Rend. Istit. Mat. Univ. Trieste
Volume 51 (2019), 125–166
DOI: 10.13137/2464-8728/29333

Convexity, topology and nonlinear differential systems with nonlocal boundary conditions: a survey[1]

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Abstract. *This paper is a survey of recent existence results for solutions of first and second order nonlinear differential systems with nonlocal boundary conditions using methods based upon convexity, topological degree and maximum-principle like techniques.*

Keywords: systems of differential equations, nonlocal boundary conditions, convexity, Leray-Schauder degree, maximum principle.
MS Classification 2010: 34B10, 34B15, 47H11, 52A20.

1. Introduction

For a first order ordinary differential system

$$x' = f(t,x),$$

with $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ continuous, a natural generalization of the homogeneous two–point **boundary conditions** (**BC**)

$$A_1 x(a) = A_2 x(b),$$

where A_1, A_2 are $(n \times n)$–matrices, is the m–point BC

$$\sum_{j=1}^{m} A_j x(t_j) = 0,$$

where $a = t_1 < t_2 < \ldots < t_m = b$ and $A_1, \ldots, A_m$ are $(n \times n)$–matrices. Such a multi–point boundary condition is itself a special case of the **nonlocal or**

[1]Based on lectures given by Jean Mawhin at the Dipartimento di Matematica e Geoscienze of the University of Trieste in April 2019.

integral BC

$$\int_a^b d\mathcal{A}(s)x(s) = 0,$$

where $\mathcal{A} : [a,b] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ has bounded variation and the integral is of Riemann–Stieltjes type.

Nonlocal boundary conditions of the type

$$\sum_{j=1}^{m} A_j x(t_j) + \int_0^1 B(s)x(s)\,ds = 0$$

for some $(n \times n)$–matrix–valued function $B : [0,1] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, were first introduced for linear differential equations by Picone [91] in 1908, and already applied to physics by von Mises [111] in 1912. Using Riesz representation theorem, those conditions are themselves contained in the more general form

$$\int_0^1 d\mathcal{A}(s)x(s) = 0,$$

where $\mathcal{A} : [0,1] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is a $(n \times n)$–matrix–valued functions with bounded variation. They were first introduced for linear systems in 1931 by Cioranescu [12] and by Smorgorshewsky [98] in 1940. A good survey of the linear theory is given by Krall in [57].

Nonlocal boundary conditions can be considered also for second order differential systems of the form

$$x'' = f(t, x, x'), \tag{1}$$

where $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

Without searching the maximum of generality, the most useful homogeneous **two–point BC** for system (1) are obtained by choosing two of the following expressions

$$x(a) = Ax(b) + Bx'(b), \quad x'(a) = Cx(b) + Dx'(b),$$
$$x(b) = Ex(a) + Fx'(a), \quad x'(b) = Gx(a) + Hx'(a),$$

where A, B, C, D, E, F, F, H are $(n \times n)$–matrices. The corresponding **nonlocal BC** are obtained by taking two of the following conditions

$$x(a) = \int_a^b d\mathcal{A}(s)x(s) + \int_a^b d\mathcal{B}(s)x'(s), \tag{2}$$

$$x'(a) = \int_a^b d\mathcal{C}(s)x(s) + \int_a^b d\mathcal{D}(s)x'(s), \tag{3}$$

$$x(b) = \int_a^b d\mathcal{E}(s)x(s) + \int_a^b d\mathcal{F}(s)x'(s), \tag{4}$$

$$x'(b) = \int_a^b d\mathcal{G}(s)x(s) + \int_a^b d\mathcal{H}(s)x'(s), \tag{5}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H} : [a, b] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ are $(n \times n)$–matrix valued functions having bounded variation.

Choosing $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ constant, the conditions (2)–(3) reduce to the **initial type conditions**, and choosing $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ constant, the conditions (4)–(5) reduce to the **terminal type conditions** for (1). Choosing $\mathcal{A}, \mathcal{B}, \mathcal{E}, \mathcal{F}$ constant, the conditions (2)–(4) become the **Dirichlet conditions**, and choosing $\mathcal{C}, \mathcal{D}, \mathcal{G}, \mathcal{H}$ constant, the conditions (3)–(5) become the **Neumann conditions**. **Mixed conditions** are obtained by choosing $\mathcal{A}, \mathcal{B}, \mathcal{G}, \mathcal{H}$ constant in (2)–(5) or $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$ constant in (3)–(4). The **periodic conditions** can be obtained by taking $\mathcal{A}$ such that $\int_a^b d\mathcal{A}(s)x(s) = x(b)$, $\mathcal{B}$ constant, $\mathcal{C}$ constant and $\mathcal{D}$ such that $\int_a^b d\mathcal{D}(s)x'(s) = x'(b)$. It suffices, for example, to take $\mathcal{A}(s) = h(s)I_n$ with $h(a) = 0$ and $h(s) = 1$ for $s \in (a, b]$.

The first paper dealing with nonlinear differential equations with integral boundary conditions seems to be due to Birkhoff and Kellogg [7] in 1922, as an application of their famous extension of Brouwer's fixed point theorem to some function spaces. Interesting surveys of the nonlinear theory have been given by Whyburn [112], Conti [13], Krall [57], Ma [66] and Ntouyas [87]. They mostly deal with scalar problems and cover the period 1908–2005.

In this survey, we concentrate on differential systems of first and second order (excluding specific results for scalar equations and for higher-order equations), and on methods based upon convexity, topological degree and maximum principle–like techniques to obtain pointwise estimates for the possible solutions. Because of the nonlocal character of the boundary conditions, those methods are more delicate to use than for two–point boundary value problems. We first deal with first order systems, and then with second order systems, discuss the sharpness of the obtained existence conditions and compare them with some well known classical ones for standard two-point boundary conditions like the initial, terminal, periodic, Dirichlet, mixed and Neumann ones. Let us mention also that the nonlocal boundary conditions presented are not by far the most general ones to which the methods apply, but have been chosen in order to associate a minimum of technical complication with a maximum of significancy.

Various other methods have been used to study nonlocal boundary value problems and various other classes of conditions have been imposed to the nonlinearities to obtain existence and multiplicity results. Let us mention iteration methods for Lipschitizian nonlinearities with sufficiently small coefficients [9, 15, 23, 76, 80, 81], topological methods for nonlinearities satisfying suitable growth and/or sign conditions [2, 4, 5, 8, 11, 14, 43, 44, 50, 52, 53, 55, 56, 65, 68, 71, 72, 77, 82, 83, 90, 95, 96, 97, 102, 105, 106, 107, 108, 109], maximal principle type arguments for monotone nonlinearities [17, 18, 22, 23, 40], fixed point theorems and index on cones for positive solutions [3, 16, 20, 24, 25, 26, 35, 36, 37, 38, 39, 41, 42, 45, 46, 47, 48, 49, 51, 61, 62, 63, 64, 94, 99, 113, 114, 115, 117], variational methods for potential nonlinearities [1, 21, 27, 28, 29, 33, 67, 84, 85, 86, 110, 116]. Those methods and results will not be considered here.

2. First order systems

2.1. Boundary conditions

Let us consider a first order system of ordinary differential equations

$$x' = f(t, x) \tag{6}$$

with $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ continuous. We choose $[0, 1]$ for the independent variable without loss of generality.

The homogeneous **two point BC** have the form

$$Ax(0) = Bx(1)$$

where A and B are $(n \times n)$–matrices.

Notice that for $f \equiv 0$ in (6) the BVP becomes

$$x' = 0 \ \Leftrightarrow x(t) = c \in \mathbb{R}^n, \ (A - B)c = 0.$$

Two cases are possible. If

$$\det(A - B) \neq 0,$$

0 is the unique solution and we say that the BC is **non–resonant**. If

$$\det(A - B) = 0,$$

the problem has infinitely any solutions and the BC is called **resonant**.

Standard examples of "two–point boundary conditions" for (6) are given by the **initial value condition** on $[0,1]$ $x(0) = 0$ ($A = I_n$, $B = 0_n$, non–resonant), the **terminal value condition** on $[0,1]$ $x(1) = 0$ ($A = 0_n$, $B = I_n$, non–resonant), the **anti–periodic BC** $x(0) + x(1) = 0$ ($A = -B = I_n$, non–resonant), and the **periodic BC** $x(0) = x(1)$ ($A = B = I_n$, resonant).

Given $0 = t_1 < t_2 < \ldots < t_m = 1$, one can consider also the **m–point BC**

$$\sum_{j=1}^{m} A_j x(t_j) = 0. \tag{7}$$

For $f \equiv 0$ in (6), the solutions are $x(t) = c$ with c such that $(\sum_{j=1}^{m} A_j)c = 0$. Again, if $\det(\sum_{j=1}^{m} A_j) \neq 0$, the BC (7) is called **non–resonant**, and if this determinant is equal to zero, the BC is called **resonant**.

2.2. Nonlocal initial or terminal type BC

For simplicity of exposition and of the statements, we restrict ourself to the special but representative cases of the **nonlocal initial type condition**

$$x(0) = \int_0^1 x(s)dh(s) \tag{8}$$

and of the **nonlocal terminal type condition**

$$x(1) = \int_0^1 x(s)\, dh(s), \tag{9}$$

where

(h0) $h : [0,1] \to \mathbb{R}$ *is non–decreasing.*

Recall that, for any continuous functions $x : [0,1] \to \mathbb{R}^n$, the corresponding Riemann–Stieltjes integrals always exist. Without loss of generality, we can assume that

$$h(0) = 0.$$

We first discuss the situation where

(h1) $\displaystyle\int_0^1 dh(s) = h(1) < 1.$

This is a non–resonant situation because each problem

$$x' = 0,\ x(0) = \int_0^1 x(s)\,dh(s), \qquad x' = 0,\ x(1) = \int_0^1 x(s)\,dh(s)$$

has the solution $x(t) = c$ with c verifying the equation

$$c = h(1)c,$$

which has only the trivial solution. This case contains of course the initial and terminal null conditions.

Then, we consider the case where

(h2) $\displaystyle\int_0^1 dh(s) = h(1) = 1.$

In this situation, which is clearly a resonant one, the second members of (8) and (9) can be seen as some average of the values of $x(s)$ on the interval $[0,1]$.

In order to prevent the right–hand member in (8) to be $x(0)$, which would reduce (8) to an identity, we must prevent in (8) h to have the form

$$h(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x \in (0,1] \end{cases}, \tag{10}$$

which corresponds to assume that

(h3) *there exists* $\tau \in (0,1)$ *such that* $h(\tau) < 1$.

Similarly, in order to prevent (9) to become an identity, we exclude in (9) h of the form

$$h(x) = \begin{cases} 0 & \text{if} \quad x \in [0,1) \\ 1 & \text{if} \quad x = 1 \end{cases}, \tag{11}$$

which corresponds to assume that

(h4) *there exists* $\tau \in (0,1)$ *such that* $h(\tau) > 0$.

Example 2.1: For h given by (11), we have $\int_0^1 x(s)\,dh(s) = x(1)$, and (8) reduces to the periodic BC $x(0) = x(1)$.

For h given by (10), we have $\int_0^1 x(s)\,dh(s) = x(0)$, and (9) reduces again to the periodic BC.

Example 2.2: For

$$h(x) = \begin{cases} 0 & \text{if} \quad x \in [0,\alpha) \\ \gamma & \text{if} \quad x \in [\alpha,1) \\ 1 & \text{if} \quad x = 1 \end{cases}$$

where $\alpha, \gamma \in (0,1)$, we have

$$\int_0^1 x(s)\,dh(s) = \gamma x(\alpha) + (1-\gamma)x(1),$$

and (8) reduces to the **three–point BC** $x(0) = \gamma x(\alpha) + (1-\gamma)x(1)$.

EXAMPLE 2.3: For

$$h(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \gamma & \text{if} \quad x \in (0,\alpha] \\ 1 & \text{if} \quad x \in (\alpha, 1] \end{cases}$$

where $\alpha, \gamma \in (0,1)$, we have

$$\int_0^1 x(s)\,dh(s) = \gamma x(0) + (1-\gamma)x(\alpha),$$

and (8) reduces to the **three–point BC** $x(1) = (1-\gamma)x(\alpha) + \gamma x(0)$.

2.3. Linear nonlocal initial or terminal type BVP

Let C^0 be the space $C([0,1],\mathbb{R}^n)$ of all continuous mappings from $[0,1]$ into $\mathbb{R}^n$ with the uniform norm $\|x\| = \max\{\|x_1\|_\infty, \ldots, \|x_n\|_\infty\}$.

The following results are useful to reduce our problems to a fixed point form.

LEMMA 2.4. *If conditions (h0), (h1) or (h0), (h2), (h3) hold, then, for each* $z \in C^0$*, the linear nonlocal initial value problem*

$$x' + x = z(t),\ x(0) = \int_0^1 x(s)\,dh(s) \tag{12}$$

has the unique solution

$$x(t) = \left(1 - \int_0^1 e^{-s}\,dh(s)\right)^{-1} \int_0^1 \int_0^u e^{-t-u+s} z(s)\,ds\,dh(u) + \int_0^t e^{-(t-s)} z(s)\,ds. \tag{13}$$

Proof. By the variation of constants formula, the initial value problem

$$x' + x = z(t),\ x(0) = c$$

has the unique solution

$$x(t) = ce^{-t} + \int_0^t e^{-(t-s)} z(s)\,ds \quad (c \in \mathbb{R}^n).$$

It satisfies the boundary condition (8) if and only if c satisfies the linear algebraic system

$$c = c\int_0^1 e^{-t}\,dh(t) + \int_0^1\int_0^t e^{-(t-s)}z(s)\,ds\,dh(t),$$

which has the unique solution

$$c = \left(1 - \int_0^1 e^{-s}\,dh(s)\right)^{-1}\int_0^1\int_0^u e^{-(u-s)}z(s)\,ds\,dh(u)$$

if $\int_0^1 e^{-s}\,dh(s) \neq 1$. This is trivially the case if conditions (h0), (h1) hold. If conditions (h0), (h2), (h3) hold, we have

$$\begin{aligned}
\int_0^1 e^{-s}\,dh(s) &= \int_0^1 d[e^{-s}h(s)] + \int_0^\tau e^{-s}h(s)\,ds + \int_\tau^1 e^{-s}h(s)\,ds \\
&\leq e^{-1} + h(\tau)\int_0^\tau e^{-s}\,ds + \int_\tau^1 e^{-s}\,ds \\
&= e^{-1} + h(\tau)(1-e^{-\tau}) + (e^{-\tau} - e^{-1}) \\
&= (1-e^{-\tau})h(\tau) + e^{-\tau} < 1
\end{aligned}$$

and the result follows. □

Let us denote by $K_1 : C^0 \to C^0$ the linear operator mapping z into x given by (13). Notice that each $K_1 z$ is of class C^1.

COROLLARY 2.5. *There exists $C_1 > 0$ such that, for each $z \in C^0$, one has*

$$\|K_1 z\| \leq C_1\|z\|,\ \ \|(K_1 z)'\| \leq (C_1+1)\|z\|,$$

and K_1 is a compact operator.

Proof. Follows easily from (12) and (13) and Arzelà-Ascoli's theorem. □

LEMMA 2.6. *If conditions (h0), (h1) or (h0), (h2), (h4) hold, then, for each $z \in C^0$, the linear nonlocal terminal value problem*

$$x' - x = z(t),\ x(1) = \int_0^1 x(s)\,dh(s) \tag{14}$$

has the unique solution

$$x(t) = \left(1 - \int_0^1 e^{s-1}\,dh(s)\right)^{-1}\int_0^1\int_1^u e^{t-1+u-s}z(s)\,ds\,dh(u) + \int_1^t e^{t-s}z(s)\,ds. \tag{15}$$

Proof. Let $z \in C^0$. By the variation of constants formula, the terminal value problem

$$x' - x = z(t),\ x(1) = c$$

has the unique solution

$$x(t) = e^{t-1}c + \int_1^t e^{t-s} z(s)\, ds \quad (c \in \mathbb{R}^n).$$

It satisfies the boundary condition (9) if c verifies the linear algebraic system

$$c - \left(\int_0^1 e^{t-1}\, dh(t)\right) c = \int_0^1 \int_1^t e^{t-s} z(s)\, ds\, dh(t).$$

which has the unique solution

$$c = \left(1 - \int_0^1 e^{s-1}\, dh(s)\right)^{-1} \int_0^1 \int_1^u e^{u-s} z(s)\, ds\, dh(u),$$

if $1 \neq \int_0^1 e^{s-1}\, dh(s)$. This is trivially the case if conditions (h0), (h1) hold. In the second case, we have

$$\begin{aligned}\int_0^1 e^{s-1}\, dh(s) &= \int_0^1 d[e^{s-1}h(s)] - \int_0^\tau e^{s-1}h(s)\, ds - \int_\tau^1 e^{s-1}h(s)\, ds \\ &\leq 1 - h(\tau)\int_\tau^1 e^{s-1}\, ds = 1 - h(\tau)(1 - e^{\tau-1}) < 1\end{aligned}$$

and the result follows. □

Let us denote by $K_2 : C^0 \to C^0$ the linear operator mapping z into x given by (15). Notice that each $K_2 z$ is of class C^1.

Corollary 2.7. *There exists $C_2 > 0$ such that, for each $z \in C^0$, one has*

$$\|K_2 z\| \leq C_2\|z\|,\ \ \|(K_2 z)'\| \leq (C_2 + 1)\|z\|,$$

and K_2 is a compact operator.

Proof. Follows easily from (14) and (15) and Arzelà-Ascoli's theorem. □

2.4. Fixed point formulation of nonlinear nonlocal initial or terminal type BVP

Let now $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and define the mapping $N_1 : C^0 \to C^0$ by

$$N_1 x = f(\,\cdot\,, x(\cdot)) - x(\cdot),\ \ \forall\, x \in C^0$$

and the mapping $N_2 : C^0 \to C^0$ by

$$N_2 x = f(\,\cdot\,, x(\cdot)) + x(\cdot), \;\; \forall\, x \in C^0.$$

It is easy to show that N_1 and N_2 are continuous on C^0 and take bounded sets of C^0 into bounded sets of C^0. Under the conditions of Lemma 2.4,

$$G_1 := K_1 N_1 : C^0 \to C^0$$

is compact on bounded subsets of C^0, and the **nonlinear nonlocal initial type problem**

$$x' = f(t,x), \;\; x(0) = \int_0^1 x(s)\,dh(s) \tag{16}$$

is equivalent to the fixed point problem in C^0

$$x = G_1 x. \tag{17}$$

Similarly, under the conditions of Lemma 2.6,

$$G_2 := K_2 N_2 : C^0 \to C^0$$

is compact on bounded subsets of C^0, and the **nonlinear nonlocal terminal type problem**

$$x' = f(t,x), \;\; x(1) = \int_0^1 x(s)\,dh(s) \tag{18}$$

is equivalent to the fixed point problem in C^0

$$x = G_2 x. \tag{19}$$

We apply to the equations (17) and (19) the following existence result, which follows easily from Leray-Schauder continuation theorem [60, 69].

PROPOSITION 2.8. *Let X be a real normed space, $\Omega \subset X$ be an open bounded neighborhood of 0, and $T : \overline{\Omega} \to X$ be a compact operator. If $x \neq \lambda T x$ for every $(x,\lambda) \in \partial\Omega \times (0,1)$, then T has at least a fixed point in $\overline{\Omega}$.*

2.5. Some classical results for periodic BC

Let $\langle \cdot \,|\, \cdot \rangle$ denote the classical inner product in $\mathbb{R}^n$, $|\cdot|$ the corresponding Euclidian norm, and $B_R \subset \mathbb{R}^n$ the closed ball of center 0 and radius $R > 0$.

A classical existence theorem for the periodic BVP associated to (6) is the following one.

THEOREM 2.9. *If there exists $R > 0$ such that either*

$$\langle u | f(t,u) \rangle \geq 0,\ \forall\, (t,u) \in [0,1] \times \partial B_R, \tag{20}$$

or

$$\langle u | f(t,u) \rangle \leq 0,\ \forall\, (t,u) \in [0,1] \times \partial B_R, \tag{21}$$

then the problem

$$x' = f(t,x), \quad x(0) = x(1) \tag{22}$$

has at least one solution taking values in $\overline{B}_R$.

Notice that the two statements in Theorem 2.9 are equivalent : each one implies the other one through the change of variables $\tau = 1 - t$. The full statement can be seen as a nonlinear version of the following linear elementary result

PROPOSITION 2.10. *For each $\lambda \in \mathbb{R} \setminus \{0\}$ and each $e \in C^0$, the problem*

$$x' = \lambda x + e(t),\ x(0) = x(1)$$

has a solution.

Quite strangely, it is difficult to locate the first appearance of Theorem 2.9 in the literature. It is a special case (not directly mentioned !) of Theorem 3.2 in Krasnosel'skii's monograph [58] of 1966. On the other hand, it is explicitely mentioned by Gustafson and Schmitt in 1974 [30] (with strict inequalities in (20) or (21)) as a special case of the following theorem.

Let C be an open convex neighborhood of 0 in $\mathbb{R}^n$. It is well known that $\forall\, u \in \partial C,\ \exists\, \nu(u) \in \mathbb{R}^n \setminus \{0\} : \langle \nu(u) | u \rangle > 0$ and $C \subset \{v \in \mathbb{R}^n : \langle \nu(u) | v - u \rangle < 0\}$. The mapping $\nu : \partial C \to \mathbb{R}^n \setminus \{0\}$ is called an **outer normal field** on ∂C.

THEOREM 2.11. *If there exists a bounded convex open neighborhood C of 0 in $\mathbb{R}^n$, and an outer normal field ν on ∂C such that either*

$$\langle \nu(u) | f(t,u) \rangle > 0,\ \forall\, (t,u) \in [0,1] \times \partial C$$

or

$$\langle \nu(u) | f(t,u) \rangle < 0,\ \forall\, (t,u) \in [0,1] \times \partial C,$$

then the problem (22) has at least one solution taking values in C.

Notice that Krasnosel'skii's monograph is not quoted by Gustafson and Schmitt. In [69], the connexion between Krasnosel'skii's results and Gustafson-Schmitt's ones is made explicit, the Gustafson-Schmitt's theorem is extended to the case of weak inequalities and Krasnosel'skii's theorem is shown to be a special case of this extension of Gustafson-Schmitt's theorem.

2.6. Nonlocal initial type BVP

The following theorem essentially comes from [73]. The special case of a global initial value problem can be found in [70].

THEOREM 2.12. *If $h : [0,1] \to \mathbb{R}$ satisfies conditions (h0), h(1) or conditions (h0), h(2), (h3), and if there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν on ∂C such that*

$$\langle \nu(u) | f(t,u) \rangle \leq 0, \ \forall \, (t,u) \in [0,1] \times \partial C, \tag{23}$$

then the problem (16) *has at least one solution taking values in $\overline{C}$ for all $t \in [0,1]$.*

Proof. Let us consider the equation (17) and define the open bounded neighborhood Ω of 0 in C^0 by

$$\Omega = \{x \in C^0 : x(t) \in C, \ \forall \, t \in [0,1]\}. \tag{24}$$

Notice that

$$\begin{aligned} \overline{\Omega} &= \{x \in C^0 : x([0,1]) \subset \overline{C}\}, \\ \partial\Omega &= \{x \in \overline{\Omega} : \exists \, t_0 \in [0,1] : x(t_0) \in \partial C\}. \end{aligned} \tag{25}$$

By the discussion above G_1 is compact on $\overline{\Omega}$. According to Proposition 2.8, a solution of (17) in $\overline{\Omega}$, i.e. a solution of (16) such that $x(t) \in \overline{C}$ for all $t \in [0,1]$ will exist, if we can show that, for each $\lambda \in (0,1)$, no possible solution of the problem

$$x' + x = \lambda[f(t,x) + x], \ x(0) = \int_0^1 x(s) \, dh(s), \tag{26}$$

belongs to $\partial\Omega$. Let $\lambda \in (0,1)$ and $x(t) \in \partial\Omega$ be a possible solution to (26). Then $x(t) \in \overline{C}$ for all $t \in [0,1]$ and there is some $t_0 \in [0,1]$ such that $x(t_0) \in \partial C$. Therefore, for all $t \in [0,1]$,

$$\xi_{t_0}(t) := \langle \nu(x(t_0)) | x(t) \rangle \leq \langle \nu(x(t_0), x(t_0) \rangle = \xi_{t_0}(t_0), \ \forall \, t \in [0,1],$$

which means that the real function $\xi_{t_0} : [0,1] \to \mathbb{R}$ reaches its maximum at t_0. If $t_0 \in (0,1]$,

$$\begin{aligned} 0 &\leq \xi'_{t_0}(t_0) = \langle \nu(x(t_0)) | x'(t_0) \rangle \\ &= -(1-\lambda)\langle \nu(x(t_0)) | x(t_0) \rangle + \lambda \langle \nu(x(t_0)) | f(t_0, x(t_0)) < 0, \end{aligned}$$

a contradiction. If $t_0 = 0$ and conditions (h0), (h1) hold, then

$$\begin{aligned}\xi_0(0) &= \langle \nu(x(0)), x(0)\rangle = \int_0^1 \langle \nu(x(0)), x(s)\, dh(s)\rangle \\ &\leq \max_{[0,1]} \langle \nu(x(0)), x(s)\rangle \int_0^1 dh(s) < \max_{[0,1]} \langle \nu(x(0), x(s)\rangle \\ &= \max_{s\in[0,1]} \xi_0(s),\end{aligned}$$

a contradiction. If $t_0 = 0$ and conditions (h0), (h2), (h3) hold, it remains only to consider the case where 0 is the only value of t at which $x(t) \in \partial C$, i.e. the case where ξ_0 reaches its maximum only at 0. Then

$$\xi_0(s) < \xi_0(0),\ \forall s \in (0,1],$$

and hence, using the boundary condition and assumptions (h2) and (h3),

$$\begin{aligned}\xi_0(0) &= \left\langle \nu(x(0)) \,\middle|\, \int_0^1 x(s)\, dh(s) \right\rangle = \int_0^1 \langle \nu(x(0)) | x(s)\, dh(s)\rangle \\ &= \int_0^1 \langle \nu(x(0)) | x(s)\rangle\, dh(s) = \int_0^1 \xi_0(s)\, dh(s) \\ &= \int_0^\tau \xi_0(s)\, dh(s) + \int_\tau^1 \xi_0(s)\, dh(s) \\ &\leq \xi_0(0) h(\tau) + (\max_{[\tau,1]} \xi_0)[1 - h(\tau)] < \xi_0(0),\end{aligned}$$

a contradiction. Consequently the assumptions of Proposition 2.8 are satisfied for G_1 on $\overline{\Omega}$, and the conclusion follows. □

Corollary 2.13. *If $h : [0,1] \to \mathbb{R}$ satisfies conditions (h0), h(1) or conditions (h0), (h2), (h3), and if there exists $R > 0$ such that*

$$\langle u | f(t,u)\rangle \leq 0,\ \forall\, (t,u) \in [0,1] \times \partial B_R, \tag{27}$$

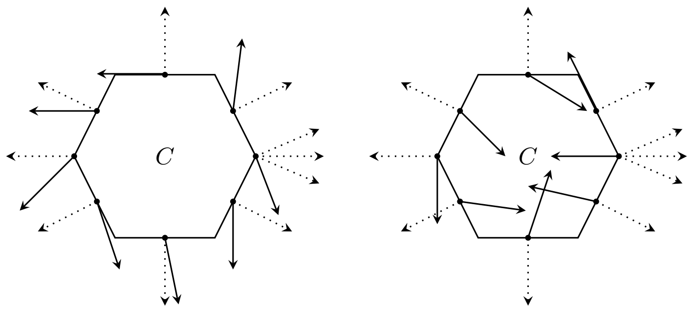
then the problem (16) has at least one solution taking values in $\overline{B}_R$.

Proof. Take $C = B_R$ and, for each $u \in \partial B_R$, $\nu(u) = u$. □

Corollary 2.14. *If $h : [0,1] \to \mathbb{R}$ satisfies conditions (h0), h(1) or conditions (h0), (h2), (h3), and if there exists $R_j > 0$ $(1 \leq j \leq n)$ such that*

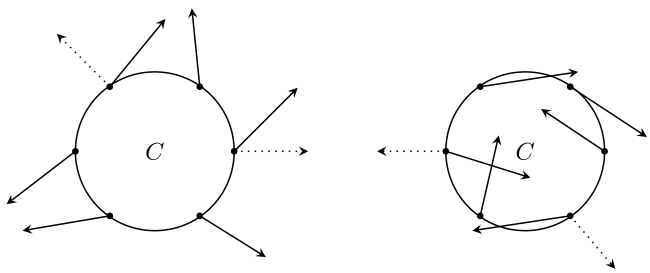
$$u_i f_i(t,u) \leq 0,\ \forall\, (t,u) \in [0,1] \times \prod_{j=1}^n [-R_j, R_j] \,:\, |u_i| = R_i\ (1 \leq i \leq n), \tag{28}$$

then the problem (16) has at least one solution taking values in $\Pi_{j=1}^n [-R_j, R_j]$.



$\langle \nu(u) | f(t,u) \rangle \geq 0$ $\quad$ $\langle \nu(u) | f(t,u) \rangle \leq 0$

Figure 1: The case when C is a hexagon.



$\langle u | f(t,u) \rangle \geq 0$ $\quad$ $\langle u | f(t,u) \rangle \leq 0$

Figure 2: The case when C is a ball.

Proof. Take $C = \prod_{j=1}^{n}(-R_j, R_j)$ and, for each $u \in \prod_{j=1}^{n}[-R_j, R_j]$ and $|u_i| = R_i$, $\nu(u) = u_i e^i$, where $e^i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the i^{th} element of the canonical basis of $\mathbb{R}^n$ $(i = 1, \ldots, n)$. □

2.7. Nonlocal terminal type BVP

The following theorem essentially comes from [73].

THEOREM 2.15. *If* $h : [0,1] \to \mathbb{R}$ *satisfies conditions (h0), h(1), or conditions (h0), (h2),(h4), and if there exists an open, bounded, convex neighborhood* C *of* 0 *in* $\mathbb{R}^n$ *and an outer normal field* ν *on* ∂C *such that*

$$\langle \nu(u) | f(t,u) \rangle \geq 0, \ \forall\, (t,u) \in [0,1] \times \partial C, \tag{29}$$

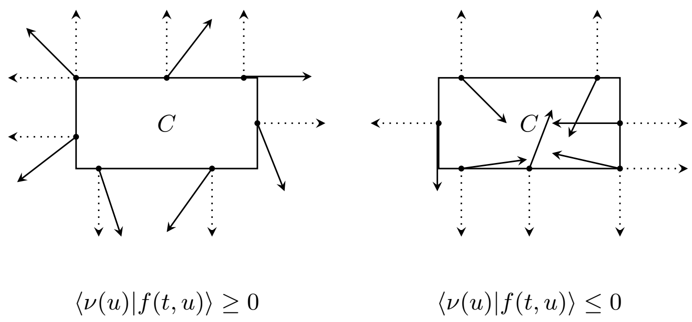


Figure 3: The case when C is a rectangle.

then the problem (18) *has at least one solution taking values in* $\overline{C}$.

Proof. Let us consider the equation (19) and let Ω the open bounded neighborhood of 0 in C^0 defined by (24). By the discussion above G_2 is compact on $\overline{\Omega}$. According to Proposition 2.8, a solution of (19) in $\overline{\Omega}$, i.e. a solution of (16) such that $x(t) \in \overline{C}$ for all $t \in [0,1]$ will exist if we can show that for each $\lambda \in (0,1)$, no possible solution of the problem

$$x' - x = \lambda[f(t,x) - x], \; x(1) = \int_0^1 x(s)\, dh(s), \tag{30}$$

belongs to $\partial\Omega$. Let $\lambda \in (0,1)$ and $x(t) \in \partial\Omega$ be a possible solution to (26). Then $x(t) \in \overline{C}$ for all $t \in [0,1]$ and there is some $t_0 \in [0,1]$ such that $x(t_0) \in \partial C$. Therefore, for all $t \in [0,1]$,

$$\xi_{t_0}(t) := \langle \nu(x(t_0)) | x(t) \rangle \leq \langle \nu(x(t_0) | x(t_0) \rangle = \xi_{t_0}(t_0), \; \forall\, t \in [0,1],$$

which means that the real function $\xi_{t_0} : [0,1] \to \mathbb{R}$ reaches its maximum at t_0. If $t_0 \in [0,1)$,

$$\begin{aligned} 0 \;&\geq\; \xi'_{t_0}(t_0) = \langle \nu(x(t_0)) | x'(t_0) \rangle \\ &=\; (1-\lambda)\langle \nu(x(t_0)) | x(t_0) \rangle + \lambda \langle \nu(x(t_0)) | f(t_0, x(t_0)) > 0, \end{aligned}$$

a contradiction. If $t_0 = 1$, and conditions (h0), (h1) holds, then, because of the boundary condition,

$$\begin{aligned} \xi_1(1) \;&=\; \langle \nu(x(1)) | x(1) \rangle = \int_0^1 \langle \nu(x(1)) | x(s)\, dh(s) \rangle \\ &\leq\; \max_{[0,1]} \langle \nu(x(1)) | x(s) \rangle \int_0^1 dh(s) < \max_{[0,1]} \langle \nu(x(1) | x(s) \rangle \\ &=\; \max_{s \in [0,1]} \xi_1(s), \end{aligned}$$

a contradiction. If $t_0 = 1$, and conditions (h0), (h2), (h4) hold, it remains only to consider the case where 1 is the only value of t at which $x(t) \in \partial C$, i.e. the case where ξ_1 reaches its maximum only at 1. Then

$$\xi_1(s) < \xi_1(1),\ \forall s \in [0,1),$$

and hence, using the boundary condition and Assumptions (h3), (h4),

$$\begin{aligned}
\xi_1(1) &= \left\langle \nu(x(1)) \,\middle|\, \int_0^1 x(s)\,dh(s) \right\rangle = \int_0^1 \langle \nu(x(1)) | x(s)\,dh(s)\rangle \\
&= \int_0^1 \langle \nu(x(1)) | x(s)\rangle\,dh(s) = \int_0^1 \xi_1(s)\,dh(s) \\
&= \int_0^\tau \xi_1(s)\,dh(s) + \int_\tau^1 \xi_1(s)\,dh(s) \\
&\le (\max_{[0,\tau]} \xi_1) h(\tau) + \xi_1(1)[1 - h(\tau)] < \xi_1(1),
\end{aligned}$$

a contradiction. Consequently the assumptions of Proposition 2.8 are satisfied for G_2 and $\overline{\Omega}$, and the conclusion follows. □

Corollary 2.16. *If $h : [0,1] \to \mathbb{R}$ satisfies conditions (h0), h(1) or conditions (h0), h(2), h(4), and if there exists $R > 0$ such that*

$$\langle u | f(t,u)\rangle \ge 0,\ \forall\,(t,u) \in [0,1] \times \partial B_R, \tag{31}$$

then the problem (18) has at least one solution taking values in $\overline{B}_R$.

Proof. Take $C = B_R$ and, for each $u \in \partial B_R$, $\nu(u) = u$. □

Corollary 2.17. *If $h : [0,1] \to \mathbb{R}$ satisfies conditions (h0), h(1) or conditions (h0), (h2), (h4), and if there exists $R_j > 0$ $(1 \le j \le n)$ such that*

$$u_i f_i(t,u) \ge 0,\ \forall\,(t,u) \in [0,1] \times \prod_{j=1}^n [-R_j, R_j] \ :\ |u_i| = R_i\ (1 \le i \le n), \tag{32}$$

then the problem (18) has at least one solution taking values in $\Pi_{j=1}^n [-R_j, R_j]$.

Proof. Take $C = \prod_{j=1}^n (-R_j, R_j)$ and, for each $u \in \prod_{j=1}^n [-R_j, R_j]$ and $|u_i| = R_i$, $\nu(u) = u_i e^i$, where $e^i = (0,\ldots,0,1,0,\ldots,0)$ is the i^{th} element of the canonical basis of $\mathbb{R}^n$ $(i = 1,\ldots,n)$. □

2.8. Lower and upper solutions for nonlocal initial or terminal BVP

Corollaries 2.14 and 2.17 can be generalized by extending the classical concepts of lower and upper solutions to our nonlocal boundary value problems.

DEFINITION 2.18. *We say that* $\alpha \in C^1([0,1],\mathbb{R}^n)$ *is a* **lower solution** *and* $\beta \in C^1([0,1],\mathbb{R}^n)$ *an* **upper solution** *for problem* (16)*, if*

$$\alpha_i(t) \leq \beta_i(t) \ \ (1 \leq i \leq n)$$

and, for each $i \in \{1,\dots,n\}$,

$$\begin{aligned} \alpha_i'(t) &\leq f(t,u_1,\dots,u_{i-1},\alpha_i(t),u_{i+1}(t),\dots,u_n(t)), \\ \beta_i'(t) &\geq f(t,u_1,\dots,u_{i-1},\beta_i(t),u_{i+1}(t),\dots,u_n(t)), \end{aligned} \tag{33}$$

whenever $\alpha_j(t) \leq u_j \leq \beta_j(t)$, $t \in [0,1]$, $j \in \{1,\dots,n\} \setminus \{i\}$,

$$\alpha_i(0) \leq \int_0^1 \alpha_i(s)\,dh(s), \ \beta_i(0) \geq \int_0^1 \beta_i(s)\,dh(s).$$

DEFINITION 2.19. *We say that* $\alpha \in C^1([0,1],\mathbb{R}^n)$ *is a* **lower solution** *and* $\beta \in C^1([0,1],\mathbb{R}^n)$ *an* **upper solution** *for problem* (18)*, if*

$$\alpha_i(t) \leq \beta_i(t) \ (1 \leq i \leq n)$$

and, for each $i \in \{1,\dots,n\}$,

$$\begin{aligned} \alpha_i'(t) &\geq f(t,u_1,\dots,u_{i-1},\alpha_i(t),u_{i+1}(t),\dots,u_n(t)), \\ \beta_i'(t) &\leq f(t,u_1,\dots,u_{i-1},\beta_i(t),u_{i+1}(t),\dots,u_n(t)), \end{aligned} \tag{34}$$

whenever $\alpha_j(t) \leq u_j \leq \beta_j(t)$, $t \in [0,1]$, $j \in \{1,\dots,n\} \setminus \{i\}$,

$$\alpha_i(1) \geq \int_0^1 \alpha_i(s)\,dh(s), \ \beta_i(1) \leq \int_0^1 \beta_i(s)\,dh(s).$$

For the initial value problem and a scalar equation, the concept and the corresponding theorem was introduced by Peano [88] in 1895, rediscovered by Perron [89] in 1912, and extended to systems by Müller in 1927 [79]. The case of periodic solutions was first considered by Moretto [78] in the scalar case, by Knobloch [54] in 1962 for locally Lipschitzian systems, and generalized to continuous systems in 1974 [69].

THEOREM 2.20. *If conditions (h0), (h1) or conditions (h0), (h2), (h3) hold, and if a couple of lower and upper solutions* α, β *exists for* (16)*, then the problem (16) has a solution* x *such that* $\alpha_i(t) \leq x_i(t) \leq \beta_i(t)$ *for all* $t \in [0,1]$ *and all* $i \in \{1,\dots,n\}$.

Proof. For each $i \in \{1, \ldots, n\}$, define the continuous and bounded function $\gamma_i : [0, T] \times \mathbb{R} \to \mathbb{R}$ by

$$\gamma_i(t, u_i) := \begin{cases} \alpha_i(t) & \text{if} \quad u_i < \alpha_i(t) \\ u_i & \text{if} \quad \alpha_i(t) \leq u_i \leq \beta_i(t) \\ \beta_i(t) & \text{if} \quad u_i > \beta_i(t) \end{cases} \tag{35}$$

and consider the modified problem

$$\begin{aligned} x_i' = -[x_i - \gamma_i(t, x_i)] \quad &+ \quad f_i(t, \gamma_1(t, x_1), \ldots, \gamma_n(t, x_n)) := g_i(t, x) \ \ (1 \leq i \leq n) \\ x(0) \quad &= \quad \int_0^1 x(s)\, dh(s). \end{aligned} \tag{36}$$

As each γ_i and $f_i(\cdot, \gamma_1(\cdot, \cdot), \ldots, \gamma_n(\cdot, \cdot))$ are bounded, for each $i \in \{1, \ldots, n\}$, there exists $R_i > 0$ such that $g_i(t, u) \geq 0$ for all $u \in \prod_{j=1}^n [-R_j, R_j]$ verifying $u_i = -R_i$ and such that $g_i(t, u) \leq 0$ for all $u \in \prod_{j=1}^n [-R_j, R_j]$ verifying $u_i = R_i$. Using Corollary 2.14, we have a solution x to (36) such that $x(t) \in \prod_{j=1^n} [-R_j, R_j]$ for all $t \in [0, 1]$. We now show that $\alpha_i(t) \leq x_i(t) \leq \beta_i(t)$ for all $t \in [0, 1]$ and all $i \in \{1, \ldots, n\}$, so that x is a solution to (16). Fix some $i \in \{1, \ldots, n\}$ and assume that there is some $\tau \in [0, 1]$ such that $x_i(\tau) < \alpha_i(\tau)$. Then $x_i - \alpha_i$ reaches a negative minimum at some $t_0 \in [0, 1]$. If $t_0 \in (0, 1]$, then $x_i'(\tau) - \alpha_i'(\tau) \leq 0$, and hence

$$\begin{aligned} \alpha_i'(\tau) \quad &\geq \quad x_i'(\tau) = -[x_i(\tau) - \alpha_i(\tau)] \\ &\qquad + f_i(\tau, \gamma_1(\tau, x_1(\tau)), \ldots, \alpha_i(\tau), \ldots, \gamma_n(\tau, x_n(\tau))) \\ &> \quad f_i(\tau, \gamma_1(\tau, x_1(\tau)), \ldots, \alpha_i(\tau), \ldots, \gamma_n(\tau, x_n(\tau))), \end{aligned}$$

a contradiction with the definition (33) of lower solution for (16). If $t_0 = 0$, then, using the previous contradiction, we can assume that

$$x_i(0) - \alpha_i(0) < x_i(t) - \alpha_i(t) \ \forall\, t \in (0, 1]$$

and hence, integrating over $[0, 1]$ and using the boundary conditions for x_i and α_i,

$$[x_i(0) - \alpha_i(0)]h(1) \quad \leq \quad \int_0^1 x_i(t)\, dh(t) - \int_0^1 \alpha_i(t)\, dh(t) \leq x_i(0) - \alpha_i(0)$$

so that

$$[1 - h(1)][x_i(0) - \alpha_i(0)] \geq 0,$$

and, using (h1),

$$x_i(0) - \alpha_i(0) \geq 0,$$

a contradiction. We leave to the reader the proof in the case where conditions (h0), (h2), (h3) hold. The reasoning is similar to show that $x_i(t) \leq \beta_i(t)$ for all $t \in [0,1]$ and all $i \in \{1,\dots,n\}$. Hence, the solution x to (36) is also a solution to problem (16). □

A similar proof provides the corresponding result for the nonlocal terminal type BVP.

THEOREM 2.21. *If conditions (h0), (h1) or conditions (h0), (h2), (h4) hold, and if a couple of lower and upper solutions* α, β *exists for* (18), *then the problem (18) has a solution* x *such that* $\alpha_i(t) \leq x_i(t) \leq \beta_i(t)$ *for all* $t \in [0,1]$ *and all* $i \in \{1,\dots,n\}$.

Extensions of Knobloch's theorem to some multipoint boundary value problems have been given by Ponomarev [92, 93].

2.9. Periodic vs resonant nonlocal initial or terminal type BC

The special case of Theorem 2.12 with h given by (11) (which satisfies assumptions (h0), (h2), (h3)), together with the special case of Theorem 2.15 with h given by (10) (which satisfies assumptions (h0), (h2), (h4)) provide a proof of the generalized version of Theorem 2.11 with non–strict inequalities in the assumptions, and of its consequence Theorem 2.9, for periodic boundary conditions.

Comparing the statement of Theorem 2.9 for the periodic problem, with the statements of the corresponding Corollaries 2.13 and 2.16 we see that the sense of the inequality in conditions (27) and (31) depends upon the nonlocal boundary condition.

On the other hand, as it is easily verified by direct computation, the system

$$x' = \lambda x + e(t),$$

with each of the three–point boundary conditions

$$x'(0) = \frac{1}{2}[x(1/2) + x(1)], \quad x(1) = \frac{1}{2}[x(1/2) + x(0)],$$

has a solution for each $e \in C^0$ and each $\lambda \in \mathbb{R} \setminus \{0\}$. This is a consequence of the fact that the only real eigenvalue of $\frac{d}{dt}$ with each boundary condition is 0.

Hence a natural question is to know whether the conclusion of the above corollaries still holds when the sense of the corresponding inequality upon f is reversed.

We show by some counterexamples that the answer is negative in general, which of course implies that the same negative answer holds for Theorems 2.12 and 2.15. In this sense, the existence conditions given in Theorems 2.12 and 2.15 are sharp.

The construction of our counterexamples depends upon the study of some associated complex eigenvalue problem and of the corresponding Fredholm alternative for some special three–point boundary conditions. The results are taken from [74].

2.10. Nonlocal initial or terminal type eigenvalue problems

We first consider the three–point eigenvalue problem

$$z'(t) = \lambda z(t),\ z(0) = \frac{1}{2}[z(1/2) + z(1)], \tag{37}$$

where $\lambda \in \mathbb{C}$, $z : [0,1] \to \mathbb{C}$. The boundary condition is a special case of the one in Corollary 2.13 with

$$h(s) = \begin{cases} 0 & \text{if} \quad s \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{if} \quad s \in [\frac{1}{2}, 1) \\ 1 & \text{if} \quad s = 1. \end{cases}$$

Proposition 2.22. *The problem (37) has the eigenvalues*

$$\lambda_{IC,1,k} = 2k(2\pi i),\ \lambda_{IC,2,k} = \log 4 + (2k+1)(2\pi i) \ \ (k \in \mathbb{Z}).$$

They are located in the right part of the complex plane.

Proof. The eigenvalue problem (37) has a nontrivial solution if and only if $\lambda \in \mathbb{C}$ is such that

$$1 = \frac{1}{2}e^{\lambda/2} + \frac{1}{2}e^{\lambda}. \tag{38}$$

Setting $\mu := e^{\lambda/2}$, the equation (38) becomes the equation in μ

$$\frac{1}{2}\mu^2 + \frac{1}{2}\mu - 1 = 0$$

with solutions $\mu_{IC,1} = 1$ and $\mu_{IC,2} = -2$. The equation $e^{\lambda/2} = \mu_{IC,1}$ is satisfied for $\frac{\lambda}{2} = 2k\pi i$ $(k \in \mathbb{Z})$, which gives the eigenvalues $\lambda_{IC,1,k}$ $(k \in \mathbb{Z})$. The equation $e^{\lambda/2} = \mu_{IC,2} = -2$ is satisfied for $\frac{\lambda}{2} = \log 2 + (2k+1)(\pi i)$ $(k \in \mathbb{Z})$, which gives the eigenvalues $\lambda_{IC,2,k}$ $(k \in \mathbb{Z})$. □

Similarly, we consider the three–point eigenvalue problem

$$z'(t) = \lambda z(t),\ z(1) = \frac{1}{2}[z(0) + z(1/2)], \tag{39}$$

where $\lambda \in \mathbb{C}$, $z : [0,1] \to \mathbb{C}$. The boundary condition is a special case of the one in Corollary 2.16 with

$$h(s) = \begin{cases} 0 & \text{if} \quad s = 0 \\ \frac{1}{2} & \text{if} \quad s \in (0, \frac{1}{2}] \\ 1 & \text{if} \quad s \in (\frac{1}{2}, 1]. \end{cases}$$

Proposition 2.23. *The problem (39) has the eigenvalues*

$$\lambda_{TC,1,k} = 2k(2\pi i),\ \ \lambda_{TC,2,k} = -\log 4 + (2k+1)(2\pi i)\ \ (k \in \mathbb{Z}).$$

They are located in the left part of the complex plane.

Proof. The eigenvalue problem (39) has a nontrivial solution if and only if $\lambda \in \mathbb{C}$ is such that

$$e^{\lambda} = \frac{1}{2} + \frac{1}{2}e^{\lambda/2}. \tag{40}$$

Setting $\mu := e^{\lambda/2}$, the equation (40) becomes the equation in μ

$$\mu^2 - \frac{1}{2}\mu - \frac{1}{2} = 0,$$

with solutions $\mu_{TC,1} = 1$, $\mu_{TC,2} = -\frac{1}{2}$. The equation $e^{\lambda/2} = \mu_{TC,1} = 1$ is satisfied for $\frac{\lambda}{2} = 2k\pi i$ $(k \in \mathbb{Z})$, which gives the eigenvalues $\lambda_{TC,1,k}$ $(k \in \mathbb{Z})$. The equation $e^{\lambda/2} = \mu_{TC,2} = -\frac{1}{2}$ is satisfied for $\frac{\lambda}{2} = -\log 2 + (2k+1)\pi i$ $(k \in \mathbb{Z})$, which gives the eigenvalues $\lambda_{TC,2,k}$ $(k \in \mathbb{Z})$. $\square$

Remark 2.24: The eigenvalues of the periodic boundary conditions

$$z' = \lambda z,\ z(0) = z(1)$$

are, as easily seen, $\lambda_{P,k} = k(2\pi i)$ $(k \in \mathbb{Z})$. In the case of the problem (39), half of the eigenvalues of the periodic problem move to the line $\Re z = -\log 4$, and, in the case of the problem (37), the same half moves to the line $\Re z = \log 4$. In each case, the symmetry of the spectrum with respect to the imaginary axis is lost (see Fig. 3).

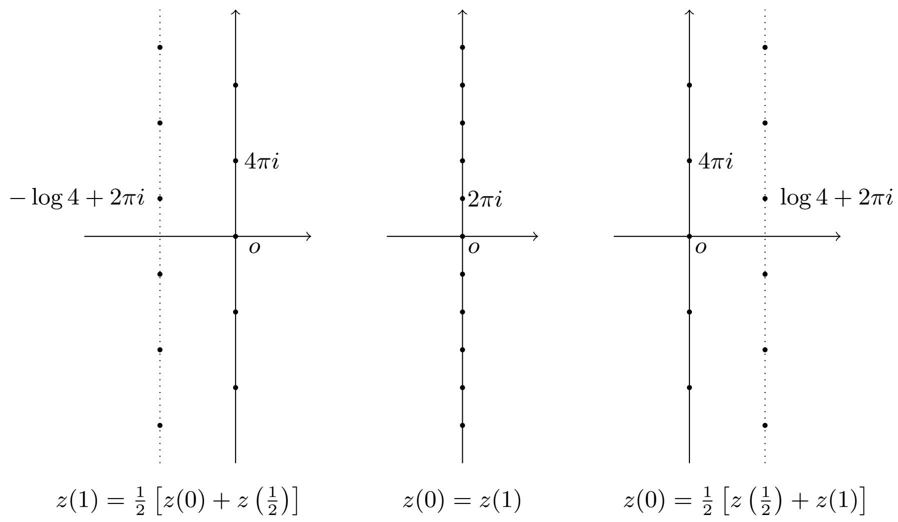


Figure 4: Eigenvalues.

2.11. Fredholm alternative for some linear nonlocal initial or terminal type BVP

To construct our counterexamples, we use of the Fredholm alternative for the corresponding forced eigenvalue problems.

Proposition 2.25. *λ is an eigenvalue of (37) (resp. (39)) if and only if there exists $e \in C([0,1],\mathbb{C})$ such that the nonhomogeneous problem (41) (resp. (42)) has no solution.*

Proof. It is shown in Lemmas 2.4 and 2.6 (or by direct verification) that the non-homogeneous problems

$$z' + z = e(t),\ z(0) = \frac{1}{2}z(1/2) + \frac{1}{2}z(1)$$

and

$$z' - z = e(t),\ z(1) = \frac{1}{2}z(1/2) + \frac{1}{2}z(0)$$

have a unique solution $z = K_1 e$ and $z = K_2 e$ for every $e \in C([0,1],\mathbb{C})$, with K_1 and K_2 compact in $C([0,1],\mathbb{C})$. Consequently, each problem

$$z' - \lambda z = e(t),\ z(0) = \frac{1}{2}z(1/2) + \frac{1}{2}z(1), \tag{41}$$

and

$$z' - \lambda z = e(t),\ z(1) = \frac{1}{2}z(1/2) + \frac{1}{2}z(0), \tag{42}$$

can be written equivalently

$$z = (\lambda + 1)K_1 z + K_1 e,\ \ z = (\lambda - 1)K_2 z + K_2 e,$$

and the Fredholm alternative follows from Riesz theory of linear compact operators. □

2.12. Counterexamples to Corollaries 2.13 and 2.16 under opposite vector fields sign conditions

We finalize the construction of our counterexamples.

In the case of a three–point boundary condition of initial type, we apply Proposition 2.25 to the eigenvalue $\lambda_{IC,2,0} = \log 4 + 2\pi i$ of (37). Using Proposition 2.25, let $e \in C([0,1], \mathbb{C})$ be such that the problem

$$z'(t) = (\log 4 + 2\pi i)z(t) + e(t),\ z(0) = \frac{1}{2}z(1/2) + \frac{1}{2}z(1)$$

has no solution. Setting $z(t) = x_1(t) + ix_2(t),\ e(t) = e_1(t) + ie_2(t)$, the equivalent planar real problem

$$x' = f(t,x),\ \ x(0) = \frac{1}{2}x(1/2) + \frac{1}{2}x(1) \tag{43}$$

with

$$f(t,u) := ((\log 4)u_1 - 2\pi u_2 + e_1(t), 2\pi u_1 + (\log 4)u_2 + e_2(t)),$$

is such that

$$\begin{aligned} \langle u | f(t,u) \rangle &= (\log 4)(u_1^2 + u_2^2) + u_1 e_1(t) + u_2 e_2(t) \\ &\geq (\log 4)|u|^2 - |e(t)||u| > 0 \end{aligned}$$

when $|u| \geq R$ for some sufficiently large R, and has no solution.

Applying Proposition 2.25 to the case of the eigenvalue $\lambda_{TC,2,0} = -\log 4 + 2\pi i$ of (39), let $e \in C([0,1], \mathbb{C})$ be such that the problem

$$z'(t) = (-\log 4 + 2\pi i)z(t) + e(t),\ z(1) = \frac{1}{2}z(0) + \frac{1}{2}z(1/2)$$

has no solution. Setting $z(t) = x_1(t) + ix_2(t)$, $e(t) = e_1(t) + ie_2(t)$, the equivalent planar real problem

$$x' = f(t,x), \quad x(1) = \frac{1}{2}x(0) + \frac{1}{2}x(1/2) \tag{44}$$

with

$$f(t,u) := (-(\log 4)u_1 - 2\pi u_2 + e_1(t), 2\pi u_1 - (\log 4)u_2 + e_2(t)),$$

is such that

$$\begin{aligned} \langle u|f(t,u)\rangle &= -(\log 4)(u_1^2 + u_2^2) + u_1 e_1(t) + u_2 e_2(t) \\ &\leq -(\log 4)|u|^2 + |e(t)||u| < 0, \end{aligned} \tag{45}$$

when $|u| \geq R$ for some sufficiently large R, and has no solution.

Remark 2.26: The symmetry–breaking for the spectra of the three–point BVP of terminal or initial type, explains the difference in the existence conditions for the nonlinear problems with the three–point boundary conditions and with the periodic conditions. The presence of the complex spectrum in the left or the right half plane influences like a ghost the existence of solutions of the real nonlinear systems. Of course, extra conditions upon f could provide existence results with the sign conditions of the counterexamples.

Remark 2.27: Our counterexamples do not cover the case of n odd. For $n = 3$, if one adds the equations

$$x_3' = (\log 4)x_3 + \frac{\log 4}{4}(x_1 + x_2), \ x_3(0) = \frac{1}{2}[x_3(1/2) + x_3(1)],$$

or

$$x_3' = -(\log 4)x_3 + \frac{\log 4}{4}(x_1 + x_2), \ x_3(1) = \frac{1}{2}[x_3(0) + x_3(1/2)]$$

to (43) or to (44) respectively, the corresponding boundary value problems have no solutions and the nonlinear parts verify the opposite sign conditions to Corollaries 2.13 and 2.16 respectively. The counterexamples for $n = 2$ and $n = 3$ easily provide counterexamples in any dimension $n \geq 2$.

Remark 2.28: As easily seen, the periodic problem

$$z' = 2\pi iz + e^{2\pi it}, \ z(0) = z(1). \tag{46}$$

has no solution. Letting $z = x_1 + ix_2$, the equivalent real planar problem

$$x' = f(t,x), \quad x(0) = x(1)$$

with

$$f_1(t,x_1,x_2) = -2\pi x_2 + \cos(2\pi t), \ f_2(t,x_1,x_2) = 2\pi x_1 + \sin(2\pi t),$$

has no solution, and is such that

$$\langle x|f(t,x)\rangle = \cos(2\pi t)x_1 + \sin(2\pi t)x_2$$

For $x = R[\cos(2\pi\theta), \sin(2\pi\theta)] \in \partial B_R$ ($\theta \in [0,1]$), we have

$$\begin{aligned}\langle x|f(t,x)\rangle &= R[\cos(2\pi t)\cos(2\pi\theta) + \sin(2\pi t)\sin(2\pi\theta)] \\ &= R\cos[2\pi(t-\theta)] \quad (t,\theta \in [0,1]),\end{aligned}$$

which implies that, for each $t \in [0,1]$, $\langle x|f(t,x)\rangle$ takes both positive and negative values on ∂B_R. Hence, the assumptions of the existence theorems for periodic problems given above are sharp.

2.13. An easy extension of Theorems 2.12 and 2.15

Let $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. By replacing f by g in the equivalent formulation as a fixed point problem and in the proofs, it is immediate to prove the following extensions of Theorems 2.12 and 2.15.

THEOREM 2.29. *If $h : [0,1] \to \mathbb{R}$ satisfies conditions (h0),(h1), or conditions h(0), h(2), h(3), and if there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν on ∂C such that*

$$\langle \nu(u)|g(t,v,u)\rangle \le 0,\ \forall\,(t,v,u) \in [0,1] \times \overline{C} \times \partial C, \tag{47}$$

then the problem

$$x'(t) = g\left(t, \int_0^t x(s)\,ds, x(t)\right) \quad (t \in [0,1]),\ x(0) = \int_0^1 dh(s)x(s) \tag{48}$$

has at least one solution taking values in $\overline{C}$.

Proof. The main difference in the proof is that the nonlinear mapping $N_1 : C^0 \to C^0$ occuring in the fixed point formulation is now defined by

$$N_1x(t) = g\left(t, \int_0^t x,\ x(t)\right) - x(t) \quad (t \in [0,1]),$$

and its value at $t \in [0,1]$ depends not only on $x(t)$ but on all values of $x(s)$ for $s \in [0,t]$. It is easily checked that it does not modify the compactness properties of the operator K_1N_1. All the other arguments of the proof remain valid *mutatis mutandis* because of the uniformity of assumption (47) with respect to v. □

THEOREM 2.30. *If $h : [0,1] \to \mathbb{R}$ satisfies conditions (h0), (h1), or conditions h(0), h(2), h(4), and if there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν on ∂C such that*

$$\langle \nu(u)|g(t,v,u)\rangle \ge 0,\ \forall\,(t,v,u) \in [0,1] \times \overline{C} \times \partial C,$$

then the problem

$$x'(t) = g\left(t, \int_0^t x(s)\,ds, x(t)\right) \quad (t \in [0,1]),\ x(1) = \int_0^1 x(s)\,dh(s) \tag{49}$$

has at least one solution taking values in $\overline{C}$.

Proof. Similar to Theorem 2.15 using the remarks in the proof of Theorem 2.29. □

Of course, the following extensions, where the value of $x'(t)$ depends this time upon the values of $x(s)$ for $s \in [t,1]$, are obtained in a similar way.

THEOREM 2.31. *If* $h : [0,1] \to \mathbb{R}$ *satisfies conditions (h0),(h1), or conditions h(0), h(2), h(3), and if there exists an open, bounded, convex neighborhood* C *of* 0 *in* $\mathbb{R}^n$ *and an outer normal field* ν *on* ∂C *such that*

$$\langle \nu(u) | g(t,v,u)\rangle \leq 0,\ \forall\,(t,v,u) \in [0,1] \times \overline{C} \times \partial C,$$

then the problem

$$x'(t) = g\left(t, \int_1^t x(s)\,ds, x(t)\right) \quad (t \in [0,1]),\ x(0) = \int_0^1 dh(s)x(s)$$

has at least one solution taking values in $\overline{C}$.

THEOREM 2.32. *If* $h : [0,1] \to \mathbb{R}$ *satisfies conditions (h0), (h1), or conditions h(0), h(2), h(4), and if there exists an open, bounded, convex neighborhood* C *of* 0 *in* $\mathbb{R}^n$ *and an outer normal field* ν *on* ∂C *such that*

$$\langle \nu(u) | g(t,v,u)\rangle \geq 0,\ \forall\,(t,v,u) \in [0,1] \times \overline{C} \times \partial C,$$

then the problem

$$x'(t) = g\left(t, \int_1^t x(s)\,ds, x(t)\right) \quad (t \in [0,1]),\ x(1) = \int_0^1 x(s)\,dh(s)$$

has at least one solution taking values in $\overline{C}$.

3. Second order systems

3.1. Boundary conditions

We now consider the case of second order (1) where $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Again there is no loss of generality in taking the independent variable in $[0,1]$.

We consider in what follows the following particular nonlocal conditions: the **Dirichlet type nonlocal conditions**

$$x(0) = 0, \ \ x(1) = \int_0^1 x(s)\, dh(s), \tag{50}$$

the **mixed type nonlocal conditions**

$$x'(0) = 0, \ \ x(1) = \int_0^1 x(s)\, dh(s), \tag{51}$$

the **nonlocal conditions of initial type**

$$x(0) = 0, \ \ x'(0) = \int_0^1 x'(s)\, dh(s),$$

the **mixed type nonlocal conditions**

$$x(0) = 0, \ \ x'(1) = \int_0^1 x'(s)\, dh(s),$$

and the **nonlocal conditions of terminal type**

$$x(1) = 0, \ \ x'(1) = \int_0^1 x'(s)\, dh(s).$$

Neumann type nonlocal conditions are considered in [73, 102, 108] using other continuation theorems.

3.2. Some nonlocal BVP for linear second order systems

We start with the Dirichlet type nonlocal BC.

LEMMA 3.1. *If conditions (h0), (h1) or (h0), (h2), (h4) hold, then, for each* $z \in C^0$*, the linear nonlocal Dirichlet type problem*

$$x'' - x = z(t),\ x(0) = 0,\ x(1) = \int_0^1 x(s)\, dh(s) \tag{52}$$

has the unique solution

$$x(t) = \left(\sinh 1 - \int_0^1 \sinh s\, dh(s)\right)^{-1} \int_0^1 \int_0^u \sinh(u-s) z(s)\, ds\, dh(u)\, \sinh t + \int_0^t \sinh(t-s) z(s)\, ds. \tag{53}$$

Proof. By the variation of constants formula for each $c \in \mathbb{R}^n$, the initial value problem

$$x'' - x = z(t),\ x(0) = 0,\ x'(0) = c,$$

has the unique solution

$$x(t) = c \sinh t + \int_0^t \sinh(t-s) z(s)\, ds.$$

It satisfies the boundary condition (50) if and only if c satisfies the linear algebraic system

$$c \sinh 1 = c \int_0^1 \sinh t\, dh(t) + \int_0^1 \int_0^t \sinh(t-s) z(s)\, ds\, dh(t),$$

which has the unique solution

$$c = \left(\sinh 1 - \int_0^1 \sinh s\, dh(s) \right)^{-1} \int_0^1 \int_0^u \sinh(u-s) z(s)\, ds\, dh(u).$$

if $\int_0^1 \sinh s\, dh(s) \neq \sinh 1$. Following the reasoning of the corresponding Lemma for first order systems, and noticing that sinh reaches its maximum on $[0,1]$ at 1, this is the case if conditions (h0), (h1) or conditions (h0), (h2), (h4) hold. The result follows. □

Let C^1 be the Banach space of mappings $x : [0,1] \to \mathbb{R}^n$ of class C^1 with the norm

$$\|x\| := \max\{\|x_1\|_\infty, \dots, \|x_n\|_\infty, \|x_1'\|_\infty, \dots, \|x_n'\|_\infty\}.$$

Like in the first order case, formula (53) defines a compact linear mapping

$$K_1 : C^0 \to C^1,\ z \mapsto x.$$

In a similar way, one can prove the corresponding results for the mixed type nonlocal BC.

LEMMA 3.2. *If conditions (h0), (h1) or (h0), (h2), (h4) hold, then, for each $z \in C^0$, the linear nonlocal mixed type problem*

$$x'' - x = z(t),\ x'(0) = 0,\ x(1) = \int_0^1 x(s)\, dh(s) \tag{54}$$

has a unique solution x and the corresponding linear mapping

$$K_2 : C^0 \to C^1,\ z \mapsto x$$

is compact.

3.3. Fixed point formulation of nonlinear nonlocal BVP of the second order

Let now $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and define the mapping $N : C^1 \to C^0$ by

$$Nx = f(\,\cdot\,, x(\cdot), x'(\cdot)) - x(\cdot).$$

It is easy to show that N is continuous on C^1 and take bounded sets of C^1 into bounded sets of C^0. Under the conditions of Lemma 3.1,

$$G_1 := K_1 N : C^1 \to C^1$$

is compact on bounded subsets of C^1, and the **nonlinear nonlocal Dirichlet type problem**

$$x'' = f(t, x, x'), \;\; x(0) = 0, \; x(1) = \int_0^1 x(s)\,dh(s) \tag{55}$$

is equivalent to the fixed point problem in C^1

$$x = G_1 x. \tag{56}$$

Similarly, under the conditions of Lemma 3.2,

$$G_2 := K_2 N : C^1 \to C^1$$

is compact on bounded subsets of C^1, and the **nonlinear nonlocal mixed type problem**

$$x'' = f(t, x, x'), \;\; x'(0) = 0, \; x(1) = \int_0^1 x(s)\,dh(s) \tag{57}$$

is equivalent to the fixed point problem in C^1

$$x = G_2 x. \tag{58}$$

We want to apply to the equations (56) and (58) the Leray–Schauder existence result given in Proposition 2.8, where now $X = C^1$, so that the a priori estimates are requested not only upon x but also upon x'.

3.4. Bernstein–Hartman lemma

In order to obtain the a priori estimates on x' requested by Proposition 2.8 when an a priori estimate on x is known, we use the following lemma, a special case of a more general result of Hartman [31, 32] for functions with values in $\mathbb{R}^n$. For $n = 1$, the result, without condition (59), was proved by Bernstein in 1912 [6].

Lemma 3.3. *Assume that* $x \in C^2([0,1], \mathbb{R}^n)$ *satisfies the following inequalities*

$$|x(t)| \leq R,$$

and

$$|x''(t)| \leq \gamma |x'(t)|^2 + K$$

for all $t \in [0,1]$ *and some* $R > 0$, $K \geq 0$ *and* $\gamma \geq 0$ *such that*

$$\gamma R < 1. \tag{59}$$

Then, there exists $M = M(R, \gamma, K)$ *such that for all* $t \in [0,1]$,

$$|x'(t)| \leq M.$$

Remark 3.4: For $n \geq 2$, the condition (59) is sharp, as shown by the example of the sequence of functions, introduced by Heinz [34],

$$x_n : [0, 2\pi] \to \mathbb{R}^2, \ t \mapsto (\cos nt, \sin nt) \quad (n \in \mathbb{N}),$$

for which, with $\langle \cdot | \cdot \rangle$ the usual inner product and $|\cdot|$ the Euclidian norm in $\mathbb{R}^2$,

$$|x_n(t)| = 1, \ |x_n''(t)| = |x_n'(t)|^2 = n^2,$$

so that that the conclusion of Lemma 3.3 does not hold for $\gamma R = 1$ and $T = 2\pi$, as $|x_n'(t)| = n$ can be arbitrary large.

3.5. Some nonlocal nonlinear BVP of Dirichlet or mixed type

We first show that conditions with respect to u on the vector field $f(t, u, v)$ similar to those introduced for first order systems also lead to the existence of solutions for second order systems.

Theorem 3.5. *Assume that conditions (h0),(h1) or conditions (h0),(h2),(h4) hold, and that there exists an open, bounded, convex neighborhood* C *of* 0 *in* $\mathbb{R}^n$*, and an outer normal vector field* ν *on* ∂C*, such that*

$$\langle \nu(u) | f(t, u, v) \rangle \geq 0, \quad \text{whenever } u \in \partial C \text{ and } \langle v | \nu(u) \rangle = 0$$

and

$$|f(t, u, v)| \leq \gamma |v|^2 + K,$$

for some $\gamma \geq 0$ *such that* $\gamma R < 1$, $K \geq 0$, *and* $R = \max_{u \in \overline{C}} |u|$. *Then the problem* (55) *has at least one solution such that* $x(t) \in \overline{C}$ *for all* $t \in [0,1]$.

Proof. Let us consider the equation (56) and define first the open bounded neighborhood Ω_1 of 0 in C^0 by

$$\Omega_1 = \{x \in C^0 : x(t) \in C,\ \forall\, t \in [0,1]\}. \tag{60}$$

As first step in applying Proposition 2.8, we show that for each $\lambda \in (0,1)$, no possible solution of the problem

$$x'' - x = \lambda[f(t,x,x') - x],\ x(0) = 0,\ x(1) = \int_0^1 x(s)\,dh(s), \tag{61}$$

belongs to $\partial\Omega_1$. Let $\lambda \in (0,1)$ and $x(t) \in \partial\Omega_1$ be a possible solution to (61). Then $x(t) \in \overline{C}$ for all $t \in [0,1]$ and there is some $t_0 \in [0,1]$ such that $x(t_0) \in \partial C$. Therefore, for all $t \in [0,1]$,

$$\xi_{t_0}(t) := \langle \nu(x(t_0))|x(t)\rangle \leq \langle \nu(x(t_0)|x(t_0)\rangle = \xi_{t_0}(t_0),\ \forall\, t \in [0,1],$$

which means that the real function $\xi_{t_0} : [0,1] \to \mathbb{R}$ reaches its maximum at t_0. Because of the first boundary condition, we cannot have $t_0 = 0$. If $t_0 \in (0,1)$, $0 = \xi'_{t_0}(t_0) = \langle \nu(x(t_0))|x'(t_0)\rangle$and

$$\begin{aligned} 0 &\geq \xi''_{t_0}(t_0) = \langle \nu(x(t_0))|x''(t_0)\rangle \\ &= (1-\lambda)\langle \nu(x(t_0))|x(t_0)\rangle + \lambda\langle \nu(x(t_0))|f(t_0, x(t_0), x'(t_0)) > 0, \end{aligned}$$

a contradiction. Finally, if $t_0 = 1$, we use the second boundary condition like in the nonlocal terminal like problem for first order systems to obtain the contradiction.

Now, as $x(t) \in \overline{C}$ for all $t \in [0,1]$, we have, for all $t \in [0,1]$,

$$\begin{aligned} |x''(t)| &= |(1-\lambda)x(t) + \lambda f(t, x(t), x'(t))| \\ &\leq R + \gamma|x'(t)|^2 + K = \gamma|x'(t)|^2 + (R+K) \end{aligned}$$

and Lemma 3.3 implies the existence of $M > 0$ depending only upon R, γ, K such that $|x'(t)| \leq M$ for all $t \in [0,1]$. If we set

$$\Omega_2 := \{x \in C^1 : |x'(t)| < M+1,\ \forall\, t \in [0,1]\},$$

and $\Omega = \Omega_1 \cap \Omega_2$, all the assumptions of Proposition 2.8 are satisfied and the conclusion follows. □

In a similar way, we can prove the following existence result for the problem (57).

THEOREM 3.6. *Assume that conditions (h0),(h1) or conditions (h0),(h2),(h4) hold, and that there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$, and an outer normal vector field ν on ∂C, such that*

$$\langle \nu(u)|f(t,u,v)\rangle \geq 0, \quad \textit{whenever } u \in \partial C,\ \langle v|\nu(u)\rangle = 0$$

and

$$|f(t,u,v)| \leq \gamma |v|^2 + K,$$

for some $\gamma \geq 0$ *such that* $\gamma R < 1$, $K \geq 0$, *and* $R = \max_{u \in \overline{C}} |u|$. *Then the problem* (57) *has at least one solution such that* $x(t) \in \overline{C}$ *for all* $t \in [0,1]$.

The choice of $C = B_R$ provide the corresponding special cases.

Corollary 3.7. *Assume that conditions (h0), (h1) or conditions (h0), (h2), (h4) hold, and that there exists* $R > 0$ *such that*

$$\langle u | f(t,u,v) \rangle \geq 0, \quad \textit{whenever } u \in \partial B_R \textit{ and } \langle v | u \rangle = 0$$

and

$$|f(t,u,v)| \leq \gamma |v|^2 + K,$$

for some $\gamma \geq 0$ *such that* $\gamma R < 1$, *and* $K \geq 0$. *Then the problem* (55) *has at least one solution such that* $x(t) \in \overline{B}_R$ *for all* $t \in [0,1]$.

Special cases of Corollary 3.7 can be found in [100].

Corollary 3.8. *Assume that conditions (h0), (h1) or conditions (h0), (h2), (h4) hold, and that there exists* $R > 0$ *such that*

$$\langle u | f(t,u,v) \rangle \geq 0, \quad \textit{whenever } u \in \partial B_R \textit{ and } \langle v | u \rangle = 0$$

and

$$|f(t,u,v)| \leq \gamma |v|^2 + K,$$

for some $\gamma \geq 0$ *such that* $\gamma R < 1$, *and* $K \geq 0$. *Then the problem* (57) *has at least one solution such that* $x(t) \in \overline{B}_R$ *for all* $t \in [0,1]$.

The assumptions in Corollaries 3.7 and 3.8 can be slightly improved by taking in account the curvature of the ball, in contrast with the general case of convex sets which may have flat curvature almost everywhere (polyhedra). The corresponding conditions were first obtained by Hartman [31, 32] for Dirichlet boundary conditions, and extended by various authors to other classical two–point BC like mixed, Neumann or periodic, and to some four-point boundary conditions and component-wise Bernstein-Nagumo conditions by Calábek [10].

Theorem 3.9. *Assume that conditions (h0),(h1) or conditions (h0),(h2),(h4) hold, and that there exists* $R > 0$ *such that*

$$|v|^2 + \langle u | f(t,u,v) \rangle \geq 0, \quad \textit{whenever } u \in \partial B_R \textit{ and } \langle v | u \rangle = 0$$

and

$$|f(t,u,v)| \leq \gamma |v|^2 + K,$$

for some $\gamma \geq 0$ *such that* $\gamma R < 1$, *and* $K \geq 0$. *Then the problem* (55) *has at least one solution such that* $x(t) \in \overline{B}_R$ *for all* $t \in [0,1]$.

Proof. Following the lines of the proof of Theorem 3.5, we define first the open bounded neighborhood Ω_1 of 0 in C^0 by

$$\Omega_1 = \{x \in C^0 : |x(t)| < R, \ \forall\, t \in [0,1]\}.$$

As first step in applying Proposition 2.8, we show that for each $\lambda \in (0,1)$, no possible solution of the problem

$$x'' - x = \lambda[f(t,x,x') - x], \ x(0) = 0, \ x(1) = \int_0^1 x(s)\, dh(s), \tag{62}$$

belongs to $\partial\Omega_1$. Let $\lambda \in (0,1)$ and $x(t) \in \partial\Omega_1$ be a possible solution to (62). Then $|x(t)|^2 \leq R^2$ for all $t \in [0,1]$ and there is some $t_0 \in [0,1]$ such that $|x(t_0)|^2 = R^2$. Therefore the function $\xi(t) := |x(t)|^2/2$ reaches its maximum at t_0. Because of the first boundary condition, we cannot have $t_0 = 0$. If $t_0 \in (0,1)$, $0 = \xi'(t_0) = \langle x(t_0)|x'(t_0)\rangle$and

$$\begin{aligned} 0 \ &\geq \ \xi''_{t_0}(t_0) = |x'(t_0)|^2 + \langle x(t_0)|x''(t_0)\rangle \\ &\geq \ \lambda|x'(t_0)|^2 + (1-\lambda)|x(t_0)|^2 + \lambda\langle x(t_0)|f(t_0,x(t_0),x'(t_0)) > 0, \end{aligned}$$

a contradiction. Finally, if $t_0 = 1$, we use the second boundary condition and its consequence

$$|x(1)| \leq \int_0^1 |x(s)|\, dh(s),$$

and the nonlocal terminal type problem for first order systems to obtain the contradiction. The remaining part of the proof is exactly similar to that of Theorem 3.5. □

In a similar way, we prove the corresponding result for the mixed case.

THEOREM 3.10. *Assume that conditions (h0),(h1) or conditions (h0),(h2),(h4) hold, and that there exists* $R > 0$ *such that*

$$|v|^2 + \langle u|f(t,u,v)\rangle \geq 0, \quad \textit{whenever } u \in \partial B_R \textit{ and } \langle v|u\rangle = 0$$

and

$$|f(t,u,v)| \leq \gamma |v|^2 + K,$$

for some $\gamma \geq 0$ *such that* $\gamma R < 1$, *and* $K \geq 0$. *Then the problem* (57) *has at least one solution such that* $x(t) \in \overline{B}_R$ *for all* $t \in [0,1]$.

3.6. Other nonlocal nonlinear BVP of mixed type

Let us consider the nonlocal BVP of initial type

$$x'' = f(t,x,x'), \ \ x(0) = 0, \ x'(0) = \int_0^1 x'(s)\,dh(s). \tag{63}$$

The following existence theorem is given in [75]. This time the vector field condition similar to the one for first order systems is made on $f(t,u,v)$ with respect to v.

THEOREM 3.11. *Assume that h satisfies conditions (h0), (h1), or conditions (h0), (h2), (h3) and that there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν to ∂C such that*

$$\langle \nu(v) | f(t,u,v)\rangle \le 0, \ \forall\,(t,u,v) \in [0,1] \times \overline{C} \times \partial C. \tag{64}$$

Then the problem (63) has at least one solution x such that $x(t) \in \overline{C}$ and $x'(t) \in \overline{C}$ for all $t \in [0,1]$.

Proof. We set $y = x'$, so that, using the first boundary condition $x(0) = 0$,

$$x(t) = \int_0^t x'(s)\,ds = \int_0^t y(s)\,ds,$$

and the problem (63) can be written, in terms of y,

$$y'(t) = f\left(t, \int_0^t y(s)\,ds, y(t)\right), \ \ y(0) = \int_0^1 y(s)\,dh(s). \tag{65}$$

The result follows then from Theorem 2.29 and the fact that, by the convexity of C, $\int_0^t y(s)\,ds \in \overline{C}$ for all $t \in [0,1]$. □

A similar result, with a similar proof using Theorem 2.30, holds for the following nonlinear BVP of mixed type.

THEOREM 3.12. *Assume that h satisfies conditions (h0), (h1), or conditions (h0), (h2), (h4) and that there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν to ∂C such that*

$$\langle \nu(v) | f(t,u,v)\rangle \ge 0, \ \forall\,(t,u,v) \in [0,1] \times \overline{C} \times \partial C. \tag{66}$$

Then the problem

$$x'' = f(t,x,x'), \ \ x(0) = 0, \ x'(1) = \int_0^1 x'(s)\,dh(s)$$

has at least one solution x such that $x(t) \in \overline{C}$ and $x'(t) \in \overline{C}$ for all $t \in [0,1]$.

Finally, using Theorems 2.31 and 2.32 and the fact that $x(t) = \int_1^t x'(s)\,ds = \int_1^t y(s)\,ds$, we obtain in a similar way the following results.

THEOREM 3.13. *Assume that h satisfies conditions (h0), (h1), or conditions (h0), (h2), (h3) and that there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν to ∂C such that condition* (64) *holds. Then the problem*

$$x'' = f(t, x, x'), \;\; x'(0) = \int_0^1 x'(s)\,dh(s), \; x(1) = 0$$

has at least one solution x such that $x(t) \in \overline{C}$ and $x'(t) \in \overline{C}$ for all $t \in [0,1]$.

THEOREM 3.14. *Assume that h satisfies conditions (h0), (h1), or conditions (h0), (h2), (h4) and that there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν to ∂C such that the condition* (66) *holds. Then the problem*

$$x'' = f(t, x, x'), \;\; x(1) = 0, \; x'(1) = \int_0^1 x'(s)\,dh(s)$$

has at least one solution x such that $x(t) \in \overline{C}$ and $x'(t) \in \overline{C}$ for all $t \in [0,1]$.

Special cases of those results when $C = B_R$ as well as results for other similar nonlocal boundary conditions can be found in [59, 101, 103, 104].

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Received October 21, 2019
Revised November 12, 2019
Accepted November 13, 2019