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Asymptotic properties of a free boundary problem for a reaction-diffusion equation with multi-stable nonlinearity

Yoshio Yamada

"Dedicated to Professor Julian López-Gómez on the occasion of his 60th birthday"

Abstract. *This paper deals with a free boundary problem for a reaction-diffusion equation with moving boundary, whose dynamics is governed by the Stefan condition. We will mainly discuss the problem for the case of multi-stable nonlinearity, which is a function with a multiple number of positive stable equilibria. The first result is concerned with the classification of solutions in accordance with large-time behaviors. As a consequence, one can observe a multiple number of spreading phenomena corresponding for each positive stable equilibrium. Here it is seen that there exists a certain group of spreading solutions whose element accompanies a propagating terrace. We will derive sharp asymptotic estimates of free boundary and profile of every spreading solution including spreading one with propagating terrace.*

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1. Introduction

This paper is concerned with the following free boundary problem for reaction-diffusion equations:

$$\text{(FBP)}\qquad \begin{cases} u_t = du_{xx} + f(u), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases}$$

where d, μ and h_0 are positive constants and $x = h(t)$ is a free boundary. Nonlinearity f is a function of class $C^1[0,\infty)$ satisfying

$$f(0) = f(u^*) = 0 \quad \text{with some} \ \ u^* > 0 \ \ \text{and} \ \ f(u) < 0 \ \ \text{for} \ \ u > u^* \tag{1}$$

and u_0 is a nonnegative function of class $C^2[0, h_0]$ such that

$$u_0'(0) = u_0(h_0) = 0 \quad \text{and} \ \ u_0 \not\equiv 0. \tag{2}$$

Since Du and Lin published a pioneer work [4] on (FBP) in 2010, a lot of authors have studied (FBP) and related free boundary problems. Among them, we should refer to the paper of Du and Lou [5], who obtained very important results on large-time behaviors of solution $(u(t,\cdot), h(t))$ of (FBP) for typical types of nonlinearity f such as monostable, bistable and combustion types. Moreover, we should also note the work of Du, Matsuzawa and Zhou [9], who derived sharp asymptotic estimates of $(u(t,\cdot), h(t))$ as $t \to \infty$ in the case $\lim_{t\to\infty} h(t) = \infty$.

The main purpose of the present paper is to study (FBP) when f is a multi-stable function, that is, f has a multiple number of positive stable equilibria. For the sake of simplicity, we assume that $f \in C^1[0,\infty)$ satisfies the following conditions:

(PB) $f(u) = 0$ has solutions $u = 0, u_1^*, u_2^*, u_3^*$ $(0 < u_1^* < u_2^* < u_3^*)$,
$f'(0) > 0,\ f'(u_1^*) < 0,\ f'(u_2^*) \geq 0,\ f'(u_3^*) < 0,\ \displaystyle\int_{u_1^*}^{u_3^*} f(u)du > 0,$
and $f(u) \neq 0$ for $u \neq 0, u_1^*, u_2^*, u_3^*$.

When f satisfies (PB), we say that f is a function of *positive bistable type*. For such nonlinearity, we will show that solutions of (FBP) exhibit interesting large-time behaviors which are different from those discussed in previous works (see, e.g., Du and Lou [5] for monostable type and bistable type). Our first aim is to investigate what kind of asymptotic behaviors can be found for (FBP) with positive bistable nonlinearity. We will classify all solutions of (FBP) into the following four types:

(i) $\displaystyle\lim_{t\to\infty} h(t) < \infty$ and $\displaystyle\lim_{t\to\infty} u(t,x) = 0$ for $x \geq 0$ (vanishing),

(ii) $\displaystyle\lim_{t\to\infty} h(t) = \infty$ and $\displaystyle\lim_{t\to\infty} u(t,x) = u_1^*$ for $x \geq 0$ (small spreading),

(iii) $\displaystyle\lim_{t\to\infty} h(t) = \infty$ and $\displaystyle\lim_{t\to\infty} u(t,x) = u_3^*$ for $x \geq 0$ (big spreading),

(iv) $\displaystyle\lim_{t\to\infty} h(t) = \infty$ and $\displaystyle\lim_{t\to\infty} u(t,x) = v_{dec}(x)$ for $x \geq 0$, where v_{dec} is a uniquely determined decreasing function such that $\lim_{x\to\infty} v_{dec}(x) = u_1^*$ (transition).

For numerical simulations of these typical types of solutions, see Figure 1. Here, if we consider (FBP) for $u_0 = \sigma u_0^*$ with parameter $\sigma \geq 0$ and any fixed nonnegative function u_0^* satisfying (2), we can prove the existence of two threshold numbers σ_1^* and σ_2^* ($\sigma_1^* < \sigma_2^*$) with the following properties:

The solution of (FBP) satisfies vanishing (i) for all $\sigma \in [0, \sigma_1^*]$, small spreading (ii) for all $\sigma \in (\sigma_1^*, \sigma_2^*)$, big spreading (iii) for all $\sigma \in (\sigma_2^*, \infty)$ and transition (iv) for exactly $\sigma = \sigma_2^*$. It should be noted that, for any stable equilibrium of f, one can observe the corresponding spreading phenomenon for (FBP).

Our second aim is to study asymptotic speed of $h(t)$ and asymptotic profile of $u(t, x)$ as $t \to \infty$ when $(u(t, x), h(t))$ exhibits spreading property (ii) or (iii) (or (iv)). It is shown by Du and Lou [5] that the study of asymptotic estimates of $u(t, x)$ and $h(t)$ is closely related with the following problem

$$\text{(SWP)} \qquad \begin{cases} dq'' - cq' + f(q) = 0, \quad q(z) > 0 \quad \text{for} \ \ z \in (0, \infty), \\ q(0) = 0, \ \ \mu q'(0) = c, \quad \lim_{z \to \infty} q(z) = u^*, \end{cases}$$

with $u^* = u_1^*$ or $u^* = u_3^*$. When $(q(z), c) = (q^*(z), c^*)$ satisfies (SWP), $q^*(z)$ is called a *semi-wave with speed* c^*. Let (u, h) be a solution of (FBP) with $\lim_{t\to\infty} u(t, x) = u^*$ ($u^* = u_1^*$ or u_3^*) and let (SWP) possess a solution pair (q^*, c^*). Then it will be proved that (q^*, c^*) gives sharp estimates in the following sense:

$$\lim_{t\to\infty} \{h(t) - c^* t\} = H^* \quad \text{with some} \ \ H^* \in \boldsymbol{R},$$
$$\lim_{t\to\infty} \sup_{0 \leq x \leq h(t)} |u(t, x) - q^*(h(t) - x)| = 0.$$

The same estimates have been obtained by Du, Matsuzawa and Zhou [9] in the case that f is monostable or bistable type of nonlinearity.

The analysis of (SWP) can be carried out by using the phase plane analysis (see, e.g. [5]). It can be shown that (SWP) with $u^* = u_1^*$ always has a unique solution pair, whereas (SWP) with $u^* = u_3^*$ does not have a solution under a certain circumstance. Numerical simulations in this situation suggest that a spreading solution accompanies a propagating terrace (see Figure 2). In order to estimate such a terrace, we will use a travelling wave for the following problem:

$$\text{(TWP)} \qquad \begin{cases} dQ'' - cQ' + f(Q) = 0, \quad Q(z) > 0 \quad \text{for} \ \ z \in (-\infty, \infty), \\ \lim_{z\to -\infty} Q(z) = u_1^*, \quad Q(0) = (u_1^* + u_3^*)/2, \quad \lim_{z\to\infty} Q(z) = u_3^*. \end{cases}$$

We will prove that a semi-wave of (SWP) with $u^* = u_1^*$ together with a travelling wave of (TWP) gives a good approximation of any spreading solution of (FBP) with $\lim_{t\to\infty} u(t, x) = u_3^*$ in the case that there exists no solution pair of (SWP) with $u^* = u_3^*$.

The contents of the present paper are as follows. In Section 2 we will prepare some basic results of solutions for (FBP) with general nonlinearity f. Section 3 is devoted to the analysis of (FBP) for positive bistable nonlinearity. We will give a classification theorem and sharp estimates of spreading solutions when the corresponding semi-wave problem has a unique solution pair. In Section 4 we will estimate any spreading solution with propagating terrace by using solutions of (SWP) and (TWP). Finally, in Section 5, we will state two related topics. The first one is concerned with a free boundary problem in a radial symmetric environment of $\boldsymbol{R}^N$ and the second is the study of (FBP) with Neumann condition at $x = 0$ replaced by zero Dirichlet condition.

2. Basic results for (FBP)

In this section, we will collect some basic results on (FBP) with general nonlinearity f. The first result is the existence and uniqueness of a global solution to (FBP) (see Du-Lin [4, Theorems 2.1, 2.3 and Lemma 2.2] and Du-Lou [5, Theorem 2.4 and Lemma 2.8]).

THEOREM 2.1. *Let f and u_0 satisfy* (1) *and* (2)*, respectively. Then* (FBP) *admits a unique solution (u, h) in the following class*

$$(u,h) \in \left\{ C^{(1+\alpha)/2,1+\alpha}(\overline{\Omega}) \cap C^{1+\alpha/2,2+\alpha}(\Omega) \right\} \times C^{1+\alpha/2}[0,\infty)$$

for any $\alpha \in (0,1)$ with $\Omega = \{(t,x) \in \boldsymbol{R}^2 |\ t > 0,\ 0 < x < h(t)\}$. Moreover, (u,h) possesses the following properties:

(i) *It holds that*

$$\begin{aligned} 0 < u(t,x) \le C_1 \quad &\text{for} \ \ t > 0 \ \ \text{and} \ \ 0 < x < h(t), \\ 0 < h'(t) \le C_2 \quad &\text{for} \ \ t > 0, \end{aligned}$$

where C_1 and C_2 are positive constants depending only on $\|u_0\|_{C[0,h_0]}$ and $\|u_0\|_{C^1[0,h_0]}$, respectively.

(ii) *$u_x(t,x) < 0$ for all $t \in (0,\infty)$ and $x \in [h_0, h(t)]$.*

The second result is the comparison theorem which is a very important tool in the analysis of dynamic behavior of solutions of (FBP) (see [4, Lemma 3.5]).

THEOREM 2.2. *For $T > 0$, let $(u^*, h^*) \in \{C^{0,1}(\overline{\Omega^*_T}) \cap C^{1,2}(\Omega^*_T)\} \times C^1[0,T]$ with $\Omega^*_T = \{(t,x) \in \boldsymbol{R}^2 |\ 0 < t < T,\ 0 < x < h^*(t)\}$ satisfy*

$$\begin{cases} u^*_t \ge d u^*_{xx} + f(u^*) & \text{for} \ \ (t,x) \in \Omega^*_T, \\ u^*_x(0,t) \le 0, \quad u^*(t,h^*(t)) = 0 & \text{for} \ \ t \in [0,T], \\ (h^*)'(t) \ge -\mu u^*_x(t,h^*(t)) & \text{for} \ \ t \in [0,T]. \end{cases} \tag{3}$$

Let $(u_*, h_*) \in \{C^{0,1}(\overline{\Omega_{*,T}}) \cap C^{1,2}(\Omega_{*,T})\} \times C^1[0,T]$ *satisfy* (3) *with inequality signs replaced by inverse inequality signs, where* $\Omega_{*,T} = \{(t,x) \in \boldsymbol{R}^2 |\ 0 < t < T,\ 0 < x < h_*(t)\}$. *If*

$$h^*(0) \geq h_*(0) \quad \textit{and} \quad u^*(0,x) \geq u_*(0,x) \quad \textit{for}\ \ 0 \leq x \leq h_*(0),$$

then

$$h^*(t) \geq h_*(t) \quad \textit{for}\ \ t \in [0,T] \quad \textit{and} \quad u^*(t,x) \geq u_*(t,x) \quad \textit{for}\ \ (t,x) \in \Omega_{*,T}.$$

Remark 2.3: If (u^*, h^*) satisfies (3), $h^*(0) \geq h_0$ and

$$u^*(0,x) \geq u_0(x) \quad \text{for}\ \ 0 \leq x \leq h_0,$$

then (u^*, h^*) is called a *super-solution* of (FBP). Similarly, a *sub-solution* of (FBP) is defined with obvious modification.

We now introduce the notion of vanishing and spreading of solutions of (FBP).

Definition 2.4. *Let* (u,h) *be a solution of* (FBP). *Then* (u,h) *is called a vanishing solution if* $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$ *and it is called a spreading solution if*

$$\lim_{t\to\infty} h(t) = \infty \quad \textit{and} \quad \liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} > 0.$$

As an application of the comparison theorem (Theorem 2.2), we will give one of sufficient conditions for the spreading.

Theorem 2.5. *For positive number* ℓ, *let* φ *be a solution of*

$$\begin{cases} d\varphi'' + f(\varphi) = 0, \quad \varphi > 0 \quad \textit{in}\ \ (0,\ell), \\ \varphi'(0) = \varphi(\ell) = 0. \end{cases} \tag{4}$$

Suppose that (u_0, h_0) *satisfies* $h_0 \geq \ell$ *and* $u_0(x) \geq \varphi(x)$ *for* $x \in [0,\ell]$. *Then the solution* (u,h) *of* (FBP) *satisfies*

$$\lim_{t\to\infty} h(t) = \infty \quad \textit{and} \quad \liminf_{t\to\infty} u(t,x) \geq v^*(x) \quad \textit{for all}\ \ x \geq 0,$$

where v^* *is a minimal solution of*

$$\text{(SP)} \qquad \begin{cases} dv'' + f(v) = 0, \quad v > 0 \quad \textit{in}\ \ (0,\infty), \\ v'(0) = 0 \end{cases}$$

satisfying $v^*(x) \geq \varphi(x)$ *for all* $x \in (0,\ell)$.

This theorem can be proved by repeating the arguments used in the proofs of Theorem 2.11 and Corollary 2.12 in [17].

The following result gives a necessary and sufficient condition for the vanishing of solutions.

THEOREM 2.6. *Assume $f'(0) \neq 0$. Then a solution (u, h) of* (FBP) *is vanishing if and only if $\lim_{t\to\infty} h(t) < \infty$. In particular, if $f'(0) > 0$, then a vanishing solution satisfies*

$$\lim_{t\to\infty} h(t) \le \ell^* := \frac{\pi}{2}\sqrt{\frac{d}{f'(0)}}.$$

Proof. Let (u, h) be a solution of (FBP) such that $\lim_{t\to\infty} h(t) < \infty$. Then it is possible to prove the vanishing of the solutions, i.e, $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$ essentially in the same way as the proof of Theorem 2.10 of [17].

As to the necessity part, we will first discuss the case $f'(0) > 0$. When (u, h) is a vanishing solution, assume $\lim_{t\to\infty} h(t) > \ell^* = (\pi/2)\sqrt{d/f'(0)}$ to derive a contradiction. Then there exists a large number $T > 0$ such that $h(T) > \ell^*$. Here it should be noted that, for every $\ell > \ell^*$ there exists a unique solution $\varphi(x;\ell)$ of (4) and that $\lim_{\ell\to\ell^*} \|\varphi(\cdot;\ell)\|_{C[0,\ell]} = 0$. Therefore, we can find a suitable $\ell \in (\ell^*, h(T))$ such that $u(T, x) \ge \varphi(x;\ell)$ for $x \in [0, \ell]$. Therefore, it follows from Theorem 2.5 that

$$\lim_{t\to\infty} h(t) = \infty \quad \text{and} \quad \liminf_{tto\infty} u(t, x) \ge v^*(x) > 0 \quad \text{for} \ \ x > 0,$$

where v^* is a suitable positive solution of (SP). This is a contradiction to the vanishing of (u, h); so that h must satisfy $\lim_{t\to\infty} h(t) \le \ell^*$.

We next consider the case $f'(0) < 0$. Note that there exist positive constants η and δ such that

$$f(u) \le -\delta u \quad \text{for all} \ \ u \in [0, \eta].$$

We define $(u^*(t, x), h^*(t))$ by

$$h^*(t) = H\left(1 - \frac{1}{2}e^{-\delta t}\right) \quad \text{and} \quad u^*(t, x) = \rho e^{-\delta t}\cos\left(\frac{\pi x}{2h^*(t)}\right),$$

where H and ρ are positive constants to be determined later. We will show that (u^*, h^*) satisfies (3). If ρ satisfies $\rho \le \eta$, then

$$\begin{aligned} u_t^* - du_{xx}^* - f(u^*) &= -\delta u^* + \rho e^{-\delta t}\cdot\frac{\pi x(h^*)'(t)}{2h^*(t)^2}\cdot\sin\left(\frac{\pi x}{2h^*(t)}\right) \\ &\qquad + \frac{\pi^2 d}{4h^*(t)^2}u^* - f(u^*) \\ &\ge -\delta u^* + \frac{\pi^2 d}{4h^*(t)^2}u^* + \delta u^* = \frac{\pi^2}{4h^*(t)^2}u^* > 0. \end{aligned}$$

Moreover, if H satisfies $H^2\delta \geq 2\mu\rho\pi$, then

$$\begin{aligned}(h^*)'(t) + \mu u_x^*(t, h^*(t)) &= \frac{H\delta}{2}e^{-\delta t} - \frac{\pi\mu\rho}{2h^*(t)}e^{-\delta t}\\ &\geq \frac{H^2\delta - 2\pi\mu\rho}{2H}e^{-\delta t} > 0.\end{aligned}$$

It is easy to see $u_x^*(t,0) = 0$ and $u^*(t, h^*(t)) = 0$. Since (u,h) is a vanishing solution, we can take a sufficiently large $T > 0$ such that $\|u(T)\|_{C[0,h(T)]} \leq \eta$. Furthermore, choose sufficiently large H satisfying $h(T) \leq h^*(0) = H/2$ and $u(T,x) \leq \rho\cos(x/H)$ for $0 \leq x \leq h(T)$. Then Theorem 2.2 allows us to conclude

$$h(t+T) \leq h^*(t) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad u(t+T, x) \leq u^*(t,x)$$

for $t \geq 0$ and $0 \leq x \leq h(t+T)$. The above estimates implies $\lim_{t\to\infty} h(t) \leq \lim_{t\to\infty} h^*(t) = H$: so that the free boundary remains bounded. □

THEOREM 2.7. *Assume $f'(0) \neq 0$ and let (u,h) be a solution of* (FBP) *satisfying $\lim_{t\to\infty} h(t) = \infty$. Then it holds taht for any $R > 0$*

$$\lim_{t\to\infty} u(t,x) = v^*(x) \quad \textit{uniformly in} \quad x \in [0,R], \tag{5}$$

where v^* is a non-increasing solution of (SP).

Proof. We consider an even extension of $u(t,x)$ for $x \in [-h(t), h(t)]$ and apply the general convergence theorem due to Du and Lou [5, Theorem 1.1] (see also [6]). It can be seen from $\lim_{t\to\infty} h(t) = \infty$ that $u(t,x)$ satisfies (5) for a nonnegative function v^*, which is a solution of

$$dv_{xx}^* + f(v^*) = 0 \;\; \text{in} \;\; I := [0,\infty) \;\; \text{and} \;\; v_x^*(0) = 0.$$

Suppose $v^*(x_0) = 0$ for some $x_o \in I$. Then $v_x^*(x_0) = 0$; so that the uniqueness of solutions for the initial value problem for second-order ordinary differential equations leads to $v^* \equiv 0$ in I. Then it follows that (u,h) must be a vanishing solution. Therefore, Theorem 2.6 implies $\lim_{t\to\infty} h(t) < \infty$, which is a contradiction to the assumption. Thus v must satisfy $v^*(x) > 0$ for all $x \in I$; so that v^* is a solution of (SP). The non-increasing property is an easy consequence of (ii) of Theorem 2.1. □

Let $\mathcal{S}$ be the set of non-increasing solutions of (SP). In order to determine the complete structure of $\mathcal{S}$, we will take two types of typical examples of f:

(M) *Monostable type*: $f \in C^1[0,\infty)$ and there exists a positive number u^* such that $f(0) = f(u^*) = 0$ with $f'(0) > 0, f'(u^*) < 0,\ f(u) > 0$ for $u \in (0,u^*)$ and $f(u) < 0$ for $u > u^*$.

(B) *Bistable type*: $f \in C^1[0,\infty)$ and there exist two positive numbers u^* and θ with $0 < \theta < u^*$ such that $f(0) = f(\theta) = f(u^*) = 0$, $f'(0) < 0, f'(u^*) < 0$, $f(u) > 0$ for $u \in (\theta, u^*)$, $f(u) < 0$ for $u \in (0,\theta) \cup (u^*,\infty)$ and $\displaystyle\int_0^{u^*} f(u)du > 0$.

When f is a monostable type of function, the phase plane analysis of (SP) enables us to prove $\mathcal{S} = \{u^*\}$. Then we can obtain the following result as in [4].

Theorem 2.8. *Let f satisfy* (M) *and let (u,h) be the solution of* (FBP). *Then (u,h) satisfies one of the following properties:*

(i) *Vanishing;* $\lim_{t\to\infty} h(t) \le (\pi/2)\sqrt{d/f'(0)}$ *and* $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$.

(ii) *Spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = u^*$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

When f is a bistable type of function, one can see from the phase plane analysis that $\mathcal{S} = \{u^*, \theta, \hat{v}\}$. Here $\hat{v}$ is a monotone decreasing solution of (SP) satisfying $\hat{v}(0) = \hat{u}$ and $\lim_{x\to\infty} \hat{v}(x) = 0$, where $\hat{u} \in (\theta, u^*)$ is a unique number satisfying $\int_0^{\hat{u}} f(u)du = 0$. Furthermore, we can exclude the possibility of $\lim_{t\to\infty} u(t,x) = \theta$ by usig the zero number arguments (for details, see the proof of Theorem 3.1) . More precisely, it is possible to prove the following (see [5]):

Theorem 2.9. *Let f satisfy* (B) *and let (u,h) be the solution of* (FBP). *Then (u,h) satisfies one of the following properties:*

(i) *Vanishing;* $\lim_{t\to\infty} h(t) < \infty$ *and* $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$.

(ii) *Spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = u^*$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

(iii) *Transition;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = \hat{v}(x)$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

3. Large-time behaviors of solutions for positive bistable nonlinearity

3.1. Positive bistable nonlinearity and classification of large-time behaviors

We will take multi-stable nonlinearity f in (FBP), that is, f has a multiple number of positive stable equilibrium points. A typical example is given by

$$f(u) = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2}, \quad \text{with} \quad q, r > 0, \tag{6}$$

which is a combination of a logistic term $ru(1 - u/q)$ and a predation term called Holling type III, $-u^2/(1+u^2)$. For ecological background of such f and its analysis, see the paper of Ludwig, Aronson and Weinberger [22]. It is known that, if q and r satisfy suitable conditions, then above f has two positive stable equilibria and satisfies (PB) given in Section 1.

In what follows, we always assume that f satisfies (PB). Note that $f(u)$ is a monostable type for $0 \le u \le u_1^*$ and is a bistable type for $u_1^* \le u \le u_3^*$. Our first result is the following classification result of solutions of (FBP) based on their large-time behaviors (see [19, Theorem 3.1]).

THEOREM 3.1. *Let* (u, h) *be the solution of* (FBP). *Then it satisfies one of the following properties:*

(i) *Vanishing;* $\lim_{t\to\infty} h(t) \le (\pi/2)\sqrt{d/f'(0)}$ *and* $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$.

(ii) *Small spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = u_1^*$ *uniformly in* $x \in [0, R]$ *for any* $R > 0$.

(iii) *Big spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = u_3^*$ *uniformly in* $x \in [0, R]$ *for any* $R > 0$.

(iv) *Transition;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = v_{dec}(x)$ *uniformly in* $x \in [0, R]$ *for any* $R > 0$, *where* v_{dec} *is a solution of* (SP) *satisfying*

$$(v_{dec})'(x) < 0 \quad \textit{for} \quad x > 0 \quad \textit{and} \quad \lim_{x\to\infty} v_{dec}(x) = u_1^*.$$

Proof. Let $\mathcal{S}$ be the set of non-increasing solutions of (SP). Using the phase plane analysis one can show

$$\mathcal{S} = \{u_1^*, u_2^*, u_3^*, v_{dec}\}.$$

By virtue of Theorems 2.6 and 2.7, it is sufficient to exclude the possibility $\lim_{t\to\infty} u(t,x) = u_2^*$ in order to complete the proof.

Assuming

$$\lim_{t\to\infty} u(t,\cdot) = u_2^* \quad \text{uniformly in} \quad [0,R] \tag{7}$$

for any $R > 0$, we will derive a contradiction. Let v be a periodic solution of (SP) satisfying $v(0) = \max_{x\geq 0} v(x) > u_2^*$. The phase plane analysis yields

$$u_1^* < \min_{x\geq 0} v(x) < u_2^* < \max_{x\geq 0} v(x) < u_3^*.$$

Set $w(t,x) = u(t,x) - v(x)$. Then

$$w_t = dw_{xx} + c(t,x)w,$$

where $c(t,x) = \displaystyle\int_0^1 f'(\theta u(t,x) + (1-\theta)v(x))d\theta$ is a bounded and continuous function. For a continuous function $\varphi(x)$ defined in a closed interval I, we denote by $\mathcal{Z}_I(\varphi)$ the number of zero-points of φ in I. Setting $I(t) = [0, h(t)]$ we consider $\mathcal{Z}_{I(t)}(w(t))$. Note that $w(t,h(t)) = -v(h(t)) < 0$ and $w(t,0) = u(t,0) - v(0) < 0$ for $t \geq T$ with sufficiently large $T > 0$. Then it follows from the zero number result of Angenent [1, Theorems C and D] that $t \to \mathcal{Z}_{I(t)}(w(t))$ is finite and non-increasing for $t \geq T$. However,

$$\mathcal{Z}_{I(t_2)}(w(t_2)) > \mathcal{Z}_{I(t_1)}(w(t_1)) \quad \text{for} \quad t_2 > t_1 \geq T$$

if $t_2 - t_1$ is large because $\lim_{t\to\infty} h(t) = \infty$, u satisfies (7) and $v(x)$ is periodic with respect to x. This result contradicts to the non-increasing property of $\mathcal{Z}_{I(t)}(w(t))$; so that (7) never happens. □

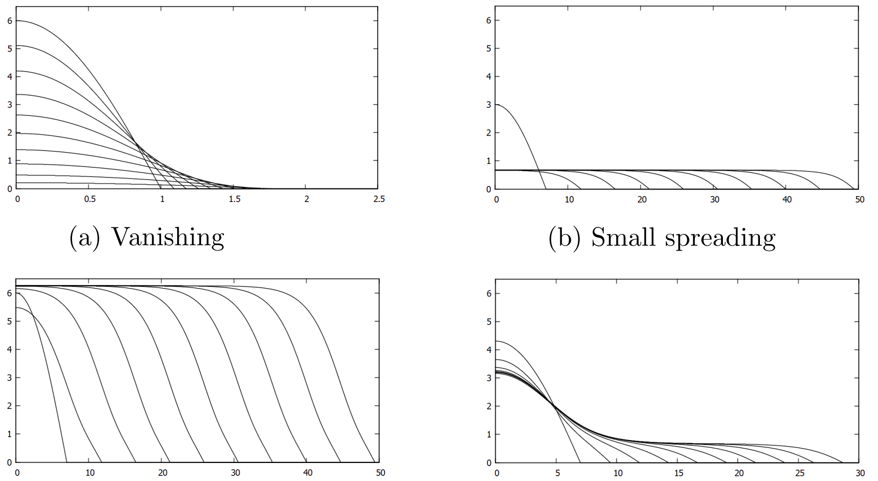
Remark 3.2: We consider (FBP) with initial condition replaced by

$$h(0) = h_0, \quad u(0,x) = \sigma u_0^*(x), \quad 0 \leq x \leq h_0,$$

where $\sigma \geq 0$ is a parameter and u_0^* is a nonnegative function satisfying (2). Denote by $(u_\sigma(t,x), h_\sigma(t))$ the solution of the above problem. By virtue of Theorems 3.7 and 3.8 in [19] there exist two threshold numbers σ_1^* and σ_2^* ($\sigma_1^* < \sigma_2^*$) with the following properties:

- For $\sigma \in [0, \sigma_1^*]$, (u_σ, h_σ) is a vanishing solution.
- For $\sigma \in (\sigma_1^*, \sigma_2^*)$, (u_σ, h_σ) is a small spreading solution.
- For $\sigma = \sigma_2^*$, (u_σ, h_σ) is a transition solution.
- For $\sigma > \sigma_2^*$, (u_σ, h_σ) is a big spreading solution.

As a result, the transition is a special solution which occurs as a borderline behavior between the small spreading and the big spreading.



(a) Vanishing (b) Small spreading

(c) Big spreading (d) Transition

Figure 1: Four types of large-time behaviors of $u(t,x)$ for (FBP) are shown as (a) vanishing, (b) small spreading, (c) big spreading and (d) transition. The right-end point of each curve represents $h(t)$ and moves forward as t goes on.

Numerical simulations for (FBP) are shown in Figure 1 for $d = 1, \mu = 0.1$ and f given by (6) with $q = 40/3$ and $r = 0.3$. As to small spreading, big spreading and transition of solutions, these simulations suggest that $u(t,x)$ proceeds like a " travelling wave " near the spreading front $x = h(t)$ for large t. We will investigate asymptotic behaviors of $(u(t,x), h(t))$ as $t \to \infty$ in the subsequent subsections.

3.2. Large-time behaviors of solutions and semi-wave problem

We will study large-time behaviors of solutions of (FBP) which possess properties (ii) and (iii) of Theorem 3.1. In the case $\lim_{t\to\infty} h(t) = \infty$, we infer from the preceding numerical simulations that such a spreading solution converges to a pair $(u(t,x), h(t))$ of the following form as $t \to \infty$:

$$h(t) = ct + H \quad (H: \;\text{ constant}), \quad u(t,x) = q(h(t) - x), \quad 0 \le x \le h(t), \quad (8)$$

where c is a positive constant and $q = q(z)$ is a positive function defined for $z \ge 0$. Substitution of (8) into the first equation of (FBP) yields

$$dq'' - cq' + f(q) = 0, \quad q(z) > 0 \;\text{ for }\; z > 0. \quad (9)$$

At $x = h(t)$ in (FBP), we get

$$q(0) = 0 \quad \text{and} \quad \mu q'(0) = c. \tag{10}$$

Moreover, since $\lim_{t\to\infty} u(t,x) = u_i^*$ $(i = 1, 3)$ uniformly in $x \in [0, R]$ for any $R > 0$, q must satisfy

$$\lim_{z\to\infty} q(z) = u_i^* \quad (i = 1, 3). \tag{11}$$

Summarizing (9), (10) and (11) we arrive at (SWP) given in Section 1. This problem was first introduced by Du and Lou [5] and it is called a *semi-wave problem*. They have shown the existence of a unique solution pair $(q, c) = (q^*, c^*)$ when f is a monostable type or a bistable type.

Let f satisfy (PB) and consider a small spreading solution or a big spreading solution of (FBP). When we discuss a small (resp. big) spreading solution, the corresponding semi-wave problem (SWP) with $u^* = u_1^*$ (resp. $u^* = u_3^*$) is denoted by (SWP-1) (resp. (SWP-3)). The solvability of these problems has been established by Kawai and Yamada [19, Theorem 4.1].

Proposition 3.3. (i) *For every $\mu > 0$,* (SWP-1) *has a unique solution pair $(q, c) = (q_S, c_S)$.*

(ii) *Case A: For every $\mu > 0$,* (SWP-3) *has a unique solution pair $(q, c) = (q_B, c_B)$.*
Case B: There exists a positive number μ^* such that (SWP-3) *has a unique solution pair $(q, c) = (q_B, c_B)$ for $\mu \in (0, \mu^*)$, whereas* (SWP-3) *has no solution for $\mu \in [\mu^*, \infty)$.*

(iii) *$q_S'(z) > 0$, $q_B'(z) > 0$ for $z \geq 0$ and $c_B > c_S$ when (q_B, c_B) exists.*

The semi-wave of (SWP) is useful for the study of asymptotic behaviors of spreading solutions as $t \to \infty$. The following result gives a rough estimate of the spreading speed of $h(t)$ (see [19, Theorem 4.2]).

Proposition 3.4. *Let c_S, c_B and μ^* be positive constants given in Proposition 3.3.*

(i) *If (u, h) is a small spreading solution of* (FBP)*, then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_S.$$

(ii) *Let (u, h) be a big spreading solution of* (FBP)*. Then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = \begin{cases} c_B & \text{if (SWP-3) has a solution pair } (q_B, c_B), \\ c_S & \text{if (SWP-3) has no solution pair.} \end{cases}$$

(iii) *If (u, h) is a transition solution of* (FBP)*, then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_S.$$

3.3. Sharp asymptotic estimates of spreading solutions

We will show that the unique solution pair of (SWP) with $u^* = u_1^*$ or u_3^* gives a good approximation of any spreading solution of (FBP) for large t whenever the corresponding semi-wave exists.

We begin with the analysis of a small spreading solution $(u(t,x), h(t))$ of (FBP). The following result gives a rough estimate of (u,h) with use of (q_S, u_S).

LEMMA 3.5. *Let (u,h) be a small spreading solution of* (FBP)*. Then there exist positive constants δ, M_1, T_1 and H_1 such that*

$$\begin{aligned} &h(t) \le c_S t + H_1, \\ &u(t,x) \le (1 + M_1 e^{-\delta t}) q_S(c_S t + H_1 - x), \end{aligned}$$

for all $t \ge T_1$ and $0 \le x \le h(t)$.

Proof. Define (u^*, h^*) by

$$\begin{cases} h^*(t) = c_S(t-T) + \rho(e^{-\delta T} - e^{-\delta t}) + H, & t \ge T, \\ u^*(t,x) = (1 + Me^{-\delta t}) q_S(h^*(t) - x), & t \ge T, \quad 0 \le x \le h^*(t), \end{cases}$$

where δ is a positive constant satisfying

$$f'(u) \le -\delta \quad \text{for} \quad u \in [u_1^* - \eta, u_1^* + \eta]$$

with some $\eta > 0$ and ρ, M, T and H are constants to be determined later. We will show that (u^*, h^*) is a super-solution of (FBP) for $t \ge T$; that is,

$$u_t^* \ge du_{xx}^* + f(u^*), \qquad t \ge T, \quad 0 \le x \le h^*(t), \tag{12}$$

$$u_x^*(t,0) \le 0, \qquad u^*(t, h^*(t)) = 0, \qquad t \ge T, \tag{13}$$

$$(h^*)'(t) \ge -\mu u_x^*(t, h^*(t)), \qquad t \ge T, \tag{14}$$

$$h^*(T) \ge h(T), \qquad u^*(T,x) \ge u(T,x), \qquad 0 \le x \le h(T). \tag{15}$$

Clearly, (13) holds and, moreover,(14) is satisfied if $\rho\delta \ge Mc_S$. If we follow the arguments in the work of Du, Matsuzawa and Zhou [9, Lemma 3.2], we can prove (12) provided that ρ is sufficiently large. Finally, taking sufficiently large T such that $u(T,x) \le u_1^* + \varepsilon$ for $0 \le x \le h(T)$ with sufficiently small $\varepsilon > 0$ and choosing sufficiently large M and H such that

$$h^*(T) = H \ge h(T) \quad \text{and} \quad u^*(T,x) = (1 + Me^{-\delta T}) q_S(H - x) \ge u(T,x),$$

for $0 \le x \le h(T)$, one can verify (15).

The application of Theorem 2.2 leads to

$$h(t) \le h^*(t) \quad \text{and} \quad u(t,x) \le u^*(t,x) \tag{16}$$

for $t \ge T$ and $0 \le x \le h(t)$. Since $q_S(z)$ is strictly increasing in $z \ge 0$, it is easy to derive the assertion from (16). □

Similarly, one can also show the following rough estimate from below.

LEMMA 3.6. *Let* (u, h) *be a small spreading solution of* (FBP). *Then there exist positive constants* δ, M_2, T_2 *and* $H_2 \in \boldsymbol{R}$ *such that*

$$h(t) \geq c_S t + H_2,$$
$$u(t,x) \geq (1 - M_1 e^{-\delta t}) q_S(c_S t + H_2 - x),$$

for all $t \geq T_2$ *and* $0 \leq x \leq c_S t + H_2$.

For the proof of this lemma, see, e.g. [9, Lemma 3.3].

We can get sharper estimates than Lemmas 3.5 and 3.6 if we repeat the arguments in [9] (see also [13, Proposition 1.3]).

THEOREM 3.7. *Let* (u, h) *be a small spreading solution of* (FBP) *and let* (q_S, c_S) *be the solution pair of* (SWP-1). *Then there exists* $H_S \in \boldsymbol{R}$ *such that*

$$\lim_{t\to\infty} (h(t) - c_S t) = H_S \quad \textit{and} \quad \lim_{t\to\infty} h'(t) = c_S$$

and

$$\lim_{t\to\infty} \sup_{0\leq x\leq h(t)} |u(t,x) - q_S(h(t) - x)| = 0.$$

Theorem 3.7 shows that (q_S, c_S) plays a very important role in the estimate of any small spreading solution (u, h) of (FBP): c_S gives an asymptotic constant speed of the free boundary $x = h(t)$ and a simple function $q(z)$ is enough to approximate $u(t, x)$ in the form of $q(h(t) - x)$ over the whole interval $[0, h(t)]$ for large t. An analogous result is also valid for any big spreading solution when (SWP-3) has a unique solution pair (q_B, c_B).

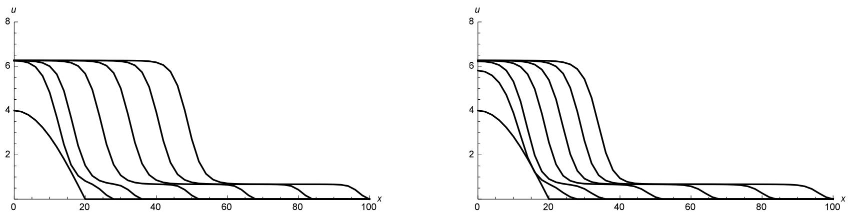
THEOREM 3.8. *Let* (u, h) *be a big spreading solution of* (FBP) *and assume that* (SWP-3) *admits a unique solution pair* (q_B, c_B). *Then there exists* $H_B \in \boldsymbol{R}$ *such that*

$$\lim_{t\to\infty} (h(t) - c_B t) = H_B \quad \textit{and} \quad \lim_{t\to\infty} h'(t) = c_B$$

and

$$\lim_{t\to\infty} \sup_{0\leq x\leq h(t)} |u(t,x) - q_B(h(t) - x)| = 0.$$

This theorem gives a sharp estimate of any big spreading solution $(u(t, x), h(t))$ over the whole interval $[0, h(t)]$ when the corresponding semi-wave exists. We will discuss its asymptotic estimate for the remaining case in the next section.



$\mu = 3$ $\qquad$ $\mu = 10$

Figure 2: Numerical simulations of (FBP) for $d = 1$ and $f(u) = u(0.5 - 0.055u) - u^2/(1 + u^2)$ with $u_1^* \approx 0.672$ and $u_3^* = 6.258$

4. Sharp asymptotic estimates of solutions with propagating terrace

We will derive asymptotic estimates of a big spreading solution (u, h) of (FBP) under the following condition

(A) Semi-wave problem (SWP-3) has no solution pair.

By Proposition 3.4 such a big spreading solution satisfies

$$\lim_{t \to \infty} \frac{h(t)}{t} = c_S,$$

where c_S is the speed of semi-wave q_S for (SWP-1). Thus (q_S, c_S) will be helpful to approximate $(u(t, x), h(t))$ around the spreading front $x = h(t)$. On the other hand,

$$\lim_{t \to \infty} u(t, x) = u_3^* \quad \text{uniformly in} \quad x \in [0, R]$$

for any $R > 0$. Taking account of these facts we guess that there must be a function like a "travelling wave", which connects u_1^* with u_3^*. Numerical simulations of such big spreading solutions are given in Figure 2 when f satisfies (A). These simulations suggest the following dynamics:

A big spreading solution proceeds like a small spreading solution around the spreading front $x = h(t)$ and a propagating function (connecting u_1^* and u_3^*) subsequently appears with slower speed in the intermediate range.

As a candidate of such a connecting function, we will take a travelling wave for (TWP) (see Section 1). It is known that (TWP) has a unique solution $(Q^*(z), c^*)$. Moreover, it follows from the result of [19, Remark 4.1] that condition (A) assures

$$c^* < c_S. \tag{17}$$

Hereafter we will study a big spreading solution (u, h) by assuming (17). We will briefly explain the arguments developed by Kaneko, Matsuzawa and

Yamada [13] to obtain sharp asymptotic estimates for $(u(t,x),h(t))$ with use of both (q_S,c_S) and (Q^*,c^*).

As the first step, define

$$u^*(t,x)=Q^*(c^*t+H-\rho e^{-\delta t}-x)+Me^{-\delta t}, \tag{18}$$

where $\delta>0$ is a constant satisfying

$$f'(u)\le -\delta \quad \text{for} \quad u\in[u_1^*-\eta,u_1^*+\eta]\cup[u_3^*-\eta,u_3^*+\eta]$$

with some $\eta>0$. Then one can choose sufficiently small $M>0$ and large positive ρ,H and T such that

$$\begin{array}{ll} u_t^*\ge du_{xx}^*+f(u^*) & \text{for } \ t\ge T, \ \ 0\le x\le h(t),\\ u_x^*(t,0)<0, \qquad u^*(t,h(t))>0 & \text{for } \ t\ge T,\\ u^*(T,x)\ge u(T,x) & \text{for } \ 0\le x\le h(T).\end{array}$$

The comparison principle for parabolic equations yields

$$u(t,x)\le u^*(t,x)$$

for $t\ge T$ and $0\le x\le h(t)$. Since $Q^*(z)$ is strictly increasing in z, the above estimate together with (18) allows us to show the following result (see [13, Lemma 3.5]).

Lemma 4.1. *Let (u,h) a big spreading solution of* (FBP). *Then there exist positive constants δ,M_1,H_1 and T_1 such that*

$$u(t,x)\le Q^*(c^*t+H_1-x)+M_1e^{-\delta t}$$

for all $t\ge T_1$ and $0\le x\le h(t)$.

The second step is to derive the following rough estimate for (u,h) from below.

Lemma 4.2. *Let (u,h) be a big spreading solution of* (FBP). *Then there exists constants $T_2>0$ and $H_2\in\boldsymbol{R}$ such that*

$$h(t)\ge c_St+H_2, \quad u(t,x)\ge q_s(c_St+H_2-x)$$

for all $t\ge T_2$ and $0\le x\le c_St+H_2$.

Proof. We define

$$h_*(t)=c_St+H, \quad u_*(t,x)=q_*(c_St+H-x),$$

where H is a number to be determined later. It is easy to verify

$$\begin{array}{ll} (u_*)_t = d(u_*)_{xx} + f(u_*), & t > 0, \quad 0 \le x \le h_*(t), \\ u_*(t, h_*(t)) = 0, & t > 0, \\ (h_*)'(t) = -\mu (u_*)_x(t, h_*(t)), & t > 0. \end{array}$$

We can choose a sufficiently large $T > 0$ such that

$$u_*(t,0) < u_1^* < u(t+T,0) \quad \text{for} \quad t \ge 0;$$

then

$$h_*(0) = H < h(T), \quad u_*(0,x) = q_S(H-x) \le u(T,x) \quad \text{for} \quad 0 \le x \le H$$

with small $H > 0$. Therefore, the comparison principle allows us to derive

$$h_*(t) \le h(t+T), \quad u_*(t,x) \le u(t+T,x)$$

for $t \ge 0$ and $0 \le x \le h_*(t)$; so that the assertion follows from the above inequalities. □

In Lemmas 4.1 and 4.2, note (17) and $\lim_{z\to-\infty} Q^*(z) = u_1^*$. Therefore, if c satisfies $c^* < c < c_S$, then $u(t, ct) \to u_1^*$ as $t \to \infty$. More precisely, it is possible to show the following result from Lemmas 4.1 and 4.2:

Proposition 4.3. *Let* (u,h) *be any big spreading solution of* (FBP). *Then*

$$\lim_{t\to\infty} \sup_{c_1 t \le x \le c_2 t} |u(t,x) - u_1^*| = 0$$

for any c_1 *and* c_2 *satisfying* $c^* < c_1 < c_2 < c_S$.

Roughly speaking, Proposition 4.3 implies that $u(t,x)$ stays at almost constant u_1^* when x lies in an intermediate range $[c_1 t, c_2 t]$ of $(0, h(t))$ with $c^* < c_1 < c_2 < c_S$. Taking account of this fact we will be able to obtain a similar result to Lemma 3.5. As the third step, one can repeat the proof of Lemma 3.5 with some modification and prove the following lemma (see also [13, Lemma 3.9]).

Lemma 4.4. *Let* (u,h) *be a big spreading solution of* (FBP). *Then for any* $c \in (c^*, c_S)$ *there exist positive constants* δ, M_3, T_3 *and* $H_3 \in \boldsymbol{R}$ *such that*

$$\begin{array}{l} h((t) \le c_S t + H_3, \\ u(t,x) \le (1 + M_3 e^{-\delta t}) q_S(c_S t + H_3 - x), \end{array}$$

for all $t \ge T_3$ *and* $ct \le x \le h(t)$.

Lemmas 4.2 and 4.4 yield rough estimates of any big spreading solution $u(t,x)$ over $[ct, h(t)]$ for any $c \in (c^*, c_S)$ if t is sufficiently large. Therefore, the arguments developed by Du, Matsuzawa and Zhou [9] are valid and allow us to get the following sharp estimate (for details, see the proofs of (1.11) and (1.12) in [13]).

Theorem 4.5. *Let* (u,h) *be any big spreading solution of* (FBP). *Then there exists* $H_s \in \boldsymbol{R}$ *such that*

$$\lim_{t\to\infty}(h(t) - c_S t) = H_S, \quad \lim_{t\to\infty} h'(t) = c_S$$

and, for any $c \in (c^*, c_S)$,

$$\lim_{t\to\infty} \sup_{ct\le x\le h(t)} |u(t,x) - q_S(h(t) - x)| = 0.$$

The final step is to estimate $u(t,x)$ from below when x lies in $[0, ct]$ for any $c \in (c^*, c_S)$.

Lemma 4.6. *Let* (u,h) *be any big spreading solution of* (FBP). *Then, for any* $c \in (c^*, c_S)$, *there exist positive constants* δ, M_4, T_4 *and* $H_4 \in \boldsymbol{R}$ *such that*

$$u(t,x) \ge Q^*(c^*t + H_4 - x) - M_4 e^{-\delta t}$$

for all $t \ge T_4$ *and* $0 \le x \le ct$.

For the proof of this lemma, see [13, Lemma 3.8].

Since we have established Lemmas 4.1 and 4.6, we are ready to approximate $u(t,x)$ over $[0, ct]$ for any $c \in (c^*, c_S)$ by using travelling wave (Q^*, c^*) of (TWP). Indeed, we have the following theorem whose proof can be found in [13, Section 5].

Theorem 4.7. *Let* (u,h) *be any big spreading solution of* (FBP). *Then there exists* $H^* \in \boldsymbol{R}$ *such that for any* $c \in (c^*, c_S)$

$$\lim_{t\to\infty} \sup_{0\le x\le ct} |u(t,x) - Q^*(c^*t + H^* - x)| = 0.$$

Owing to Theorems 4.5 and 4.7, we have obtained sharp asymptotic estimates of any big spreading solution under assumption (A). In this situation, the semi-wave of (SW-1) gives a good approximation of $u(t,x)$ near the spreading front $x = h(t)$, whereas $u(t,x)$ is sharply estimated by the travelling wave of (TWP) over the other range in $[0, h(t)]$. In particular, we can say that for large t a big spreading solution proceed at almost constant speed c_S and it is accompanied by a propagating terrace with slower speed c^*.

REMARK 4.8: It is also possible to derive sharp estimates for a transition solution. Indeed, a transition solution (u, h) satisfies the same assertion as Theorem 4.5 for any $c \in (0, c_S)$ and, furthermore,

$$\lim_{t\to\infty} \sup_{0\le x\le ct} |u(t,x) - v_{dec}(x)| = 0$$

for any $c \in (0, c_S)$ (see [13, Theorem C]).

5. Concluding remarks

5.1. Free boundary problem in $\boldsymbol{R}^N$

In this subsection we will consider a free boundary problem for a reaction-diffusion equation in $\boldsymbol{R}^N$. We focus on the problem in a radially symmetric environment. So it is formulated in the following form for a pair of unknown function $u = u(t, r)$ with $r = |x|$ ($x \in \boldsymbol{R}^N$) and $h = h(t)$:

$$\begin{cases} u_t = d\Delta u + f(u), & t > 0, \quad 0 < r < h(t), \\ u_r(t,0) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0, \quad u(0,r) = u_0(r), & 0 \le r \le h_0, \end{cases} \tag{19}$$

where d, μ and h_0 are positive constants, $\Delta u = u_{rr} + (N-1)u_r/r$ and u_0 is a nonnegative function satisfying (2). When f satisfies (1), it is shown by Du and Guo [2] that (19) admits a unique global solution which possesses similar properties to those in Theorem 2.1. Moreover, basic properties on the comparison principle and large-time behaviors of solutions hold true as in the one-dimensional case (see [2], [7] and [12]). In particular, if f satisfies (M) (resp. (B)), it is also possible to show the same classification result as Theorem 2.8 (resp. Theorem 2.9) established for $N = 1$. For the study of free boundary problems for general domain, see, for instance, [3], [7] and [8].

We will investigate (19) for positive bistable nonlinearity. In addition to (PB), we put the following condition on f:

(PB-1) $f(u)/(u - \overline{u})$ is non-increasing for $u \in (\overline{u}, u_3^*)$, where $\overline{u} \in (u_2^*, u_3^*)$ is a unique number determined by $\displaystyle\int_{u_2^*}^{\overline{u}} f(s)ds = 0$.

Then it is possible to prove the following classification theorem which corresponds to Theorem 3.1 (see [14, Theorem A]).

THEOREM 5.1. *Let f satisfy* (PB) *and* (PB-1). *Then the solution* (u, h) *of* (19) *satisfies one of the following properties:*

(i) *Vanishing;* $\lim_{t\to\infty} h(t) \le \sqrt{d\lambda_1/f'(0)}$ *and* $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$*, where* λ_1 *is the principal eigenvalue of*

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \ \Omega := \{x \in \boldsymbol{R}^N \mid \ |x| < 1\}, \\ \varphi = 0 & \text{on } \ \partial\Omega := \{x \in \boldsymbol{R}^N \mid \ |x| = 1\}. \end{cases}$$

(ii) *Small spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,r) = u_1^*$ *uniformly in* $r \in [0,R]$ *for any* $R > 0$*.*

(iii) *Big spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,r) = u_3^*$ *uniformly in* $r \in [0,R]$ *for any* $R > 0$*.*

(iv) *Transition;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,r) = V_{dec}(r)$ *uniformly in* $r \in [0,R]$ *for any* $R > 0$*, where* V_{dec} *is a decreasing solution of*

$$\begin{cases} dV_{rr} + (N-1)V_r/r + f(V) = 0, \quad V(r) > 0 \qquad \text{for } \ r > 0, \\ V_r(0) = 0, \end{cases} \tag{20}$$

and it satisfies $\lim_{r\to\infty} V_{dec}(r) = u_1^*$*.*

Note that (20) corresponds to stationary problem (SP) for $N = 1$. In the proof of Theorem 5.1, it is important to study the set of non-increasing solutions of (20). We need to take a different approach from the phase plane analysis which is efficient for $N = 1$.

As to large-time behaviors of spreading solutions $(u(t,r), h(t))$ of (19), semi-waves for (SWP) are still available in the analysis. Indeed, rough estimates of the free boundary are given by the following result (see [14, Theorem C]).

Proposition 5.2. *Assume that* f *satisfies* (PB) *and* (PB-1)*. Let* (u,h) *be the solution of* (19)*. Then the same conclusions as Proposition 3.4 hold true.*

This proposition implies that there is no difference between $N = 1$ and $N \ge 2$ in order to give rough estimates of $h(t) \to \infty$ as $t \to \infty$.

The dependence on the space dimension N appears in sharp estimates of $h(t)$ of spreading solutions. They have been obtained by Du, Matsuzawa and Zhou [10] in the case that f satisfies (M) or (B). When f is positive bistable nonlinearity satisfying (PB) and (PB-1), we can prove similar results for small spreading solutions or big spreading solutions. Let (u,h) be a small spreading solution or a big spreading solution of (19) and let the corresponding semi-wave problem (SWP) possess a solution pair $(q^*(z), c^*)$ for $u^* = u_1^*$ or $u^* = u_3^*$. Then it is possible to show the following estimate ([15]):

There exists a constant $R^* \in \boldsymbol{R}$ such that

$$\lim_{t\to\infty}\{h(t) - c^*t + (N-1)c_* \log t\} = R^*,$$
$$\lim_{t\to\infty} \sup_{0\le r\le h(t)} |u(t,r) - q^*(h(t)-r)| = 0,$$

where $c_* = 1/(\zeta c^*)$,

$$\zeta = 1 + \frac{c^*}{\mu^2 \int_0^\infty \{(q^*)'(z)\}^2 e^{-c^* z} dz}$$

(see also [10, Theorem 4.1]). For a big spreading solution (u,h) whose corresponding semi-wave problem (SWP-3) has no solution pair, we can also give sharp estimates with use of semi-wave (q_S, c_S) of (SWP-1) and travelling wave (Q^*, c^*) of (TWP):
There exist $R_S, R_B \in \boldsymbol{R}$ such that

$$\lim_{t\to\infty}\{h(t) - c_S t + (N-1)c_{S*} \log t\} = R_S$$

with $c_{S*} = 1/(\zeta c_S)$,

$$\zeta = 1 + \frac{c_S}{\mu^2 \int_0^\infty \{(q_S)'(z)\}^2 e^{-c_S z} dz}$$

and, for sufficiently large $L > 0$,

$$\lim_{t\to\infty} \sup_{c_S t - L\log t \le r \le h(t)} |u(t,r) - q_S(h(t)-r)| = 0,$$
$$\lim_{t\to\infty} \sup_{0\le r\le c_S t - L\log t} \sup \left| u(t,r) - Q^*\left(c^*t - \frac{N-1}{c^*}\log t + R_B - r\right)\right| = 0.$$

For details of these results, see [15].

5.2. Free boundary problem with Dirichlet boundary condition

In this subsection we will consider (FBP) with zero Neumann condition at $x = 0$ replaced by zero Dirichlet condition. The problem is written as follows:

$$\begin{cases} u_t = du_{xx} + f(u), & t > 0,\ 0 < x < h(t), \\ u(t,0) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases} \tag{21}$$

where f is a function satisfying (1) and u_0 is a nonnegative function of class $C^2[0, h_0]$ such that

$$u_0(0) = u_0(h_0) = 0 \quad \text{and} \quad u_0 \not\equiv 0. \tag{22}$$

Basic results such as the existence and uniqueness of global solutions (Theorem 2.1) and the comparison principle (Theorem 2.2) are valid with obvious modification (see Kaneko and Yamada [17]). The notion of vanishing and spreading of solutions to (21) is the same as Definition 2.4. It should be noted that the following theorem holds true in place of Theorem 2.6 (see [16] and [17]).

THEOREM 5.3. *Assume* $f'(0) \neq 0$. *Then a solution* (u, h) *of* (21) *is vanishing if and only if* $\lim_{t\to\infty} h(t) < \infty$. *Moreover, if* $f'(0) > 0$, *a spreading solution satisfies*

$$\lim_{t\to\infty} h(t) \leq \pi\sqrt{d/f'(0)}.$$

For the case $\lim_{t\to\infty} h(t) = \infty$, it is also possible to prove the following theorem similarly to Theorem 2.7 (see [11, Proposition 4.7]).

THEOREM 5.4. *Assume* $f'(0) \neq 0$ *and let* (u, h) *be the solution of* (21) *satisfying* $\lim_{t\to\infty} h(t) = \infty$. *Then it holds that for any* $R > 0$

$$\lim_{t\to\infty} u(t, x) = v^*(x) \quad \textit{uniformly in} \quad x \in [0, R],$$

where v^* *is a bounded solution of*

$$\begin{cases} dv'' + f(v) = 0, & v(x) > 0 \quad \textit{for} \quad x \in (0, \infty), \\ v(0) = 0. \end{cases} \tag{23}$$

We now consider positive bistable nonlinearity f, which satisfies (PB). Let $\mathcal{S}$ be the set of bounded solutions of stationary problem (23). The phase plane analysis is available to get

$$\mathcal{S} = \{v_1, v_3\},$$

where v_i is an increasing solution of (23) satisfying

$$\lim_{x\to\infty} v_i(x) = u_i^*$$

for each $i = 1, 3$. Therefore, Theorems 5.3 and 5.4 enable us to show the following classification theorem (see Endo, Kaneko and Yamada [11, Theorem 4.1]).

THEOREM 5.5. *Under assumption* (PB), *the solution* (u, h) *of* (21) *satisfies one of the following properties:*

(i) *Vanishing;* $\lim_{t\to\infty} h(t) \leq \pi\sqrt{d/f'(0)}$ *and* $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$.

(ii) *Small spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = v_1(x)$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

(iii) *Big spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = v_3(x)$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

REMARK 5.6: Differently from the classification result for the Neumann boundary condition (Theorem 3.1), a transition solution does not appear as a borderline one in Theorem 5.5. But small spreading solutions can be divided into two subgroups;

(a) small spreading solutions with $\liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} < u_2^*$,

(b) small spreading solutions with $\liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} \geq u_2^*$

(see [11, Remark 5]). We have a conjecture that a small spreading solution in the latter subgroup exhibits a borderline behavior between small spreading solutions in the former subgroup and big spreading solutions. For the related problem, see the works of Liu and Lou [20, 21]. They discussed the existence of a transition solution with a moving peak as a borderline behavior for f satisfying (B).

In the study of large-time behaviors of solutions with $\lim_{t\to\infty} h(t) = \infty$, semi-waves of (SWP) also play a crucial role. Indeed, we can obtain the following result (see [11, Theorems 5.3 and 5.5]).

THEOREM 5.7. *Under assumption* (PB)*, let* (u,h) *be a small spreading solution of* (21) *satisfying* $\liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} < u_2^*$ *and let* (q_S, c_S) *be the solution pair of* (SWP-1)*. Then there exists* $h_S \in \boldsymbol{R}$ *such that*

$$\lim_{t\to\infty} \{h(t) - c_S t\} = h_S \qquad \text{and} \qquad \lim_{t\to\infty} h'(t) = c_S$$

and

$$\lim_{t\to\infty} \sup_{h(t)/2 \leq x \leq h(t)} |u(t,x) - q_S(h(t)-x)| = 0.$$

Moreover, for any $c \in (0, c_S)$*,*

$$\lim_{t\to\infty} \sup_{0 \leq x \leq ct} |u(t,x) - v_1(x)| = 0.$$

REMARK 5.8: Let (u,h) be a small spreading solution of (21) sch that u satisfies $\liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} \geq u_2^*$. Then $u(t,x)$ has a moving peak at $x = x_t^*$ such that $u(t,x_t^*) \geq x_2^* - \delta$ with some $\delta > 0$ for sufficiently large t. On the other hand, it satisfies $\lim_{t\to\infty} u(t,x) = v_1(x) < u_1^*$ for each $x \in [0,\infty)$. Therefore, $u(t,x)$ cannot be estimated by only q_S and v_1. Approximation of such a spreading solution is an interesting open problem.

When (u, h) is a big spreading solution of (21) and (SWP-3) has a solution pair (q_B, c_B), it is seen from [18] that similar results to Theorem 5.7 hold true (see also [11, Theorems 5.4 and 5.5]).

When (SWP-3) has no solution pair, a big spreading solution will be approximated with use of semi-wave $(q_S.c_S)$ of (SWP-1), travelling wave (Q^*, c^*) of (TWP) and stationary solution v_3 of (23). We will discuss this problem elsewhere.

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Author's address:

Yoshio Yamada
Department of Pure and Applied Mathematics,
Waseda University,
3-4-1 Ohkubo, Shinjuku-ku,
Tokyo 169-8555, JAPAN
E-mail: `yamada@waseda.jp`

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