Global structure of bifurcation curves related to inverse bifurcation problems

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Dedicated to Professor Julián López-Gómez on his sixtieth birthday

Abstract. *We consider the nonlinear eigenvalue problem*

$$[D(u(t))u(t)']' + \lambda g(u(t)) = 0,$$
$$u(t) > 0, \quad t \in I := (0,1), \quad u(0) = u(1) = 0,$$

which comes from the porous media type equation. Here, $D(u) = pu^{2n} + \sin u$ ($n \in \mathbb{N}$, $p > 0$: given constants), $g(u) = u$ or $g(u) = u + \sin u$. $\lambda > 0$ is a bifurcation parameter which is a continuous function of $\alpha = \|u_\lambda\|_\infty$ of the solution u_λ corresponding to λ, and is expressed as $\lambda = \lambda(\alpha)$. Since our equation contains oscillatory term in diffusion term, it seems significant to study how this oscillatory term gives effect to the structure of bifurcation curves $\lambda(\alpha)$. We propose a question from a view point of inverse bifurcation problems and show that the simplest case $D(u) = u^2 + \sin u$ and $g(u) = u$ gives us the most impressible asymptotic formula for global behavior of $\lambda(\alpha)$.

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1. Introduction

We study the following nonlinear eigenvalue problems

$$[D(u(t))u(t)']' + \lambda g(u(t)) = 0, \qquad t \in I := (0,1), \tag{1}$$
$$u(t) > 0, \qquad t \in I, \tag{2}$$
$$u(0) = u(1) = 0, \tag{3}$$

where $D(u) := pu^{2n} + \sin u$ ($n \in \mathbb{N}$, $p > 0$: given constants), $g(u) = u$ or $g(u) = u + \sin u$, and $\lambda > 0$ is a bifurcation parameter. We assume the following condition (A.1).

(A.1) $D(u) > 0$ for $u > 0$.

Under the condition (A.1), we know from [11] that for a given $\alpha > 0$, there is a unique solution pair (u_α, λ) of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$. Moreover, λ is parameterized by $\alpha > 0$ as $\lambda(\alpha)$ and is continuous for $\alpha > 0$.

The purpose of this paper is to show how the oscillatory diffusion term $D(u)$ gives effect to the structure of bifurcation curves $\lambda(\alpha)$. To clarify our intention, let $n = p = 1$ in (1) for simplicity. Then (A.1) is satisfied and we have the equation

$$[\{u(t)^2 + \sin u(t)\}u'(t)]' + \lambda u(t) = 0, \qquad t \in I. \tag{4}$$

The other equations similar to (4) are

$$[u(t)^2 u(t)']' + \lambda(u(t) + \sin u(t)) = 0, \qquad t \in I, \tag{5}$$

$$[\{u(t)^2 + \sin u(t)\}u(t)']' + \lambda(u(t) + \sin u(t)) = 0, \qquad t \in I. \tag{6}$$

We propose the following question from a view point of inverse bifurcation problems

QUESTION A. *Consider* (4), *or* (5), *or* (6) *with* (2)–(3). *Then is it possible to distinguish* (4), (5) *and* (6) *from the asymptotic behavior of* $\lambda(\alpha)$ *for* $\alpha \gg 1$ *or not?*

We explain the back ground of Question A more precisely. Bifurcation problems with $D(u) \equiv 1$ are one of the main interest in the study of differential equations, and many results have been established concerning the asymptotic behavior of bifurcation curves from mathematical point of view. We refer to [1, 2, 3, 4, 9, 10, 12, 13, 14] and the references therein. Besides, the bifurcation problems with nonlinear diffusion appear in the various fields. The case $D(u) = u^k$ $(k > 0)$ appears as the porous media equation in material science and logistic type model equation in population dynamics. In the latter case, it implies that the diffusion rate $D(u)$ depends on both the population density u and a parameter $1/\lambda$. We refer to [11, 15, 19] and the references therein. Added to these, there are several papers studying the asymptotic behavior of oscillatory bifurcation curves. We refer to [6, 7, 8, 9, 16, 17, 18] and the references therein.

Recently, the following equation has been considered in [18].

$$[D(u(t))u(t)']' + \lambda g(u(t)) = 0, \qquad t \in I \tag{7}$$

with (2)–(3). Here, $D(u) = u^k$, $g(u) = u^{2m-k-1} + \sin u$, and $m \in \mathbb{N}$, k $(0 \le k < 2m - 1)$ are given constants. In particular, if we put $m = k = 2$, then we have the equation (5). In [18], the following result has been obtained.

THEOREM 1.1 ([18]). *Consider* (7) *with* (2)–(3). *Then as* $\alpha \to \infty$,

$$\lambda(\alpha) = 4m\alpha^{2k+2-2m} \left\{ A_{k,m}^2 - 2A_{k,m} \sqrt{\frac{\pi}{2m}} \, \alpha^{k+(1/2)-2m} \sin\left(\alpha - \frac{\pi}{4}\right) \right.$$
$$\left. + \, o(\alpha^{k+(1/2)-2m}) \right\}, \tag{8}$$

where

$$A_{k,m} = \int_0^1 \frac{s^k}{\sqrt{1-s^{2m}}} ds.$$

If $m = k = 2$ in (8), then the asymptotic formula for $\lambda(\alpha)$ as $\alpha \to \infty$ for (5) is given by

$$\lambda(\alpha) = 8\alpha^2 \left\{ A_{2,2}^2 - A_{2,2}\sqrt{\pi}\alpha^{-3/2} \sin\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-3/2}) \right\}. \tag{9}$$

Moreover, it was shown in [18] that if $m = k = 2$, then $\lambda(\alpha) = 4B_0^2\alpha^2(1+o(1))$ as $\alpha \to 0$, where B_0 is a positive constant. Therefore, the rough picture of $\lambda(\alpha)$ in (5) is depicted in Figure 1.
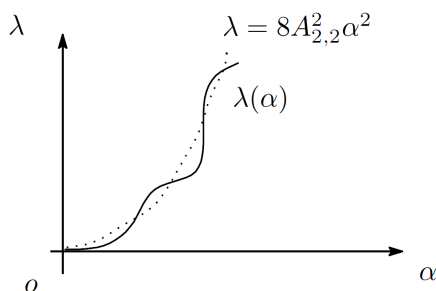

Figure 1: The graph of $\lambda(\alpha)$ for (5).

We understand from Theorem 1.1 that, by the effect of $\sin u$, $\lambda(\alpha)$ in Figure 1 oscillates and crosses the curve $\lambda = 8A_{2,2}^2\alpha^2$, which are the *original* bifurcation curve obtained from the equation (5) *without* $\sin u$, infinitely many times. Motivated by this, we would like to study how oscillatory nonlinear diffusion influences on the structure of $\lambda(\alpha)$. The question we have to ask here is whether the oscillatory term $\sin u$ in $D(u)$ gives the same influence on the asymptotic behavior of $\lambda(\alpha)$ as (8) or not.

Now we state our main results which give us the answer to this question. We begin with the typical case $n = 1$ and $g(u) = u$, since in this case, we are able to obtain up to the third term of $\lambda(\alpha)$. This fact seems to be a signifgicant progress in the study of asymptotic behavior of $\lambda(\alpha)$ as $\alpha \to \infty$.

THEOREM 1.2. *Assume (A.1). Let $n = 1$. Consider (1)–(3) with $g(u) = u$.*

(i) *As $\alpha \to \infty$,*

$$\lambda(\alpha) = 8pA_{2,2}^2\alpha^2 + \frac{8(p-1)}{p}A_{2,2}\sqrt{\pi}\alpha^{-1/2}\sin\left(\alpha - \frac{\pi}{4}\right)$$

$$+ \frac{8}{p}\left\{2A_{2,2}\left(p - 2B\right)\cos\alpha\right\}\alpha^{-1} + O(\alpha^{-3/2}), \qquad (10)$$

where

$$B := 2\int_0^{\pi/2} \frac{\sin^2\theta}{(1 + \sin^2\theta)^{5/2}}\left(1 - \frac{1}{2}\sin^2\theta\right)d\theta. \qquad (11)$$

In particular, let $p = 1$. Then (10) is represented as

$$\lambda(\alpha) = 8A_{2,2}^2\alpha^2 + (16A_{2,2}\left(1 - 2B\right)\cos\alpha)\alpha^{-1} + O(\alpha^{-3/2}). \qquad (12)$$

(ii) *As $\alpha \to 0$,*

$$\lambda(\alpha) = 6\alpha(C_0^2 + 2pC_0C_1\alpha + O(\alpha^2)),$$

where

$$C_0 := \int_0^1 \frac{s}{\sqrt{1 - s^3}}ds, \quad C_1 := \int_0^1 \frac{1}{\sqrt{1 - s^3}}\left(s^2 - \frac{3s(1 - s^4)}{8(1 - s^3)}\right)ds.$$

Therefore, our first conclusion is that the the shape of the second term of (12) is completely different from that of (9), since we are able to show by direct calculation that $B \neq 1/2$. Namely, the answer to Question A is affirmative. The global structures of the bifurcation curves for (4) and (5) do not coincide each other.

For the case $n \geq 2$, we obtain up to the second term of $\lambda(\alpha)$.

THEOREM 1.3. *Assume (A.1). Let $n \geq 2$. Consider (1)–(3) with $g(u) = u$. Then as $\alpha \to \infty$,*

$$\lambda(\alpha) = \frac{4(n+1)}{p}\left(p^2 A_{2n,n+1}^2\alpha^{2n} + 2A_{2n,n+1}(p-1)\right.$$

$$\left.\times \sqrt{\frac{\pi}{2(n+1)}}\alpha^{-1/2}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1})\right).$$

In particular, let $p = 1$. Then as $\alpha \to \infty$,

$$\lambda(\alpha) = 4(n+1)A_{2n,n+1}^2\alpha^{2n} + O(\alpha^{-1}). \qquad (13)$$

REMARK 1.4: (i) Certainly, the future direction of this study will be to obtain the exact second term of (13), although it seems very difficult to get it technically.

(ii) Theorem 1.2-(ii) is only proved for the case $n = 1$ to show that the rough picture of $\lambda(\alpha)$ is almost the same as that of Fig. 1 if $p \neq 1$. Certainly, we easily obtain Theorem 1.2-(ii) for the case $n \geq 2$.

The following Theorem 1.5 gives us the negative answer to Question A.

THEOREM 1.5. *Consider* (6) *with* (2)–(3).

(i) *The asymptotic formula* (9) *holds as* $\alpha \to \infty$.

(ii) *The following asymptotic formula holds as* $\alpha \to 0$.

$$\lambda(\alpha) = 3\alpha \left(C_0^2 + 2C_0 C_1 \alpha + O(\alpha^2) \right).$$

We find from Theorems 1.2–1.5 that $\sin u$ in diffusion term has deep influences on the global behavior of $\lambda(\alpha)$.

We prove our results by using time-map method and stationary phase method.

2. Proof of Theorem 1.3-(i)

In this section, let $D(u) = pu^{2n} + \sin u$, $g(u) = u$ and $\alpha \gg 1$. We denote by C the various positive constants independent of $\alpha \gg 1$. We put

$$\Lambda := \left\{ \alpha > 0 \mid g(\alpha) > 0, \int_u^\alpha g(t)D(t)dt > 0 \text{ for all } u \in [0, \alpha) \right\}.$$

It follows from [11, (2.7)], that if $\alpha \in \Lambda$, then $\lambda(\alpha)$ is well defined. By (A.1), we have $D(t) > 0$, $g(t) > 0$ for $t > 0$. So $g(t)D(t) > 0$ for $t > 0$ holds. Hence, $\Lambda \equiv \mathbb{R}_+$. By this and the generalized time-map in [9, (2.5)] (cf. (15) below) and the time-map argument in [10, Theorem 2.1], we find that for any given $\alpha > 0$, there is a unique solution pair $(u_\alpha, \lambda) \in C^2(I) \bigcap C(\bar{I}) \times \mathbb{R}_+$ of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$. Moreover, λ is parameterized by α as $\lambda = \lambda(\alpha)$ and is a continuous function for $\alpha > 0$. It is well known that if $(u_\alpha, \lambda(\alpha)) \in C^2(I) \bigcap C(\bar{I}) \times \mathbb{R}_+$ satisfies (1)–(3), then

$$u_\alpha(t) = u_\alpha(1 - t), \quad 0 \leq t \leq 1,$$

$$u_\alpha \left(\frac{1}{2} \right) = \max_{0 \leq t \leq 1} u_\alpha(t) = \alpha,$$

$$u_\alpha'(t) > 0, \quad 0 < t < \frac{1}{2}.$$

We put

$$G(u) := \int_0^u D(x)g(x)dx = \int_0^u (px^{2n} + \sin x)xdx \tag{14}$$

$$= \frac{p}{2n + 2}u^{2n+2} - u\cos u + \sin u,$$

$$M_1 := \alpha \cos \alpha - \alpha s \cos(\alpha s), \ \ M_2 := \sin \alpha - \sin(\alpha s) \ \ (0 \le s \le 1).$$

For $0 \le s \le 1$ and $\alpha \gg 1$, we have

$$\frac{|M_1| + |M_2|}{\alpha^{2n+2}(1 - s^{2n+2})} \le C\alpha^{-2n}. \tag{15}$$

By this, Taylor expansion and putting $u = s\alpha$, we have from [9] that

$$
\begin{aligned}
\sqrt{\frac{\lambda}{2}} &= \int_0^\alpha \frac{D(u)}{\sqrt{G(\alpha) - G(u)}} du \tag{16} \\
&= \alpha \int_0^1 \frac{p\alpha^{2n} s^{2n} + \sin(\alpha s)}{\sqrt{\frac{p}{2n+2}\alpha^{2n+2}(1 - s^{2n+2}) - (M_1 - M_2)}} ds \\
&= \sqrt{\frac{2n+2}{p}} \alpha^{-n} \int_0^1 \frac{p\alpha^{2n} s^{2n} + \sin(\alpha s)}{\sqrt{1 - s^{2n+2}}\sqrt{1 - \frac{2n+2}{p\alpha^{2n+2}(1 - s^{2n+2})}\{M_1 - M_2\}}} ds \\
&= \sqrt{\frac{2n+2}{p}} \alpha^{-n} \int_0^1 \frac{p\alpha^{2n} s^{2n} + \sin(\alpha s)}{\sqrt{1 - s^{2n+2}}} \\
&\qquad \times \left(1 + \frac{n+1}{p\alpha^{2n+2}(1 - s^{2n+2})}\{M_1 - M_2\}(1 + O(\alpha^{-2n}))\right) ds.
\end{aligned}
$$

This implies that

$$\sqrt{\frac{\lambda}{2}} = \sqrt{\frac{2n+2}{p}} \alpha^{-n}\{J_1 + J_2 + J_3 + J_4 + J_5\}(1 + O(\alpha^{-2n})), \tag{17}$$

where

$$
\begin{aligned}
J_1 &:= p\alpha^{2n} \int_0^1 \frac{s^{2n}}{\sqrt{1 - s^{2n+2}}} ds = pA_{2n,n}\alpha^{2n}, \\
J_2 &:= \int_0^1 \frac{\sin(\alpha s)}{\sqrt{1 - s^{2n+2}}} ds, \tag{18} \\
J_3 &:= \frac{n+1}{p\alpha} \int_0^1 \frac{s^{2n}}{(1 - s^{2n+2})^{3/2}}(\cos \alpha - s \cos(\alpha s)) ds, \tag{19} \\
J_4 &:= -\frac{n+1}{p\alpha^2} \int_0^1 \frac{s^{2n}}{(1 - s^{2n+2})^{3/2}}(\sin \alpha - \sin(\alpha s)) ds, \\
J_5 &:= \frac{n+1}{p\alpha^{2n+2}} \int_0^1 \frac{\sin(\alpha s)}{(1 - s^{2n+2})^{3/2}}\{M_1 - M_2\} ds. \tag{20}
\end{aligned}
$$

To calculate $J_2 \sim J_5$, we use the following equality.

LEMMA 2.1 ([7, Lemma 2], [9, Lemma 2.25]). *Assume that the function* $f(r) \in C^2[0,1]$, $w(r) = \cos(\pi r/2)$. *Then as* $\mu \to \infty$

$$\int_0^1 f(r)e^{i\mu w(r)}dr = e^{i(\mu-(\pi/4))}\sqrt{\frac{2}{\mu\pi}}f(0) + O\left(\frac{1}{\mu}\right). \tag{21}$$

In particular, by taking the real and imaginary parts of (21), as $\mu \to \infty$,

$$\int_0^1 f(r)\cos(\mu w(r))dr = \sqrt{\frac{2}{\mu\pi}}f(0)\cos\left(\mu - \frac{\pi}{4}\right) + O\left(\frac{1}{\mu}\right),$$

$$\int_0^1 f(r)\sin(\mu w(r))dr = \sqrt{\frac{2}{\mu\pi}}f(0)\sin\left(\mu - \frac{\pi}{4}\right) + O\left(\frac{1}{\mu}\right).$$

LEMMA 2.2. *As* $\alpha \to \infty$,

$$J_2 = \sqrt{\frac{\pi}{2(n+1)\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1}).$$

Proof. Putting $s = \sin\theta$, $\theta = \frac{\pi}{2}(1-x)$ and using Lemma 2.1, we have

$$
\begin{aligned}
J_2 &= \int_0^1 \frac{\sin(\alpha s)}{\sqrt{1-s^2}\sqrt{1+s^2+\cdots+s^{2n}}}ds \tag{22}\\
&= \int_0^{\pi/2} \frac{1}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2n}\theta}}\sin(\alpha\sin\theta)d\theta\\
&= \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\sqrt{1+\cos^2\left(\frac{\pi}{2}x\right)+\cdots+\cos^{2n}\left(\frac{\pi}{2}x\right)}}\sin\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right)dx\\
&= \sqrt{\frac{\pi}{2(n+1)\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1}).
\end{aligned}
$$

Thus the proof is complete. □

LEMMA 2.3. *As* $\alpha \to \infty$,

$$J_3 = -\frac{1}{p}\sqrt{\frac{\pi}{2(n+1)\alpha}}\left\{\sin\left(\alpha - \frac{\pi}{4}\right) - \alpha^{-1}\cos\left(\alpha - \frac{\pi}{4}\right)\right\} + O(\alpha^{-1}). \tag{23}$$

Proof. We put $J_3 = (n+1)J_{31}/(p\alpha)$, $s = \sin\theta$ and $K(\theta) := \sin^{2n}\theta/(1+\sin^2\theta+$

$\cdots + \sin^{2n}\theta)^{3/2}$. Then by integration by parts,

$$J_{31} = \int_0^1 \frac{s^{2n}}{(1-s^2)^{3/2}(1+s^2+\cdots+s^{2n})^{3/2}}(\cos\alpha - s\cos(\alpha s))ds \qquad (24)$$

$$= \int_0^{\pi/2} \frac{1}{\cos^2\theta}K(\theta)(\cos\alpha - \sin\theta\cos(\alpha\sin\theta))d\theta$$

$$= [\tan\theta K(\theta)(\cos\alpha - \sin\theta\cos(\alpha\sin\theta))]_0^{\pi/2}$$

$$- \int_0^{\pi/2} \tan\theta K'(\theta)(\cos\alpha - \sin\theta\cos(\alpha\sin\theta))d\theta$$

$$- \int_0^{\pi/2} \tan\theta K(\theta)\{-\cos\theta\cos(\alpha\sin\theta) + \alpha\sin\theta\cos\theta\sin(\alpha\sin\theta)\}d\theta$$

$$:= J_{311} - J_{312} + J_{313}.$$

By using l'Hôpital's rule, we have

$$\lim_{\theta\to\pi/2} \frac{\cos\alpha - \sin\theta\cos(\alpha\sin\theta)}{\cos\theta}$$

$$= \lim_{\theta\to\pi/2} \frac{-\cos\theta\cos(\alpha\sin\theta) + \alpha\sin\theta\cos\theta\sin(\alpha\sin\theta)}{-\sin\theta} = 0.$$

We see from this that $J_{311} = 0$. Moreover, by direct calculation, we see that $J_{312} = O(1)$. Now, putting $\theta = \frac{\pi}{2}(1-x)$ and using Lemma 2.1, we have

$$J_{313} = \int_0^{\pi/2} \sin\theta K(\theta)\cos(\alpha\sin\theta)d\theta \qquad (25)$$

$$- \alpha \int_0^{\pi/2} K(\theta)\sin^2\theta\sin(\alpha\sin\theta)d\theta$$

$$= \frac{\pi}{2}\int_0^1 \frac{\cos^{2n+1}\left(\frac{\pi}{2}x\right)}{\left(1+\cos^2\left(\frac{\pi}{2}x\right)+\cdots+\cos^{2n}\left(\frac{\pi}{2}x\right)\right)^{3/2}}\cos\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right)dx$$

$$- \frac{\pi}{2}\alpha \int_0^1 \frac{\cos^{2n+2}\left(\frac{\pi}{2}x\right)}{\left(1+\cos^2\left(\frac{\pi}{2}x\right)+\cdots+\cos^{2n}\left(\frac{\pi}{2}x\right)\right)^{3/2}}\sin\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right)dx$$

$$= (n+1)^{-3/2}\sqrt{\frac{\pi}{2\alpha}}\cos\left(\alpha-\frac{\pi}{4}\right) - (n+1)^{-3/2}\sqrt{\frac{\pi\alpha}{2}}\sin\left(\alpha-\frac{\pi}{4}\right) + O(1).$$

This implies (23). Thus the proof is complete. □

LEMMA 2.4. *As* $\alpha \to \infty$,

$$J_4 = -\frac{1}{p}\sqrt{\frac{\pi}{2(n+1)\alpha}}\alpha^{-1}\cos\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-2}). \qquad (26)$$

Proof. We put $J_4 = -(n+1)J_{41}/(p\alpha^2)$ and $s = \sin\theta$. Then by the same argument as that to obtain J_{31} in (25), we obtain

$$
\begin{aligned}
J_{41} &= \int_0^1 \frac{s^{2n}}{(1-s^{2n+2})^{3/2}}(\sin\alpha - \sin(\alpha s))ds \\
&= \int_0^{\pi/2} \frac{1}{\cos^2\theta}K(\theta)(\sin\alpha - \sin(\alpha\sin\theta))d\theta \\
&= [\tan\theta K(\theta)(\sin\alpha - \sin(\alpha\sin\theta))]_0^{\pi/2} \\
&\quad - \int_0^{\pi/2}\tan\theta K'(\theta)(\sin\alpha - \sin(\alpha\sin\theta))d\theta \\
&\quad + \alpha\int_0^{\pi/2}\sin\theta K(\theta)\cos(\alpha\sin\theta)d\theta \\
&= \frac{\pi}{2}\alpha\int_0^1 \frac{\cos^{2n+1}\left(\frac{\pi}{2}x\right)}{\left(1+\cos^2\left(\frac{\pi}{2}x\right)+\cdots+\cos^{2n}\left(\frac{\pi}{2}x\right)\right)^{3/2}}\cos\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right)dx \\
&\quad + O(1) \\
&= (n+1)^{-3/2}\sqrt{\frac{\pi\alpha}{2}}\cos\left(\alpha - \frac{\pi}{4}\right) + O(1).
\end{aligned}
$$

By this, we obtain (26). Thus the proof is complete. $\qquad\square$

Proof of Theorem 1.3-(i). By (15) and (20), we see that $J_5 = O(\alpha^{-2n})$. By this, (17) and Lemmas 2.2–2.4, we obtain

$$
\sqrt{\frac{\lambda}{2}} = \sqrt{\frac{2n+2}{p}}\alpha^{-n}\left[pA_{2n,n}\alpha^{2n} + \left(1 - \frac{1}{p}\right)\right.
$$
$$
\left. \times \sqrt{\frac{\pi}{2(n+1)\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1})\right].
$$

By this, we obtain

$$
\lambda = \frac{4(n+1)}{p}\alpha^{-2n}\left[p^2 A_{2n,n+1}^2\alpha^{4n} + 2(p-1)A_{2n,n+1}\right. \tag{27}
$$
$$
\left. \times \alpha^{2n-(1/2)}\sqrt{\frac{\pi}{2(n+1)}}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{2n-1})\right].
$$

This implies Theorem 1.3-(i). Thus the proof is complete. $\qquad\square$

3. Proof of Theorem 1.2-(i)

Let $n = 1$ in this section. Based on the calculation in the previous section and the argument of stationary phase method (cf. [9, Lemmas 2.24 and 2.25]), we calculate the third term of (10).

LEMMA 3.1. *Let $\alpha \gg 1$. Then*

$$\Phi(\alpha) := \int_0^1 e^{-i\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} e^{-i\pi/4} + \frac{i}{2\alpha} e^{-i\alpha} + O(\alpha^{-3/2}). \qquad (28)$$

Proof. We put $t = \sqrt{\alpha}x$. Then by integral by parts, we obtain

$$\Phi(\alpha) = \frac{1}{\sqrt{\alpha}} \int_0^{\sqrt{\alpha}} e^{-it^2} dt \qquad (29)$$

$$= \frac{1}{\sqrt{\alpha}} \left\{ \int_0^{\infty} e^{-it^2} dt - \int_{\sqrt{\alpha}}^{\infty} e^{-it^2} dt \right\}$$

$$=: \frac{1}{\sqrt{\alpha}} \left\{ \frac{\sqrt{\pi}}{2} e^{-i\pi/4} - \Phi_0(\alpha) \right\}.$$

Then by integration by parts,

$$\Phi_0(\alpha) = \int_{\sqrt{\alpha}}^{\infty} \frac{1}{-2it} (e^{-it^2})' dt \qquad (30)$$

$$= \left[\frac{1}{-2it} e^{-it^2} \right]_{\sqrt{\alpha}}^{\infty} + \frac{i}{2} \int_{\sqrt{\alpha}}^{\infty} \left(-\frac{1}{t} \right)' e^{-it^2} dt$$

$$=: \frac{1}{2i\sqrt{\alpha}} e^{-i\alpha} + \frac{i}{2} \Phi_1(\alpha).$$

Then

$$\Phi_1(\alpha) = \int_{\sqrt{\alpha}}^{\infty} \frac{1}{t^2} e^{-it^2} dt = \int_{\sqrt{\alpha}}^{\infty} \frac{1}{t^2} \left(\frac{1}{-2it} \right) (e^{-it^2})' dt \qquad (31)$$

$$= \left[\frac{i}{2} \frac{1}{t^3} e^{-it^2} \right]_{\sqrt{\alpha}}^{\infty} + \frac{3i}{2} \int_{\sqrt{\alpha}}^{\infty} \frac{1}{t^4} e^{-it^2} dt = O(\alpha^{-3/2}).$$

By (29)–(31), we obtain (28). Thus the proof is complete. $\qquad \square$

LEMMA 3.2. *Let $\alpha \gg 1$, $g(x) := \cos(\pi x/2)$ and $k(x) \in C^3([0,1])$. Then as $\alpha \to \infty$,*

$$I := \int_0^1 k(x) e^{i\alpha g(x)} dx \qquad (32)$$

$$= \frac{4}{\pi} e^{i\alpha} \int_0^1 k \left(\frac{2}{\pi} \cos^{-1}(1 - t^2) \right) \frac{1}{\sqrt{2 - t^2}} e^{-i\alpha t^2} dt.$$

Proof. We put $t = \sqrt{1 - \cos\left(\frac{\pi}{2}x\right)}$. Then by direct calculation, we obtain (32).
$\qquad \square$

LEMMA 3.3. *Let* $f(x) \in C^3[0,1]$. *Then as* $\alpha \to \infty$,

$$II := \int_0^1 f(x)e^{-i\alpha x^2}\,dx \tag{33}$$

$$= f(0)\left\{\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}e^{-i\pi/4} + \frac{i}{2\alpha}e^{-i\alpha}\right\} + \frac{i}{2\alpha}(h(1)e^{-i\alpha} - h(0)) + O(\alpha^{-3/2}),$$

where $h(x) := (f(x) - f(0))/x$.

Proof. We have

$$II = \int_0^1 (f(0) + h(x)x)e^{-i\alpha x^2}\,dx \tag{34}$$

$$= f(0)\int_0^1 e^{-i\alpha x^2}\,dx + III$$

$$:= f(0)\int_0^1 e^{-i\alpha x^2}\,dx + \int_0^1 h(x)xe^{-i\alpha x^2}\,dx.$$

By Lemma 3.1, we have

$$III = \int_0^1 h(x)\left(\frac{1}{-2i\alpha}e^{-i\alpha x^2}\right)'\,dx$$

$$= \left[h(x)\left(\frac{1}{-2i\alpha}e^{-i\alpha x^2}\right)\right]_0^1 + \frac{1}{2i\alpha}\int_0^1 h'(x)e^{-i\alpha x^2}\,dx$$

$$= \frac{i}{2\alpha}(h(1)e^{-i\alpha} - h(0)) + \frac{1}{2i\alpha}\left(h'(0)\int_0^1 e^{-i\alpha x^2}\,dx + O(\alpha^{-1})\right)$$

$$= \frac{i}{2\alpha}(h(1)e^{-i\alpha} - h(0)) + O(\alpha^{-3/2}).$$

By this, Lemma 3.1 and (34), we obtain (33). Thus the proof is complete. $\square$

LEMMA 3.4. *Consider* J_2 *defined in* (18). *Then as* $\alpha \to \infty$,

$$J_2 = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + \frac{1}{\alpha} + O(\alpha^{-3/2}). \tag{35}$$

Proof. Since $n = 1$, by (22), we have

$$J_2 = \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1 + \cos^2\left(\frac{\pi}{2}x\right)}}\sin\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right)\,dx := \frac{\pi}{2}J_{21}. \tag{36}$$

We put $g(x) = \cos\left(\frac{\pi}{2}x\right)$ and $k(x) = 1/\sqrt{1 + \cos^2\left(\frac{\pi}{2}x\right)}$. Then by using Lemma 3.2, we have

$$J_{21} = \operatorname{Im}\left(\frac{4}{\pi}e^{i\alpha}\int_0^1 \frac{1}{\sqrt{2 - 2t^2 + t^4}}\frac{1}{\sqrt{2 - t^2}}e^{-i\alpha t^2}dt\right). \qquad (37)$$

We put $m(t) := 4 - 6t^2 + 4t^4 - t^6$. Then we have

$$K(t) := \frac{1}{\sqrt{2 - 2t^2 + t^4}}\frac{1}{\sqrt{2 - t^2}} = \frac{1}{\sqrt{m(t)}}, \qquad (38)$$

$$h(t) := \frac{K(t) - K(0)}{t} = \frac{6t - 4t^3 + t^5}{2\sqrt{m(t)}(\sqrt{m(t)} + 2)}. \qquad (39)$$

Clearly, $h(t) \in C^3[0, 1]$. By (38) and (39), we have

$$m(0) = 4, \ m(1) = 1, \ K(0) = \frac{1}{2}, \ h(0) = 0, \ h(1) = \frac{1}{2}.$$

By this and Lemma 3.2, we obtain

$$J_{21} = \frac{4}{\pi}e^{i\alpha}\left[\frac{\sqrt{\pi}}{4\sqrt{\alpha}}e^{-i\pi/4} + \frac{i}{2\alpha}e^{-i\alpha} + O(\alpha^{-3/2})\right],$$

$$\operatorname{Im}J_{21} = \frac{1}{\sqrt{\alpha\pi}}\sin\left(\alpha - \frac{\pi}{4}\right) + \frac{2}{\alpha\pi} + O(\alpha^{-3/2}).$$

By this and (36), we obtain (35). Thus the proof is complete. □

LEMMA 3.5. *Consider J_3 defined in (19). Let B be the constant defined in (11). Then as $\alpha \to \infty$,*

$$J_3 = -\frac{1}{2p}\sqrt{\frac{\pi}{\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) - \frac{2B}{p\alpha}\cos\alpha + O(\alpha^{-3/2}).$$

Proof. We use the same notation as those in Lemma 2.3. We have $J_3 = \frac{2}{p\alpha}J_{31}$ and $J_{31} = -J_{312} + J_{313}$. Since $n = 1$, we have $K(\theta) = \sin^2\theta/(1 + \sin^2\theta)^{3/2}$. We have

$$K'(\theta) = \frac{2\sin\theta\cos\theta}{(1 + \sin^2)^{5/2}}\left(1 - \frac{1}{2}\sin^2\theta\right).$$

By this and Lemma 2.1, we obtain

$$J_{312} = \int_0^1 \frac{2\sin^2\theta}{(1 + \sin^2\theta)^{5/2}}\left(1 - \frac{1}{2}\sin^2\theta\right)(\cos\alpha - \sin\theta\cos(\alpha\sin\theta))d\theta$$

$$= B\cos\alpha + O(\alpha^{-1/2}).$$

Next, by (24) and (25), we have

$$J_{313} = 2^{-3/2}\sqrt{\frac{\pi}{2\alpha}}\cos\left(\alpha - \frac{\pi}{4}\right)$$
$$- \frac{\pi}{2}\alpha\int_0^{\pi/2}\frac{\cos^4\left(\frac{\pi\theta}{2}\right)}{\left(1 + \cos^2\left(\frac{\pi\theta}{2}\right)\right)^{3/2}}\sin\left(\alpha\cos\left(\frac{\pi\theta}{2}\right)\right)d\theta.$$

For $g(\theta) = \cos(\pi\theta/2)$, let

$$N := \int_0^1 \frac{\cos^4\left(\frac{\pi\theta}{2}\right)}{\left(1 + \cos^2\left(\frac{\pi\theta}{2}\right)\right)^{3/2}}e^{-i\alpha g(\theta)}d\theta.$$

Then by the same calculation as that in (37), we have

$$N = \frac{4}{\pi}e^{i\alpha}\int_0^1 \frac{1}{\sqrt{m(t)}}\frac{(1-t^2)^4}{2 - 2t^2 + t^4}e^{-i\alpha t^2}dt.$$

We put

$$K(t) := \frac{1}{\sqrt{m(t)}}M(t), \quad M(t) := \frac{(1-t^2)^4}{2 - 2t^2 + t^4}, \quad H(t) := \frac{K(t) - K(0)}{t}. \quad (40)$$

Let $X := (1 - t^2)^2$. Then by direct calculation, we have

$$H(t) = M(t)h(t) + \frac{(2X + 1)(-2t + t^3)}{4(1 + X)},$$

where $h(t)$ is a function defined in (39). By this and (40), we have

$$K(1) = 0, \quad K(0) = \frac{1}{4}, \quad X(1) = 0, \quad X(0) = 1, \quad M(1) = 0, \quad M(0) = \frac{1}{2},$$
$$h(1) = \frac{1}{2}, \quad h(0) = 0, \quad H(1) = -\frac{1}{4}, \quad H(0) = 0.$$

By this and Lemmas 3.2 and 3.3, we obtain

$$N = \frac{1}{2\sqrt{\alpha\pi}}e^{i(\alpha-\pi/4)} + O(\alpha^{-3/2}),$$
$$\mathrm{Im}N = \frac{1}{2\sqrt{\alpha\pi}}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-3/2}),$$
$$J_{313} = -\frac{\pi\alpha}{2}\left(\frac{1}{2\sqrt{\alpha\pi}}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-3/2})\right).$$

Then

$$J_3 = \frac{2}{p\alpha}(-J_{312} + J_{313}) = -\frac{1}{2p}\sqrt{\frac{\pi}{\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right)$$
$$-\frac{2B}{p\alpha}\cos\alpha + O(\alpha^{-3/2}).$$

Thus the proof is complete. □

Proof of Theorem 1.2-(i). We know from Lemma 2.4 and the first line of the proof of Theorem 1.3-(i) that $J_4 = O(\alpha^{-3/2})$ and $J_5 = O(\alpha^{-2})$. Then by (17) and Lemmas 3.4 and 3.5, we obtain

$$\sqrt{\frac{\lambda}{2}} = \frac{2}{\sqrt{p}}\alpha^{-1}\left[pA_{2,2}\alpha^2 + \frac{p-1}{2p}\sqrt{\frac{\pi}{\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right)\right.$$
$$\left.+\frac{1}{p\alpha}(p - 2B\cos\alpha) + O(\alpha^{-3/2})\right].$$

By this and direct calculation, we obtain (10). Thus the proof of Theorem 1.2-(i) is complete. □

4. Proof of Theorem 1.5-(i)

In this section, let $D(u) = u^2 + \sin u$ and $g(u) = u + \sin u$. It follows from [11, (2.7)], we also find, as in Section 2, that for any given $\alpha > 0$, there is a unique classical solution pair (λ, u_α) of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$. Moreover, λ is parameterized by α as $\lambda = \lambda(\alpha)$ and is a continuous function for $\alpha > 0$. Let $u \geq 0$. We put

$$G(u) := \int_0^u g(y)D(y)dy$$
$$= \frac{1}{4}u^4 - u\cos u + \sin u + (2u\sin u - (u^2 - 2)\cos u - 2)$$
$$+ \frac{1}{2}\left(u - \frac{1}{2}\sin 2u\right)$$
$$:= \frac{1}{4}u^4 + G_1(u).$$

For $0 \leq s \leq 1$ and $\alpha \gg 1$, we have

$$
\begin{aligned}
G(\alpha) - G(\alpha s) &= \frac{1}{4}\alpha^4(1 - s^4) + G_1(\alpha) - G_1(\alpha s) \qquad\qquad (41)\\
&= \frac{1}{4}\alpha^4(1 - s^4) - (\alpha\cos\alpha - \alpha s\cos(\alpha s)) + (\sin\alpha - \sin(\alpha s))\\
&\quad + 2(\alpha\sin\alpha - \alpha s\sin(\alpha s)) - (\alpha^2\cos\alpha - \alpha^2 s^2\cos(\alpha s))\\
&\quad + 2(\cos\alpha - \cos(\alpha s)) + \frac{1}{2}\alpha(1 - s) - \frac{1}{4}(\sin 2\alpha - \sin 2\alpha s))\\
&:= \frac{1}{4}\alpha^4(1 - s^4) - I_1 + I_2 + I_3 - I_4 + I_5 + I_6 - I_7.
\end{aligned}
$$

It is easy to see that for $0 \leq s \leq 1$,

$$
\left|\frac{I_4}{\alpha^4(1 - s^4)}\right| \leq C\alpha^{-1}, \qquad\qquad (42)
$$

$$
\left|\frac{I_1}{\alpha^4(1 - s^4)}\right|, \left|\frac{I_3}{\alpha^4(1 - s^4)}\right| \leq C\alpha^{-2}, \qquad\qquad (43)
$$

$$
\left|\frac{I_2}{\alpha^4(1 - s^4)}\right|, \left|\frac{I_5}{\alpha^4(1 - s^4)}\right|, \left|\frac{I_6}{\alpha^4(1 - s^4)}\right|, \left|\frac{I_7}{\alpha^4(1 - s^4)}\right| \leq C\alpha^{-3}. \qquad (44)
$$

By putting $u = \alpha s$, (42)–(44) and Taylor expansion, we have from [11, (2.5)] that

$$
\begin{aligned}
\sqrt{\frac{\lambda(\alpha)}{2}} &= \int_0^\alpha \frac{D(u)}{\sqrt{G(\alpha) - G(u)}}\,du \qquad\qquad (45)\\
&= \int_0^\alpha \frac{u^2 + \sin u}{\sqrt{\frac{1}{4}(\alpha^4 - u^4) + G_1(\alpha) - G_1(u)}}\,du\\
&= \alpha\int_0^1 \frac{\alpha^2 s^2 + \sin\alpha s}{\sqrt{\alpha^4(1 - s^4)/4 + G_1(\alpha) - G_1(\alpha s)}}\,ds\\
&= 2\alpha^{-1}\int_0^1 \frac{\alpha^2 s^2 + \sin\alpha s}{\sqrt{1 - s^4}\sqrt{1 + \frac{4}{\alpha^4(1 - s^4)}(G_1(\alpha) - G_1(\alpha s))}}\,ds\\
&= 2\alpha^{-1}\int_0^1 \frac{1}{\sqrt{1 - s^4}}(\alpha^2 s^2 + \sin\alpha s)\\
&\quad \times \left\{1 - \frac{2}{\alpha^4(1 - s^4)}(G_1(\alpha) - G_1(\alpha s))(1 + O(\alpha^{-1}))\right\}ds.
\end{aligned}
$$

Now we show that the leading and second terms of the right hand side of

(45) are

$$L_1 := 2\alpha \int_0^1 \frac{s^2}{\sqrt{1-s^4}} ds = 2A_{2,2}\alpha, \tag{46}$$

$$L_4 := 4\alpha^{-5} \int_0^1 \frac{s^2}{(1-s^4)^{3/2}} I_4 ds.$$

Indeed, by (42)–(44), we obtain

$$2\alpha^{-1} \int_0^1 \frac{|\alpha^2 s^2 + \sin \alpha s|}{\sqrt{1-s^4}} \cdot \frac{1}{\alpha^4(1-s^4)} |I_1 + I_3 + I_5 + I_6 + I_7| ds = O(\alpha^{-1}).$$

Furthermore, by Lemma 2.2,

$$L_2 := 2\alpha^{-1} \int_0^1 \frac{\sin(\alpha s)}{\sqrt{1-s^4}} ds = (1+o(1))\sqrt{\pi}\alpha^{-3/2} \sin\left(\alpha - \frac{\pi}{4}\right).$$

We calculate L_4 by Lemma 2.1.

LEMMA 4.1. *As $\alpha \to \infty$,*

$$L_4 = -\sqrt{\pi}\alpha^{-1/2} \sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1}). \tag{47}$$

Proof. We put $s = \sin\theta$. Then

$$L_4 = 4\alpha^{-1} \int_0^1 \frac{s^2(\cos\alpha - s^2\cos(\alpha s))}{(1-s^2)^{3/2}(1+s^2)^{3/2}} ds$$

$$= 4\alpha^{-1} \int_0^{\pi/2} \frac{1}{\cos^2\theta} Y(\theta)(\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta)) d\theta,$$

where $Y(\theta) = \sin^2\theta/(1+\sin^2\theta)^{3/2}$. By Integration by parts, we have

$$L_4 = 4\alpha^{-1} \left[\tan\theta Y(\theta)(\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta))\right]_0^{\pi/2} \tag{48}$$

$$- 4\alpha^{-1} \int_0^1 \tan\theta \left\{Y(\theta)(\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta))\right\}' d\theta$$

$$= 4\alpha^{-1}(L_{41} - L_{42}).$$

By using l'Hôpital's rule, we have

$$\lim_{\theta\to\pi/2} \frac{\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta)}{\cos\theta}$$

$$= \lim_{\theta\to\pi/2} \frac{-2\sin\theta\cos\theta\cos(\alpha\sin\theta) + \alpha\sin^2\theta\cos\theta\sin(\alpha\sin\theta)}{-\sin\theta} = 0.$$

By this, we see that $L_{41} = 0$. It is easy to see that

$$\int_0^1 \tan\theta \left\{ Y(\theta)'(\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta)) \right\} d\theta = O(1).$$

By this, putting $\theta = \frac{\pi}{2}(1-x)$ and using Lemma 2.1, we have

$$
\begin{aligned}
L_{42} &= \int_0^1 \tan\theta\, Y(\theta)(\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta))' d\theta + O(1) \\
&= \alpha \int_0^{\pi/2} \frac{\sin^5\theta}{(1+\sin^2\theta)^{3/2}} \sin(\alpha\sin\theta) d\theta + O(1) \\
&= \frac{\pi}{2}\alpha \int_0^1 \frac{\cos^5\left(\frac{\pi}{2}x\right)}{(1+\cos^2\left(\frac{\pi}{2}x\right))^{3/2}} \sin\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right) dx + O(1) \\
&= \frac{\sqrt{\pi\alpha}}{4} \sin\left(\alpha - \frac{\pi}{4}\right) + O(1).
\end{aligned}
$$

By this and (48), we obtain (47). Thus the proof is complete. $\qquad\square$

Proof of Theorem 1.5-(i). By (45), (46), Lemma 4.1, we obtain

$$\sqrt{\frac{\lambda}{2}} = 2A_{2,2}\alpha - \sqrt{\pi}\alpha^{-1/2}\sin\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-1/2}).$$

This implies (9). Thus the proof of Theorem 1.5-(i) is complete. $\qquad\square$

5. Proofs of Theorems 1.2-(ii) and 1.5-(ii)

In this section, let $0 < \alpha \ll 1$.

Proof of Theorem 1.2-(ii). Let $n = 1$, namely, $D(u) = pu^2 + \sin u$. By (14), Taylor expansion and direct calculation, for $0 \le s \le 1$, we have

$$G(\alpha) - G(\alpha s) = \frac{1}{3}\alpha^3(1-s^3) + \frac{p}{4}\alpha^4(1-s^4) + O(\alpha^5)(1-s^5).$$

By this, putting $\theta = \alpha s$ and (16), we obtain

$$\sqrt{\frac{\lambda}{2}} = \alpha \int_0^1 \frac{\alpha s + p\alpha^2 s^2 + O(\alpha^3)}{\sqrt{\frac{1}{3}\alpha^3(1 - s^3) + \frac{p}{4}\alpha^4(1 - s^4) + O(\alpha^5)(1 - s^5)}} ds \qquad (49)$$

$$= \sqrt{3\alpha} \int_0^1 \frac{s + p\alpha s^2 + O(\alpha^2)}{\sqrt{(1 - s^3) + \frac{3p}{4}\alpha(1 - s^4) + O(\alpha^2)(1 - s^5)}} ds$$

$$= \sqrt{3\alpha} \int_0^1 \frac{1}{\sqrt{1 - s^3}}(s + p\alpha s^2 + O(\alpha^2)) \left(1 - \frac{3p(1 - s^4)}{8(1 - s^3)}\alpha + O(\alpha^2)\right) ds$$

$$= \sqrt{3\alpha} \left\{\int_0^1 \frac{s}{\sqrt{1 - s^3}} ds + p\alpha \int_0^1 \frac{1}{\sqrt{1 - s^3}}\left(s^2 - \frac{3s(1 - s^4)}{8(1 - s^3)}\right) ds + O(\alpha^2)\right\}.$$

By this, we obtain Theorem 1.2-(ii). Thus the proof is complete. $\qquad \square$

Proof of Theorem 1.5-(ii). By (41), Taylor expansion and direct calculation, for $0 \leq s \leq 1$, we have

$$G(\alpha) - G(\alpha s) = \frac{2}{3}\alpha^3(1 - s^3) + \frac{1}{2}\alpha^4(1 - s^4) + O(\alpha^5)(1 - s^5).$$

By this, putting $\theta = \alpha s$ and (49), we obtain

$$\sqrt{\frac{\lambda}{2}} = \alpha \int_0^1 \frac{\alpha^2 s^2 + \sin(\alpha s)}{\sqrt{\frac{2}{3}\alpha^3(1 - s^3) + \frac{1}{2}\alpha^4(1 - s^4) + O(\alpha^5)(1 - s^5)}} ds$$

$$= \sqrt{\frac{3}{2\alpha}} \int_0^1 \frac{\alpha s + \alpha^2 s^2 + O(\alpha^3)s^3}{\sqrt{1 - s^3}\sqrt{1 + \frac{3}{4}\alpha\frac{1 - s^4}{1 - s^3} + O(\alpha^2)\frac{1 - s^5}{1 - s^3}}} ds$$

$$= \sqrt{\frac{3\alpha}{2}} \int_0^1 \frac{1}{\sqrt{1 - s^3}}(s + \alpha s^2 + O(\alpha^2)) \left(1 - \frac{3}{8}\alpha\frac{1 - s^4}{1 - s^3} + O(\alpha^2)\right) ds$$

$$= \sqrt{\frac{3\alpha}{2}} \left(\int_0^1 \frac{s}{\sqrt{1 - s^3}} ds + \alpha \left\{\int_0^1 \frac{s^2}{\sqrt{1 - s^3}} ds - \frac{3}{8}\int_0^1 \frac{s(1 - s^4)}{(1 - s^3)^{3/2}} ds\right\}\right.$$
$$\left. + O(\alpha^2)\right).$$

By this, we obtain Theorem 1.5-(ii). Thus the proof is complete. $\qquad \square$

Acknowledgements

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