\frac14}(\|\nabla\tilde u(s)\|\|\nabla u(s)\|)^{\frac34}\Big)ds\\
&+ C\int_0^t (t-s)^{-\frac34}\|\nabla\tilde u(s)\|_{H^1}ds + C\int_0^t (t-s)^{-\frac34}\|\tau(s)\|_{H^1}ds\\
\leqslant{}& \|e^{-tA}\tilde u(0)\| + Cr^{\frac12}\int_0^t (t-s)^{-\frac12}\|\nabla\tilde u(s)\|^{\frac32}ds\\
&+ 2(r\|u\|_{C_b})^{\frac14}\int_0^t (t-s)^{-\frac12}(\|\nabla\tilde u(s)\|\|\nabla u(s)\|)^{\frac34}ds\\
&+ C\int_0^t (t-s)^{-\frac34}\|\nabla\tilde u(s)\|_{H^1}ds + C\int_0^t (t-s)^{-\frac34}\|\tau(s)\|_{H^1}ds\\
=:{}& \|e^{-tA}\tilde u(0)\| + I(t) + II(t) + III(t) + IV(t).
\end{aligned}
\tag{15}
$$

By Theorem 2.3, $\nabla\tilde u\in L^2(\mathbb{R}_+, H^2(\Omega))$ and hence $\|\nabla\tilde u(\cdot)\|^{3/2}\in L^{4/3}(\mathbb{R}_+)$. Setting $h(t) := t^{-1/2}$ and $g_1(t) := \int_0^t h(t-s)\|\nabla\tilde u(s)\|^{3/2}ds$, we see by Lemma 3.2b) that

$$
\|g_1\|_{L^4_w(\mathbb{R}_+)} \leqslant C\|h\|_{L^2_w(\mathbb{R}_+)}\|\nabla\tilde u\|_{L^2(\mathbb{R}_+;L^2(\Omega))}.
$$

Therefore, by (8)

$$
\frac1t\int_0^t g_1(s)ds \leqslant \frac{Ct^{3/4}}{t}\|g_1\|_{L^4_w(\mathbb{R}_+)} = \frac{C_1}{t^{1/4}},\quad t>0.
$$

for suitable constants $C, C_1 > 0$. Next, since $\|\nabla u(\cdot)\|$ and $\|\tilde u(\cdot)\|$ belong to $L^2(\mathbb{R}_+)$, Hölder's inequality implies $\|\nabla u(\cdot)\|\|\tilde u(\cdot)\|\in L^1(\mathbb{R}_+)$ and hence $(\|\nabla\tilde u(\cdot)\|\|\nabla u(\cdot)\|)^{\frac34}\in L^{4/3}(\mathbb{R}_+)$. Setting $h(t) := t^{-1/2}$ and $g_2(t) := \int_0^t h(t-s)(\|\nabla\tilde u(s)\|\|\nabla u(s)\|)^{\frac34}ds$ we see that $g_2\in L^4_w(\mathbb{R}_+)$ and satisfies

$$
\|g_2\|_{L^4_w(\mathbb{R}_+)} \leqslant C\|h\|_{L^2_w(\mathbb{R}_+)}\|\nabla\tilde u\|_{L^2(\mathbb{R}_+;L^2(\Omega))}\|\nabla u\|_{L^2(\mathbb{R}_+;L^2(\Omega))}.
$$

Thus, again by (8)

$$\frac{1}{t}\int_0^t g_2(s)ds \leqslant \frac{Ct^{3/4}}{t}\|g_2\|_{L^4_w(\mathbb{R}_+)} = \frac{C_2}{t^{1/4}}, \quad t>0.$$

Theorem 2.3 implies $\|\nabla\tilde{u}(\cdot)\|_{H^1} \in L^2(\mathbb{R}_+)$ and hence for h_3 and g_3 given by $h_3(t) := t^{-3/4}$ and $g_3(t) := \int_0^t h_3(t-s)\|\nabla\tilde{u}(s)\|_{H^1}ds$ we obtain

$$\|g_3\|_{L^4_w(\mathbb{R}_+)} \leqslant C\|h_3\|_{L^{4/3}_w(\mathbb{R}_+)}\|\nabla\tilde{u}\|_{L^2(\mathbb{R}_+;H^1(\Omega))}.$$

This yields

$$\frac{1}{t}\int_0^t g_3(s)ds \leqslant \frac{Ct^{3/4}}{t}\|g_3\|_{L^4_w(\mathbb{R}_+)} = \frac{C_3}{t^{1/4}}, \quad t>0.$$

Similarly, for $IV(t)$ in (15), we have $\|\tilde{\tau}(\cdot)\|_{H^1} \in L^2(\mathbb{R}_+)$. Therefore the function g_4 given by $g_4(t) := \int_0^t (t-s)^{-3/4}\|\nabla\tilde{u}(s)\|_{H^1}ds$ belongs to $L^4_w(\mathbb{R}_+)$ and satisfies

$$\|g_4\|_{L^4_w(\mathbb{R}_+)} \leqslant C\|h_3\|_{L^{4/3}_w(\mathbb{R}_+)}\|\tilde{\tau}\|_{L^2(\mathbb{R}_+;H^1(\Omega))}.$$

As above

$$\frac{1}{t}\int_0^t g_4(s)ds \leqslant \frac{Ct^{3/4}}{t}\|g_4\|_{L^4_w(\mathbb{R}_+)} = \frac{C_4}{t^{1/4}}, \quad t>0.$$

Summing up we see that

$$\frac{1}{t}\int_0^t \|\tilde{u}(s)\|ds \leqslant \frac{1}{t}\int_0^t \|e^{-sA}\tilde{u}(0)\|ds + \frac{\tilde{C}}{t^{1/4}}, \quad t>0. \tag{16}$$

Since the Stokes semigroup on exterior domain is strongly stable in the sense that

$$\lim_{t\to\infty}\|e^{-tA}\tilde{u}(0)\| = 0,$$

it follows that $\lim_{t\to\infty}\frac{1}{t}\int_0^t\|\tilde{u}(s)\|ds = 0$. Combining this with estimate (11) we finally obtain

$$\lim_{t\to\infty}\|\tilde{u}(t)\| = 0.$$

Finally, we prove that $\lim_{t\to\infty}\|\tilde{\tau}(t)\| = 0$. To this end, assume that $f, f' \in L^2(0,\infty); L^2(\Omega))$. Then the inequality

$$\|f(t)\|_2^2 \leqslant \|f(t_n)\|_2^2 + 2\Big(\int_{t_n}^t \|f(s)\|_2^2\Big)^{1/2}\Big(\int_{t_n}^t \|f'(s)\|_2^2\Big)^{1/2} \tag{17}$$

yields that $\|f(t)\|_2 \to 0$ as $t\to\infty$ provided $(t_n) \subset (0,\infty)$ is an unbounded sequence satisfying $\|f(t_n)\|_2 \to 0$ as $(t_n)\to\infty$. By Theorem 2.3, the function $\tilde{\tau}$ satisfies (17) and we thus obtain $\|\tilde{\tau}(t)\|_2 \to 0$ as $t\to\infty$. The proof is complete. □

REMARK 3.3: Taking into account that $\tilde{u}(0) \in D(A) \subset H_0^1(\Omega) \subset \mathrm{Rg}(A^{\frac{1}{2}})$ we see that $\frac{1}{t}\int_0^t \|e^{-sA}\tilde{u}(0)\|ds$ satisfies a decay rate of the form

$$\frac{1}{t}\int_0^t \|e^{-sA}\tilde{u}(0)\|ds = \frac{1}{t}\int_0^t \|A^{\frac{1}{2}}e^{-sA}(v_0-u_0)\|ds \leqslant \frac{1}{t}\int_0^t \frac{1}{s^{\frac{1}{2}}}\|v_0-u_0\|ds = \frac{C}{t^{\frac{1}{2}}}$$

for $t > 0$. In a similar way we obtain a decay rate for $\tilde{\tau}$ of the form

$$\|\tilde{\tau}(t)\| \leqslant \Big(\frac{C_1}{t^{1/2}} + \frac{C_2}{t^{1/4}}\Big)\|\tau(0) - \mu(0)\|, \ t > 0.$$

Let us also note that combining Theorem 3.1 on the stability of (u, τ) with respect to the $\|\cdot\|_2$-norm with Theorem 2.3 and with the estimate (17) yields a stability result for equation (7) with respect to the $\|\cdot\|_q$-norm for $q \in (2, 6]$. More precisely, the following holds true.

COROLLARY 3.4. *Let $q \in (2, 6]$. Then there exist constants $A, R > 0$ such that any solution (u, τ) to equation (7) with $\alpha \leqslant A$ and with initial data $(u_0, \tau_0) \in B(0, 0, r)$ with $r \leqslant R$ satisfies*

$$\lim_{t\to\infty} \|u(t)\|_{L^q} = \lim_{t\to\infty} \|\tau(t)\|_{L^q} = 0.$$

Proof. Due to Gagliardo-Nirenberg inequality we have

$$\|u\|_q \leqslant c\|\nabla u\|_2^{3(\frac{1}{2}-\frac{1}{q})}\|u\|_2^{\frac{3}{q}-\frac{1}{2}} \text{ for } 2 < q \leqslant 6.$$

By Theorem 2.3, $\nabla u, \nabla u_t \in L^2((0,\infty); L^2(\Omega))$ and hence $\|\nabla u(t)\|_2 \to 0$ as $t \to \infty$ by estimate (17). Since $u \in L^\infty((0,\infty); L^2(\Omega))$, the assertion for u follows. The assertion for $\tilde{\tau}$ follows similarly by noting that $\tilde{\tau}' \in L^\infty((0,\infty); H^1(\Omega))$. □

4. Periodic Solutions

In this section we show that the above stability result, Theorem 3.1, implies also the existence of periodic solutions to (3). More precisely, the following assertion holds.

THEOREM 4.1. *Assume in addition to the assumptions in Theorem 2.3 and 3.1 the function F is time T-periodic for some $T > 0$. Then, if $\|F\|_{L^\infty(H^2)}$ and $\|F'\|_{L^\infty(H^1)}$ are small enough, there exists a T-periodic solution to (3) and this T-periodic solution is stable in the sense of Theorem 3.1.*

Proof. Due to Theorem 2.3, we consider a bounded and small solution

$$(u, \tau) \in C_b([0,\infty); D(A)) \times C_b([0,\infty); H^2(\Omega))$$

of equation (3). In the following, we prove that $\big(u(nT),\tau(nT)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $X := C_b([0,\infty);L^2(\Omega))\times C_b([0,\infty);L^2(\Omega))$.

To this end, for $m,n\in\mathbb{N}$ with $m>n$ we set

$$(w(t),\mu(t)) := (u(t+(m-n)T),\tau(t+(m-n)T).$$

The periodicity of F implies that $(w(t),\mu(t))$ is again a solution to (3) with the initial data $(w(0),\mu(0)) = (u((m-n)T),\tau((m-n)T)$. Theorem 3.1 and Remark 3.3 imply

$$\|w(t)-u(t)\| + \|\mu(t)-\tau(t)]\| \leqslant \frac{\tilde{C}_1}{t^{1/2}} + \frac{\tilde{C}_2}{t^{1/4}}, \quad t>0.$$

Hence, by taking $t := nT$ in the above inequality we obtain

$$\|u(mT)-u(nT)\| + \|\mu(mT)-\tau(nT)]\| \leqslant \frac{\tilde{C}_1}{(nT)^{1/2}} + \frac{\tilde{C}_2}{(nT)^{1/4}}.$$

Therefore, $\big(u(nT),\tau(nT))\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in X with limit

$$(u^*,\tau^*) := \lim_{n\to\infty}(u(nT),\tau(nT)) \text{ in } X.$$

Choosing (u^*,τ^*) as initial data, we claim that the solution $(\hat{u}(t),\hat{\tau}(t))$ of equation (3) with $(\hat{u}(0),\hat{\tau}(0)) = (u^*,\tau^*)$ is T-periodic. To this end, for (u,τ) as above and $n\in\mathbb{N}$ we set

$$(v(t),\eta(t)) := (u(t+nT),\tau(t+nT))$$

The periodicity of F implies that $(v(t),\eta(t))$ is a solution of (3) with $(v(0),\eta(0)) = (u(nT),\tau(nT))$. We further see that

$$\begin{aligned}&\|\hat{u}(t)-v(t)\| + \|\hat{\tau}(t)-\eta(t)\| \\ &\qquad\leqslant \Big(\frac{C_1}{t^{\frac{1}{2}}}+\frac{C_2}{t^{\frac{3}{4}}}\Big)\|\hat{u}(0)-v(0))\| + \Big(\frac{C_3}{t^{\frac{1}{2}}}+\frac{C_4}{t^{\frac{1}{4}}}\Big)\|\hat{\tau}(0)-\eta(0)\|.\end{aligned}$$

for $t>0$. Taking $t=T$ in the above inequality yields

$$\begin{aligned}&\|\hat{u}(T)-u((n+1)T)\| + \|\hat{\tau}(T)-\tau((n+1)T)\| \\ &\qquad\leqslant \Big(\frac{C_1}{T^{\frac{1}{2}}}+\frac{C_2}{T^{\frac{3}{4}}}\Big)\|\hat{u}(0)-u(nT))\| + \Big(\frac{C_3}{T^{\frac{1}{2}}}+\frac{C_4}{T^{\frac{1}{4}}}\Big)\|\hat{\tau}(0)-\tau(nT)\|\end{aligned}$$

for all $n\in\mathbb{N}$. Letting $n\to\infty$ and using the fact that $\lim_{n\to\infty}(u(nT),\tau(nT)) = (u^*,\tau^*) = (\hat{u}(0),\hat{\tau}(0))$ in X, we obtain

$$(\hat{u}(T),\hat{\tau}(T)) = (\hat{u}(0),\hat{\tau}(0)).$$

Hence, $(\hat{u}(t),\hat{\tau}(t))$ is $T-$periodic and the proof is complete. □

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