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Population dynamics in hostile neighborhoods

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*Dedicated to Julián López-Gómez,
a mathematical friend for many years.*

Abstract. *A new class of quasilinear reaction-diffusion equations is introduced for which the mass flow never reaches the boundary. It is proved that the initial value problem is well-posed in an appropriate weighted Sobolev space setting.*

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1. Introduction

It has been known since long that spacial interactions in population dynamics can be adequately modeled by systems of reaction-diffusion equations (see E.E. Holmes et al. [10], A. Okubo and S.A. Levin [17], or J.D. Murray [15], [16], for instance). In general, these systems possess a quasilinear structure and show an extremely rich qualitative behavior depending on the various structural assumptions which can meaningfully be imposed.

Reaction-diffusion equations are of great importance also in many other scientific areas as, for example, physics, chemistry, mechanical and chemical engineering, and the social sciences. Thus our mathematical results are not restricted to population dynamics. It is just a matter of convenience to describe the phenomenological background and motivation in terms of populations.

Throughout this paper, Ω is a bounded domain in $\mathbb{R}^m$ with a smooth boundary Γ lying locally on one side of Ω. (In population dynamics, $m = 1$, 2, or 3. But this is not relevant for what follows.) By ν we denote the inner (unit) normal vector field on Γ and use $\cdot$ or $(\cdot|\cdot)$ for the Euclidean inner product in $\mathbb{R}^m$.

We assume that Ω is occupied by n different species described by their densities $u_1, \dots, u_n$ and set $u := (u_1, \dots, u_n)$. The spacial and temporal change of u, that is, the (averaged) movement of the individual populations, is mathematically encoded in the form of conservation laws

$$\partial_t u_i + \operatorname{div} j_i(u) = f_i(u) \quad \text{in} \quad \Omega \times \mathbb{R}_+, \qquad 1 \le i \le n. \tag{1}$$

Here $j_i(u)$ is the (mass) flux vector, $f_i(u)$ the production rate of the i-th species, and (1) is a mass balance law (e.g., S.R. de Groot and P. Mazur [7]).

In order to get a significant model we have to impose constitutive assumptions on the n-tuple $j(u) = \big(j_1(u), \ldots, j_n(u)\big)$ of flux vectors. In population dynamics it is customary to build on phenomenological laws which are basically variants and extensions of Fick's law, and we adhere in this paper to that practice. Thus we assume that

$$j_i(u) = -a_i(u) \operatorname{grad} u_i,$$

where the 'diffusion coefficient' $a_i(u) \in C^1(\Omega)$ may depend on the interaction of some, or all, species, hence on u. The fundamental assumption is then that

$$a_i(u)(x) > 0, \qquad x \in \Omega. \tag{2}$$

Besides of modeling the behavior of the populations in Ω, their conduct on the boundary Γ has also to be analyzed. We restrict ourselves to homogeneous boundary conditions. Then there are essentially two cases which are meaningful, namely the Dirichlet boundary condition

$$u_i = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+$$

or the no-flux condition

$$\nu \cdot j_i(u) = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+$$

for population i, or combinations thereof.

In the standard mathematical theory of reaction-diffusion equations it is assumed that $a_i(u)$ is uniformly positive on Ω. The focus of this paper is on the nonuniform case where $a_i(u)$ may tend to zero as we approach Γ.

In population dynamics nonuniformly positive one-population models have been introduced by W.S.C. Gurney and R.M. Nisbet [8] and M.E. Gurtin and R.C. MacCamy [9]. Arguing that the population desires to avoid overcrowding, they arrive at flux vectors of the form $j(u) = -u^k \operatorname{grad} u$ with $k \ge 1$. Thus their models are special instances of the porous media equation. Here the diffusion coefficient degenerates, in particular, near the Dirichlet boundary.

Ever since the appearance of the pioneering papers [8], [9], there have been numerous studies of the (weak) solvability of reaction-diffusion equations and systems exhibiting porous media type degenerations. We do not go into detail, since we propose a different approach.

In a series of papers, J. López-Gómez has studied (partly with coauthors) qualitative properties of one- and two-population models which he termed 'degenerate' (see [1, 6, 11, 12, 13] and the references therein). In those works the term 'degenerate', however, refers to the vanishing on some open subset of Ω of the 'logistic coefficient', which is part of $f(u)$.

For the following heuristic discussion, which describes the essence of our approach, we can assume that $n = 1$ and that a is independent of u.

If the Dirichlet boundary condition holds, then the population gets extinct if it reaches Γ. In the case of the no-flux condition, Γ is impenetrable, that is, the species can neither escape through the boundary nor can it get replenishment from the outside. In this sense we can say that the 'population lives in a hostile neighborhood'.

No slightly sensible species will move toward places where it is endangered to get killed, nor will it run head-on against an impenetrable wall. Instead, it will slow down drastically if it comes near such places. In mathematical terms this means that the flux in the normal direction has to decrease to zero near Γ. To achieve this, the diffusion coefficient a has to vanish sufficiently rapidly at Γ.

To describe more precisely what we have in mind, we study the motion of the population in a normal collar neighborhood of Γ. This means that we fix $0 < \varepsilon \leq 1$ such that, setting

$$S := \{\, q + y\nu(q) \ ; \ 0 < y \leq \varepsilon,\ q \in \Gamma \,\},$$

the map

$$\varphi\colon \overline{S} \to [0,\varepsilon] \times \Gamma, \quad q + y\nu(q) \mapsto (y,q) \tag{3}$$

is a smooth diffeomorphism. Note that

$$y = \operatorname{dist}(x,\Gamma), \quad x = q + y\nu(q) \in S.$$

We extend ν to a smooth vector field on S, again denoted by ν, by setting

$$\nu(x) := \nu(q), \qquad x = q + y\nu(q) \in S. \tag{4}$$

Then the normal derivative

$$\partial_\nu u(x) := \nu(x) \cdot \operatorname{grad} u(x) \tag{5}$$

is well-defined for $x \in S$.

As usual, we denote by φ_* the push-forward and by φ^* the pull-back by φ of functions and tensors, in particular of vector fields. Then

$$\varphi_*(a \operatorname{grad}) = (\varphi_* a)\frac{\partial}{\partial y} \oplus (\varphi_* a) \operatorname{grad}_\Gamma \quad \text{on} \quad N := (0,\varepsilon] \times \Gamma, \tag{6}$$

where $\operatorname{grad}_\Gamma$ is the surface gradient on Γ with respect to the metric induced by the Euclidean metric on $\overline{\Omega}$. Note that, by (4) and (5),

$$\partial_y v = \partial_\nu u, \quad v = \varphi_* u. \tag{7}$$

Set $\Gamma_y := \varphi^{-1}\big(\{y\} \times \Gamma\big)$. Then $\operatorname{grad}_{\Gamma_y} = \operatorname{grad}_\Gamma$. Hence we obtain from (6) and (7) that

$$a \operatorname{grad} u = (a\partial_\nu u)\nu \oplus a \operatorname{grad}_\Gamma u \quad \text{on } S. \tag{8}$$

Thus, if we want to achieve that the flux $j(u) = -a \operatorname{grad} u$ decays in the normal direction, but not necessarily in directions parallel to the boundary, we have to replace (8) by

$$(a_1\partial_\nu u)\nu \oplus a \operatorname{grad}_\Gamma u,$$

where a_1 tends to zero as x approaches Γ. This we effectuate by replacing $j(u)$ by

$$j^s(u) := -\big((a\rho^{2s}\partial_\nu u)\nu \oplus a \operatorname{grad}_\Gamma u\big), \qquad u \in C^1(S), \tag{9}$$

for some $s \geq 1$, where $0 < \rho \leq 1$ on S and $\rho(x) = \operatorname{dist}(x, \Gamma)$ for x near Γ. Then, irrespective of the size of $\operatorname{grad} u$, $j^s(u)$ decays to zero as we approach Γ. The 'speed' of this decay increases if s gets bigger. Note, however, that the component orthogonal to ν is the same as in (8). This reflects the fact, known to everyone who has been hiking in high mountains—in the Swiss Alps, for example(!)—that one can move forward along a level line path in front of a steep slope with essentially the same speed as this can be done in the flat country. On the other hand, one slows down drastically—and eventually gives up—if one tries to go to the top along a line of steepest ascent.

In the next section we give a precise definition of the class of degenerate equations which we consider. Section 3 contains the definition of the appropriate weighted Sobolev spaces. In addition, we present the basic maximal regularity theorem for linear degenerate parabolic initial value problems.

The main result of this paper is Theorem 4.5 which is proved in Section 4. It guarantees the local well-posedness of quasilinear degenerate reaction-diffusion systems. In the last section we present some easy examples, discuss the differences between the present and the classical approach, and suggest possible directions of further research.

2. Degenerate reaction-diffusion operators

Let (M, g) be a Riemannian manifold. Then grad_g, resp. div_g, denotes the gradient, resp. divergence, operator on (M, g). The Riemannian metric on Γ, induced by the Euclidean metric on $\overline{\Omega}$, is written h. Then $\overline{N} = [0, \varepsilon] \times \Gamma$ is endowed with the metric $g_N := dy^2 + h$.

We fix $\chi \in C^\infty\big([0,\varepsilon],[0,1]\big)$ satisfying

$$\chi(y) = \begin{cases} 1, & 0 \le y \le \varepsilon/3, \\ 0, & 2\varepsilon/3 \le y \le \varepsilon. \end{cases}$$

Then

$$r(y) := \chi(y)y + 1 - \chi(y), \qquad 0 \le y \le \varepsilon.$$

We set

$$S(j) := \varphi^{-1}\big((0, j\varepsilon/3] \times \Gamma\big), \qquad j = 1, 2,$$

and $\rho := \varphi^* r = r \circ \varphi^{-1}$. Then $\rho \in C^\infty\big(S,(0,1]\big)$ and

$$\rho(x) = \begin{cases} \operatorname{dist}(x,\Gamma), & x \in S(1), \\ 1, & x \in S \backslash S(2). \end{cases} \tag{10}$$

Given a linear differential operator $\mathcal{B}$ on S, we denote by $\varphi_*\mathcal{B}$ its 'representation in the variables $(y,q) \in N$'. Thus $\varphi_*\mathcal{B}$, the push-forward of $\mathcal{B}$, is the linear operator on N defined by

$$(\varphi_*\mathcal{B})w := \varphi_*\big(\mathcal{B}(\varphi^* w)\big), \qquad w \in C^\infty(N).$$

First we consider a single linear operator, that is, $n = 1$ and

$$\mathcal{A}v := -\operatorname{div}(a \operatorname{grad} v)$$

with

$$a \in C^1(\Omega), \quad a(x) > 0 \text{ for } x \in \overline{\Omega}. \tag{11}$$

We set $\overline{a} := \varphi_* a \in C^1(N)$ and $\overline{\mathcal{A}} := \varphi_*\mathcal{A}$. Then we find

$$\overline{\mathcal{A}}w = -\operatorname{div}_{g_N}(\overline{a} \operatorname{grad}_{g_N} w) = -\partial_y(\overline{a}\partial_y w) - \operatorname{div}_h(\overline{a} \operatorname{grad}_h w)$$

for $w \in C^2(N)$. By pulling $\overline{\mathcal{A}}$ back to S we obtain the representation

$$\mathcal{A}u = -\partial_\nu(a\partial_\nu u) - \operatorname{div}_h(a \operatorname{grad}_h u), \qquad u \in C^2(S), \tag{12}$$

of $\mathcal{A}|S$.

We put

$$U := \Omega\backslash S(2) \tag{13}$$

and fix $s \in [1,\infty)$. Then we define a linear operator $\mathcal{A}_s$ on Ω by setting

$$\mathcal{A}_s v := -\operatorname{div}_s(a \operatorname{grad}_s v), \qquad v \in C^2(\Omega), \tag{14}$$

where

$$\operatorname{div}_s(a \operatorname{grad}_s v) := \begin{cases} \operatorname{div}(a \operatorname{grad} v), & v \in C^2(U), \\ \rho^s \partial_\nu(a\rho^s \partial_\nu v) + \operatorname{div}_h(a \operatorname{grad}_h v), & v \in C^2(S). \end{cases}$$

It follows from (10), (12), and (13) that $\mathcal{A}_s v$ is well-defined for $v \in C^2(\Omega)$. The map $\mathcal{A}_s$ is said to be a linear *s-degenerate reaction-diffusion* (or *divergence form*) *operator on* Ω.

Remark 2.1: It has been shown in [5] that the right approach to study differential operators which are s-degenerate, is to endow S with the metric

$$g_s := \varphi^*(r^{-2s} dy^2 \oplus h).$$

Then

$$a \operatorname{grad}_{g_s} u = (a\rho^{2s}\partial_\nu u)\nu \oplus a \operatorname{grad}_h u, \qquad u \in C^1(S),$$

which equals $-j^s(u)$ of (9). Furthermore,

$$\mathcal{A}_s u = -\operatorname{div}_{g_s}(a \operatorname{grad}_{g_s} u), \qquad u \in C^2(S).$$

Thus $\mathcal{A}_s$ is a 'standard' linear reaction-diffusion operator if S is endowed with the metric g_s.

3. The Isomorphism Theorem

The natural framework for an efficient theory of strongly degenerate reaction-diffusion systems are weighted function spaces which we introduce now. We assume throughout that

- $1 < p < \infty$.

Suppose $s \geq 1$ and $k \in \mathbb{N}$. For $u \in C^k(S)$ set

$$v(y,q) := \varphi_* u(y,q) = u\big(q + y\nu(q)\big)$$

and

$$\|u\|_{W_p^k(S;s)} := \sum_{i=0}^{k} \Big(\int_0^\varepsilon \big\| \big(r(y)^s \partial_y\big)^i v(y,\cdot)\big\|^p_{W_p^{k-i}(\Gamma)} r(y)^{-s}\, dy \Big)^{1/p}.$$

Then the weighted Sobolev space $W_p^k(S;s)$ is the completion in $L_{1,\mathrm{loc}}(S)$ of the subspace of smooth compactly supported functions with respect to the norm $\|\cdot\|_{W_p^k(S;s)}$. The *weighted Sobolev space* $W_p^k(\Omega;s)$ consists of all u belonging to $L_{1,\mathrm{loc}}(\Omega)$ with

$$u\,|\,S \in W_p^k(\Omega;s), \quad u\,|\,U \in W_p^k(\Omega).$$

It is a Banach space with the norm

$$u \mapsto \|u\,|\,S\|_{W_p^k(S;s)} + \|u\,|\,U\|_{W_p^k(U)},$$

and $L_p(\Omega; s) := W_p^0(\Omega; s)$. Of course, $W_p^k(U)$ is the usual Sobolev space.

To define weighted spaces of bounded C^k functions we set

$$\|u\|_{BC^k(S;s)} := \sum_{i=0}^{k} \sup_{0<y<\varepsilon} \big\| \big(r(y)^s \partial_y\big)^i v(y,\cdot) \big\|_{C^k(\Gamma)}. \tag{15}$$

The weighted space $BC^k(S; s)$ is the linear subspace of all $u \in C^k(S)$ for which the norm (15) is finite. Then $BC^k(\Omega; s)$ is the linear space of all $u \in C^k(\Omega)$ with

$$u\,|\,S \in BC^k(S; s), \quad u\,|\,U \in BC^k(U).$$

It is a Banach space with the norm

$$u \mapsto \|u\,|\,S\|_{BC^k(S;s)} + \|u\,|\,U\|_{BC^k(U)}. \tag{16}$$

The topologies of the weighted spaces $W_p^k(\Omega; s)$ and $BC^k(\Omega; s)$ are independent of the particular choice of S (that is, of $\varepsilon > 0$) and the cut-off function χ.

Let $0 < T < \infty$ and set $J := [0, T]$. We introduce anisotropic weighted Sobolev spaces on $\Omega \times J$ by

$$W_p^{(2,1)}(\Omega \times J; s) := L_p\big(J, W_p^2(\Omega; s)\big) \cap W_p^1\big(J, L_p(\Omega; s)\big).$$

We denote by $(\cdot,\cdot)_{\theta,p}$ the real interpolation functor of exponent $\theta \in (0,1)$. Then we institute a Besov space by

$$B_p^{2-2/p}(\Omega; s) := \big(L_p(\Omega; s), W_p^2(\Omega; s)\big)_{1-1/p,p}.$$

Lemma 3.1. *The weighted spaces possess the same embedding and interpolation properties as their non-weighted versions. In particular,*

$$BC^1(\Omega) \hookrightarrow BC^1(\Omega; s) \hookrightarrow BC(\Omega). \tag{17}$$

Proof. The first assertion is a consequence of Theorems 3.1 and 6.1 of [5]. The first embedding of (17) is obvious from $|\rho^s \partial_\nu u| \le |\partial_\nu u|$ and (16). It remains to observe that $BC(\Omega; s) = BC(\Omega)$. □

The following theorem settles the well-posedness problem for the linear initial value problem

$$\begin{aligned} \partial_t u + \mathcal{A}_s u &= f \quad \text{on } \Omega \times J, \\ \gamma_0 u &= u_0 \quad \text{on } \Omega \times \{0\}, \end{aligned}$$

where γ_0 is the trace operator at $t = 0$ and $\mathcal{A}_s$ is given by (14).

THEOREM 3.2. *Let $1 \leq s < \infty$ and $0 < T < \infty$. Assume that there exists $\underline{\alpha} > 0$ such that*

$$a \in BC^1(\Omega; s) \quad \textit{and} \quad a \geq \underline{\alpha}. \tag{18}$$

Then the map $(\partial_t + \mathcal{A}_s,\, \gamma_0)$ is a topological isomorphism from

$$W_p^{(2,1)}(\Omega \times J; s) \quad \textit{onto} \quad L_p(\Omega \times J; s) \times B_p^{2-2/p}(\Omega; s).$$

Proof. Note that

$$\operatorname{div}(a \operatorname{grad} v) = a\Delta v + \langle da, \operatorname{grad} v\rangle \quad \text{on } U, \tag{19}$$

where $\langle\cdot,\cdot\rangle$ stands for duality pairings. On S we get

$$\rho^s\partial_\nu(a\rho^s\partial_\nu v) = a(\rho^s\partial_\nu)^2 v + (\rho^s\partial_\nu a)\rho^s\partial_\nu v \tag{20}$$

and

$$\operatorname{div}_h(a \operatorname{grad}_h v) = a\Delta_h v + \langle da, \operatorname{grad}_h v\rangle, \tag{21}$$

where Δ_h is the Laplace–Beltrami operator on Γ.

Set $R(t) := t^s$ for $0 \leq t \leq 1$. Then we deduce from (18)–(21) and from Theorems 6.1 and 7.2 of [5] that $\mathcal{A}_s$ is a bc-regular R-degenerate uniformly strongly elliptic differential operator on Ω in the sense of [5, (1.6)]. (Observe that the regularity condition (1.9) in that paper is only sufficient and stronger than (18).) Hence the assertion follows from Theorem 1.3 of [5]. □

COROLLARY 3.3. *$\mathcal{A}_s$ has maximal $L_p(\Omega; s)$ regularity.*

Proof. [4] or [5]. □

4. Quasilinear degenerate systems

Now we turn to systems and consider quasilinear differential operators. Thus we assume:

$$\begin{aligned} &\text{(i)} \quad 1 \leq s < \infty.\\ &\text{(ii)} \quad X \text{ is a nonempty open subset of } \mathbb{R}^n.\\ &\text{(iii)} \quad a_i \in C^2\big(\overline{\Omega} \times X,\, (0,\infty)\big),\ 1 \leq i \leq n. \end{aligned} \tag{22}$$

Given $u \in C^1(\Omega, X)$,

$$a_i(u)(x) := a_i\big(x, u(x)\big), \qquad x \in \Omega. \tag{23}$$

Then

$$\mathcal{A}_{i,s}(u)v_i := -\operatorname{div}_s\big(a_i(u) \operatorname{grad}_s v_i\big), \qquad v_i \in C^2(\Omega),$$

and, setting $a := (a_1, \ldots, a_n)$,

$$\mathcal{A}_s(u)v := -\operatorname{div}_s\big(a(u) \operatorname{grad}_s v\big) := \big(\mathcal{A}_{1,s}(u)v_1, \ldots, \mathcal{A}_{n,s}(u)v_n\big)$$

for $v = (v_1, \dots, v_n) \in C^2(\Omega, \mathbb{R}^n)$. Note that $\mathcal{A}_s(u)$ is a diagonal operator whose diagonal elements are coupled by the u-dependence of their coefficients.

If $\mathfrak{F}(\Omega; s)$ stands for one of the spaces

$$W_p^k(\Omega; s),\ BC^k(\Omega; s),\ \text{or}\ B_p^{2-2/p}(\Omega; s),\ \text{then}\ \mathfrak{F}(\Omega, \mathbb{R}^n; s) := \mathfrak{F}(\Omega; s)^n.$$

Given subsets A and B of some topological spaces, $A \Subset B$ means that $\overline{A}$ is compact and contained in the interior of B.

We define

$$V := \{\, v \in B_p^{2-2/p}(\Omega, \mathbb{R}^n; s)\ ;\ v(\Omega) \Subset X \,\}. \tag{24}$$

LEMMA 4.1. *If $p > m+2$, then V is open in $B_p^{2-2/p}(\Omega, \mathbb{R}^n; s)$.*

Proof. It follows from Lemma 3.1 that

$$B_p^{2-2/p}(\Omega, \mathbb{R}^n; s) \hookrightarrow BC^1(\Omega, \mathbb{R}^n; s) \hookrightarrow BC(\Omega, \mathbb{R}^n). \tag{25}$$

Denote the embedding operator which maps the leftmost space into the rightmost one by ι. Let $u_0 \in V$ so that $u_0(\Omega) \Subset X$.

Fix[1] $0 < r < \operatorname{dist}\big(u_0(\Omega), \partial X\big)$ and set

$$K := \{\, x \in X\ ;\ \operatorname{dist}\big(x, u_0(\Omega)\big) < r \,\} \Subset X \tag{26}$$

and

$$B(u_0, r) := \{\, u \in BC(\Omega, \mathbb{R}^n)\ ;\ \|u - u_0\|_\infty < r \,\}.$$

Then $u(\Omega) \subset K$ for $u \in B(u_0, r)$. Hence $B(u_0, r)$ is a neighborhood of u_0 in $BC(\Omega, K)$. Thus $\iota^{-1}\big(BC(\Omega, K)\big)$ is a neighborhood of u_0 in $B_p^{2-2/p}(\Omega, \mathbb{R}^n; s)$ and it is contained in V. This proves the claim. □

For abbreviation,

$$E_0 := L_p(\Omega, \mathbb{R}^n; s), \quad E_1 := W_p^2(\Omega, \mathbb{R}^n; s), \quad E := B_p^{2-2/p}(\Omega, \mathbb{R}^n; s).$$

As usual, $\mathcal{L}(E_1, E_0)$ is the Banach space of bounded linear operators from E_1 into E_0, and C^{1-} means 'locally Lipschitz continuous'.

LEMMA 4.2. *Suppose $p > m+2$. Then*

(i) *$\mathcal{A}_s(u_0)$ has maximal $L_p(\Omega, \mathbb{R}^n; s)$ regularity for $u_0 \in V$.*

(ii) $\big(u \mapsto \mathcal{A}_s(u)\big) \in C^{1-}\big(V, \mathcal{L}(E_1, E_0)\big)$.

[1] $\operatorname{dist}\big(u_0(\Omega), \emptyset\big) := \infty$.

Proof. We denote by c constants ≥ 1 which may be different from occurrence to occurrence and write $BC_s^1 := BC^1(\Omega, \mathbb{R}^n; s)$ and $BC := BC(\Omega, \mathbb{R}^n)$.

Let $u_0 \in V$. Fix a bounded neighborhood V_K of u_0 in $\iota^{-1}\big(BC(\Omega, K)\big) \subset E$. This is possible by the preceding lemma.

Step 1. It is a consequence of (22-iii) and (26) that

$$1/c \leq a_i(u) \leq c, \qquad i = 1, \dots, n, \quad u \in V_K. \tag{27}$$

Moreover, (22-iii) also implies that a_i and its Fréchet derivative ∂a_i are locally Lipschitz continuous. Hence

$$\begin{array}{l} a_i \text{ and } \partial a_i \text{ are bounded and uniformly} \\ \text{Lipschitz continuous on } \overline{\Omega} \times K \end{array} \tag{28}$$

(e.g., [2, Proposition 6.4]). From this and (25) we infer that

$$\|a_i(u) - a_i(v)\|_\infty \leq c\, \|u - v\|_E, \qquad 1 \leq i \leq n, \quad u, v \in V_K. \tag{29}$$

Step 2. Let i and j run from 1 to n and α from 1 to m. Then, using the summation convention writing $u = (u^1, \dots, u^n)$,

$$\partial_\alpha\big(a_i(u)\big)(x) = (\partial_\alpha a_i)\big(x, u(x)\big) + (\partial_{m+j} a_i)\big(x, u(x)\big)\partial_\alpha u^j(x) \tag{30}$$

for $x \in \Omega$. By (25), V_K is bounded in BC_s^1. From this, (30), and (28) it follows

$$\sup_U \big|\partial_\alpha\big(a_i(u)\big)\big| \leq c, \qquad u \in V_K. \tag{31}$$

Similarly, by employing local coordinates on Γ,

$$\sup_S \big|\operatorname{grad}_\Gamma\big(a_i(u)\big)\big|_{T\Gamma} \leq c, \qquad u \in V_K, \tag{32}$$

where $|\cdot|_{T\Gamma}$ is the vector bundle norm on the tangent bundle $T\Gamma$ of Γ.

Let $x \in S$ and $\alpha = 1$ in (30). Then we get from $0 < \rho^s(x) \leq 1$, the boundedness of V_K in BC_s^1, and (29) that

$$\sup_S \big|\rho^s \partial_\nu\big(a_i(u)\big)\big| \leq c(1 + \sup_S |\rho^s \partial_\nu u|) \leq c, \qquad u \in V_K. \tag{33}$$

By collecting (27) and (31)–(33), we find (cf. (15) and (16))

$$a(u) \in BC_s^1, \quad \|a(u)\|_{BC_s^1} \leq c, \qquad u \in V_K. \tag{34}$$

Now (i) follows from (27) and Corollary 3.3.

Step 3. Let $u, v \in V$. Then

$$\begin{aligned}\partial_\alpha\big(a_i(u) - a_i(v)\big) = {} & \partial_\alpha(a_i)(u) - (\partial_\alpha a_i)(v) \\ & + \big((\partial_{m+j} a)(u) - (\partial_{m+j} a)(v)\big)\partial_\alpha u^j \\ & + (\partial_{m+j} a)(v)(\partial_\alpha u^j - \partial_\alpha v^j).\end{aligned}$$

Using (28) and (25), we obtain

$$\sup_U |(\partial_\alpha a)(u) - (\partial_\alpha a)(v)| \le c\, \|u - v\|_E, \qquad u, v \in V_K.$$

Similarly, employing also the boundedness of V_K in E and (34),

$$\sup_U \big|\big(\partial_{m+j} a)(u) - (\partial_{m+j} a)(v)\big)(\partial_\alpha u^j)\big| \le c\, \|u - v\|_E$$

and

$$\sup_U |(\partial_{m+j} a)(v)(\partial_\alpha u^j - \partial_\alpha v^j)| \le c\, \|u - v\|_E$$

for $u, v \in V_K$. Consequently,

$$\sup_U \big|\partial_\alpha\big(a(u) - a(v)\big)\big| \le c\, \|u - v\|_E, \qquad u, v \in V_K.$$

By analogous arguments we obtain, as in step (1),

$$\sup_S \big|\operatorname{grad}_\Gamma\big(a(u) - a(v)\big)\big|_{T\Gamma} \le c\, \|u - v\|_E$$

and

$$\sup_S \big|\rho^s \partial_\nu\big(a(u) - a(v)\big)\big| \le c\, \|u - v\|_E$$

for $u, v \in V_K$. In summary and recalling (29),

$$\|a(u) - a(v)\|_{BC_s^1} \le c\, \|u - v\|_E, \qquad u, v \in V_K.$$

This implies claim (ii). □

We also suppose

$$g \in C^1(\overline{\Omega} \times X, \mathbb{R}^{n \times n}) \tag{35}$$

and define $g(u)$ analogously to (23). Then, using obvious identifications,

$$f(u) := g(u)u, \qquad u \in C(\Omega, X). \tag{36}$$

Remark 4.3: This assumption on the production rate in (1) is motivated by models from population dynamics. It means that the reproduction (birth or death) rate is proportional to the size of the actually present crowd. Already the diagonal form

$$f_i(u) = g_i(u)u_i, \qquad 1 \le i \le n,$$

comprises the most frequently studied ecological models, namely the standard (two-population) models with competing (predator–prey or cooperative) species, for example. In those cases the g_i are affine functions of u.

Lemma 4.4. *Let $p > m+2$. Then*

$$\big(u \mapsto f(u)\big) \in C^{1\text{-}}\big(V, L_p(\Omega \times \mathbb{R}^n; s)\big).$$

Proof. Let $u_0 \in V$ and fix V_K as in the preceding proof. Then it is obvious from (25) and (35) that

$$\|g(u)\|_\infty \le c, \qquad u \in V_K,$$

and

$$\|g(u) - g(v)\|_\infty \le c\,\|u-v\|_E, \qquad u,v \in V_K.$$

Thus, since $E \hookrightarrow L_p(\Omega, \mathbb{R}^n; s) =: L_{p,s}$,

$$\|f(u)\|_{L_{p,s}} \le \|g(u)\|_\infty\,\|u\|_{L_{p,s}} \le c, \qquad u \in V_K,$$

and

$$\begin{aligned}\|f(u) - f(v)\|_{L_{p,s}} &\le \|g(u) - g(v)\|_\infty\,\|u\|_{L_{p,s}} + \|g(v)\|_\infty\,\|u-v\|_{L_{p,s}}\\ &\le c\,\|u-v\|_E\end{aligned}$$

for $u, v \in V_K$. □

Now we can prove the main result of this paper, a general well-posedness theorem for strong $L_p(\Omega, \mathbb{R}^n; s)$ solutions, by simply referring to known results. The reader may consult [2] or [3] for definitions and the facts on semiflows to which we appeal.

Theorem 4.5. *Let* (22), (35), *and* (36) *be satisfied and assume $p > m+2$. Define V by* (24). *Then the initial value problem for the s-degenerate quasilinear reaction-diffusion system*

$$\begin{aligned}\partial_t u - \operatorname{div}_s\big(a(u)\operatorname{grad}_s u\big) &= f(u) \quad \text{on } \Omega \times \mathbb{R}_+,\\ \gamma_0 u &= u_0 \qquad \text{on } \Omega \times \{0\},\end{aligned} \tag{37}$$

has for each $u_0 \in V$ a unique maximal solution

$$u(\cdot, u_0) \in W_p^{(2,1)}\big(\Omega \times [0, t^+(u_0)), X; s\big).$$

The map $(t, u_0) \mapsto u(t, u_0)$ *is a locally Lipschitz continuous semiflow on* V. *The exit time* $t^+(u_0)$ *is characterized by the following three (non mutually exclusive) alternatives:*

(i) $t^+(u_0) = \infty$.

(ii) $\liminf\limits_{t \to t^+(u_0)} \operatorname{dist}\bigl(u(t, u_0)(\Omega), \partial X\bigr) = 0$.

(iii) $\lim\limits_{t \to t^+(u_0)} u(t, u_0)$ *does not exist in* $B_p^{2-2/p}(\Omega, \mathbb{R}^n; s)$.

Proof. Due to Lemmas 4.1, 4.2, and 4.4, this follows from Theorem 5.1.1 and Corollary 5.1.2 in J. Prüss and G. Simonett [18]. □

REMARKS 4.6: (a) It is obvious from the above proofs that the regularity assumptions for a and g, concerning the variable $x \in \Omega$, are stronger than actually needed. We leave it to the interested reader to find out the optimal assumptions.

(b) Suppose that $a_i(u)$ is independent of u_j for $j \neq i$. Then the theorem remains true, with the obvious definitions of the weighted spaces, if we replace $\mathcal{A}_{i,s}(u)$ by $\mathcal{A}_{i,s_i}(u_i)$ with $1 \leq s_i < \infty$ for $1 \leq i \leq n$.

Proof. This follows by an inspection of the proof of Lemma 4.2. □

(c) For simplicity, we have assumed that $\Gamma = \partial\Omega$. It is clear that we can also consider the case where Γ is a proper open and closed subset of $\partial\Omega$ and regular boundary conditions are imposed on the remaining part.

(d) Similar results can be proved for strongly coupled systems, so-called cross-diffusion equations.

5. Examples and remarks

We close this paper by presenting some easy examples. In addition, we include some remarks on open problems and suggestions for further research. Throughout this section,

- $1 \leq s < \infty$ and $p > m + 2$.

EXAMPLE 5.1: (Two-population models) Let

$$a, b \in C^2(\overline{\Omega}, \mathbb{R}_+), \quad a_i, b_i \in C^1(\overline{\Omega}),\ i = 0, 1, 2, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}.$$

Consider the s-degenerate quasilinear system

$$\begin{aligned} \partial_t u - \operatorname{div}_s\bigl((a + u^\alpha v^\beta) \operatorname{grad}_s u\bigr) &= (a_0 + a_1 u + a_2 v)u, \\ \partial_t v - \operatorname{div}_s\bigl((b + u^\gamma v^\delta) \operatorname{grad}_s v\bigr) &= (b_0 + b_1 v + b_2 u)v \end{aligned} \tag{38}$$

on $\Omega \times \mathbb{R}_+$.

Suppose $\varepsilon > 0$ and

$$(u_0, v_0) \in W_p^2(\Omega; s), \qquad u_0, v_0 \geq \varepsilon.$$

Then there exist a maximal $t^+ = t^+(u_0, v_0) \in (0, \infty]$ and a unique solution

$$(u, v) \in W_p^{(2,1)}\big(\Omega \times [0, t^+), \mathbb{R}^2; s\big)$$

of (38) satisfying $u(t)(x) > 0$ and $v(t)(x) > 0$ for $x \in \Omega$ and $0 \leq t < t^+$.

Proof. Theorem 4.5 with $X = (0, \infty)^2$. □

Observe that the right side of (38) encompasses standard predator–prey as well as cooperation models, depending on the signs of the coefficient functions.

In the following examples we restrict ourselves to scalar equations.

EXAMPLES 5.2: (a) (Porous media equations) Let $\alpha \in \mathbb{R}\backslash\{0\}$ and assume that $g \in C^1(\overline{\Omega} \times \mathbb{R})$. Then

$$\partial_t u - \operatorname{div}_s(u^\alpha \operatorname{grad}_s u) = g(u)u \quad \text{in } \Omega \times \mathbb{R}_+$$

has for each $u_0 \in W_p^2(\Omega; s)$ with $u_0 \geq \varepsilon > 0$ a unique maximal solution

$$u \in W_p^{(2,1)}\big(\Omega \times (0, t^+); s\big)$$

satisfying $u(t)(x) > 0$ for $x \in \Omega$ and $0 \leq t < t^+$.

Proof. Theorem 4.5 with $X := (0, \infty)$. □

(b) (Diffusive logistic equations) Assume $\alpha, \lambda \in \mathbb{R}$ with $\alpha > 0$. Let $a \geq 0$. Set

$$\Delta_s := \operatorname{div}_s \operatorname{grad}_s .$$

If

$$u_0 \in W_p^2(\Omega; s), \qquad u_0 \geq \varepsilon > 0,$$

then there exist a maximal $t^+ \in (0, \infty]$ and a unique solution

$$u \in W_p^{(2,1)}\big(\Omega \times [0, t^+); s\big)$$

of

$$\partial_t u - \alpha \Delta_s u = (\lambda - au)u \quad \text{on } \Omega \times \mathbb{R}_+, \tag{39}$$

satisfying $u(t)(x) > 0$ for $x \in \Omega$ and $0 \leq t < t^+$.

Proof. This is essentially a subcase of Example 5.2. □

The most natural question which now arises is:

- How can we prove global existence?

An attempt to tackle this challenge, which is already hard in the case of standard boundary value problems, is even more demanding in the present setting. To point out where some of the difficulties originate from, we review in the following remarks some of the well-known techniques, which have successfully been applied to parabolic boundary value problems, and indicate why they do not straightforwardly apply to s-degenerate problems.

Remark 5.3: (Maximum principle techniques) First we look at the diffusive logistic equation (39) and contrast it with the simple classical counterpart

$$\begin{aligned} \partial_t u - \alpha \Delta u = (\lambda - u)u \quad & \text{on } \overline{\Omega} \times \mathbb{R}_+, \\ u = 0 \qquad\qquad & \text{on } \Gamma \times \mathbb{R}_+. \end{aligned} \tag{40}$$

Suppose $\lambda > 0$. Then (40) has for each sufficiently smooth initial value u_0 satisfying $0 \leq u_0 \leq \lambda$ a unique global solution obeying the same bounds as u_0. This is a consequence of the maximum principle, since $(0, \lambda)$ is a pair of sub- and supersolutions. For the validity of this argument it is crucial that we deal with a boundary value problem.

In the s-degenerate case there is no boundary. Hence the preceding argument does not work, since there is no appropriate maximum principle.

Remark 5.4: (Methods based on spectral properties) A further important technique, which is useful in the case of boundary value problems, rests on spectral properties of the linearization of associated stationary elliptic equations. The most prominent case is supposedly the ‘principle of linearized stability’ (and its generalizations to non-isolated equilibria, see [18, Chapter 5]). In addition, the better part of the qualitative studies on semi- and quasilinear parabolic boundary value problems, as well for a single equation as for systems, is based on spectral properties, in particular on the existence and the nature of eigenvalues.

In the case of boundary value problems, the associated linear elliptic operators have compact resolvents, due to the compactness of $\overline{\Omega}$. In our situation, Ω, more precisely, the Riemannian manifold $\Omega_s = (\Omega, g_s)$ introduced in Remark 2.1, is not compact. If $s = 1$, then it is a manifold with cylindrical ends in the sense of R. Melrose [14] and others[2]. For manifolds of this type—and more general ones—much is known about the (L_2) spectrum of the Laplace–Beltrami operator. In particular, the essential spectrum is not empty.

Nevertheless, we are in a simpler situation. In fact, the normal collar S can be represented as a half-cylinder over the compact manifold (Γ, h) and the

[2] I am grateful to Victor Nistor for pointing this out to me.

remaining 'interior part' U is flat (cf. [5, Section 5]). Thus there is hope to get sufficiently detailed information on the L_p spectrum of linear divergence form operators.

In particular, suppose $u_0 \in V \cap W_p^2(\Omega, \mathbb{R}^n; s)$ is a stationary point of (37). If it can be shown that the spectrum of $\mathcal{A}(u_0)$ is contained in an interval $[\alpha, \infty)$ with $\alpha > 0$, then, using the decay properties of the analytic semigroup generated by $-\mathcal{A}(u_0)$, it can be shown that u_0 is an asymptotically stable critical point of (37).

Remark 5.5: (The technique of a priori estimates) The general version of [18, Theorem 5.1.1] exploits the regularization properties of analytic semigroups. Suppose that alternative (ii) of Theorem 4.5 does not occur. Also assume that there can be established a uniform a priori bound in a Besov space $B_{p,s}^{\sigma-2/p} := B_p^{\sigma-2/p}(\Omega, \mathbb{R}^n; s)$ with $2/p < \sigma < 2$. Then we have global existence, *provided* the embedding $B_{p,s}^{2-2/p} \hookrightarrow B_{p,s}^{\sigma-2/p}$ is compact (see [18, Theorem 5.7.1]). Hence this technique is also not applicable to our equations.

However, we still have the possibility to use the interpolation-extrapolation techniques developed in [3] and [4] to switch to weak formulations. Then it might be possible to prove global existence by more classical techniques using a priori estimates with respect to suitable integral norms.

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