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Change of variables' formula for the integration of the measurable real functions over infinite-dimensional Banach spaces

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Abstract. *In this paper we study, for any subset I of $\mathbf{N}^*$ and for any strictly positive integer k, the Banach space E_I of the bounded real sequences $\{x_n\}_{n\in I}$, and a measure over $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$ that generalizes the k-dimensional Lebesgue one. Moreover, we recall the main results about the differentiation theory over E_I. The main result of our paper is a change of variables' formula for the integration of the measurable real functions on $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$. This change of variables is defined by some functions over an open subset of E_J, with values on E_I, called (m,σ)-general, with properties that generalize the analogous ones of the finite-dimensional diffeomorphisms.*

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1. Introduction

In the mathematical literature, some articles introduced infinite-dimensional measures analogue of the Lebesgue one: see for example the pioneering paper of Jessen [10], that one of Léandre [13], in the context of the noncommutative geometry, that one of Tsilevich et al. [19], which studies a family of σ-finite measures in the space of distributions, that one of Baker [7], which defines a measure on $\mathbf{R}^{\mathbf{N}^*}$ that is not σ-finite, that one of Henstock et al. [9], and that one of Tepper et al. [15]. However, the results obtained do not include an infinite-dimensional change of variables' formula for the integration of the measurable real functions, analogous to that which applies in the finite-dimensional case. For example, in the paper of Accardi et al. [1], the authors describe the transformations of generalized measures on locally convex spaces under smooth transformations of these spaces, but these measures have no connection with the Lebesgue one. The problem that arises is essentially the following. Consider the integration formula with respect to an image measure, that is

$$\int_E f d(\varphi(\mu)) = \int_S f(\varphi) d\mu,$$

where (S, Σ, μ) and $(E, \mathcal{E})$ are a measure space and a measurable space, respectively, $\varphi : (S, \Sigma) \longrightarrow (E, \mathcal{E})$ and $f : (E, \mathcal{E}) \longrightarrow (\mathbf{R}, \mathcal{B})$ are measurable functions. In the particular case in which E and S are open sets, suitably constructed, of two infinite-dimensional measurable spaces Ω_1 and Ω_2, respectively, on which we can define two families $\mathcal{M}_1$ and $\mathcal{M}_2$ of measures analogue of the Lebesgue one, and φ has properties that generalize the analogous ones of the standard finite-dimensional diffeomorphisms, we expect existence of two measure λ_1 in $\mathcal{M}_1$ and λ_2 in $\mathcal{M}_2$ such that $\varphi(\mu) = \lambda_1$, while μ has density $|\det J_\varphi|$ (properly defined) with respect to λ_2.

In order to achieve this result, in the articles [4], [5] and [6], for any subset I of $\mathbf{N}^*$, we define the Banach space $E_I \subset \mathbf{R}^I$ of the bounded real sequences $\{x_n\}_{n \in I}$, the σ-algebra $\mathcal{B}_I$ given by the restriction to E_I of $\mathcal{B}^{(I)}$ (defined as the product indexed by I of the same Borel σ-algebra $\mathcal{B}$ on $\mathbf{R}$), and a class of functions over an open subset of E_J, with values on E_I, called (m, σ)-general, with properties similar to those of the finite-dimensional diffeomorphisms. Moreover, for any strictly positive integer k, we introduce over the measurable space $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$ a family of infinite-dimensional measures $\lambda_{N,a,v}^{(k,I)}$, dependent on appropriate parameters N, a, v, that in the case $I = \{1, ..., k\}$ coincide with the k-dimensional Lebesgue measure on $\mathbf{R}^k$. More precisely, in the paper [4], we define some particular linear functions over E_J, with values on E_I, called (m, σ)-standard, while in the article [5] we present some results about the differentiation theory over E_I, and we remove the assumption of linearity for the (m, σ)-standard functions. In the last two papers, we provide a change of variables' formula for the integration of the measurable real functions on $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$; this change of variables is defined by some particular (m, σ)-standard functions. In the paper [6], we introduce a class of functions, called (m, σ)-general, that generalizes the set of the (m, σ)-standard functions given in [5]. The main result is the definition of the determinant of a linear (m, σ)-general function, as the limit of a sequence of the determinants of some standard matrices.

In this paper, we prove that the change of variables' formula given by the standard finite-dimensional theory and in the papers [4] and [5] can be extended by using the (m, σ)-general functions. In Section 2, we recall the construction of the infinite-dimensional Banach space E_I, with its σ-algebra $\mathcal{B}_I$ and its topologies τ_I and $\tau_{\|\cdot\|_I}$; moreover, we expose the main results about the differentiation theory over this space. In Section 3, we recall some properties of the (m, σ)-general functions defined in [6], and we expose some additional results about these functions. In Section 4, we present the main theorem of our paper, that is a change of variables' formula for the integration of the measurable real functions on $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$; this change of variables is defined by the bijective, C^1 and (m,σ)-general functions, with further properties (Theorem 4.5). In Section 5, we expose some ideas for further study in the probability theory.

2. Differentiation theory over infinite-dimensional Banach spaces

Let $I \neq \emptyset$ be a set and let $k \in \mathbf{N}^*$; indicate by τ, by $\tau^{(k)}$, by $\tau^{(I)}$, by $\mathcal{B}$, by $\mathcal{B}^{(k)}$, by $\mathcal{B}^{(I)}$, by Leb, and by $Leb^{(k)}$, respectively, the euclidean topology on $\mathbf{R}$, the euclidean topology on $\mathbf{R}^k$, the topology $\bigotimes_{i\in I}\tau$, the Borel σ-algebra on $\mathbf{R}$, the Borel σ-algebra on $\mathbf{R}^k$, the σ-algebra $\bigotimes_{i\in I}\mathcal{B}$, the Lebesgue measure on $\mathbf{R}$, and the Lebesgue measure on $\mathbf{R}^k$. Moreover, for any set $A \subset \mathbf{R}$, indicate by $\mathcal{B}(A)$ the σ-algebra induced by $\mathcal{B}$ on A, and by $\tau(A)$ the topology induced by τ on A; analogously, for any set $A \subset \mathbf{R}^I$, define the σ-algebra $\mathcal{B}^{(I)}(A)$ and the topology $\tau^{(I)}(A)$. Finally, if $S = \prod_{i\in I} S_i$ is a Cartesian product, for any $(x_i : i \in I) \in S$ and for any $\emptyset \neq H \subset I$, define $x_H = (x_i : i \in H) \in \prod_{i\in H} S_i$, and define the projection $\pi_{I,H}$ on $\prod_{i\in H} S_i$ as the function $\pi_{I,H} : S \longrightarrow \prod_{i\in H} S_i$ given by $\pi_{I,H}(x_I) = x_H$.

THEOREM 2.1. *Let $I \neq \emptyset$ be a set and, for any $i \in I$, let (S_i, Σ_i, μ_i) be a measure space such that μ_i is finite. Moreover, suppose that, for some countable set $J \subset I$, μ_i is a probability measure for any $i \in I \backslash J$ and $\prod_{j\in J}\mu_j(S_j) \in \mathbf{R}^+$. Then, over the measurable space $\left(\prod_{i\in I} S_i, \bigotimes_{i\in I}\Sigma_i\right)$, there is a unique finite measure μ, indicated by $\bigotimes_{i\in I}\mu_i$, such that, for any $H \subset I$ such that $|H| < +\infty$ and for any $A = \prod_{h\in H} A_h \times \prod_{i\in I\backslash H} S_i \in \bigotimes_{i\in I}\Sigma_i$, where $A_h \in \Sigma_h$ $\forall\, h \in H$, we have $\mu(A) = \prod_{h\in H}\mu_h(A_h) \prod_{j\in J\backslash H}\mu_j(S_j)$. In particular, if I is countable, then $\mu(A) = \prod_{i\in I}\mu_i(A_i)$ for any $A = \prod_{i\in I} A_i \in \bigotimes_{i\in I}\Sigma_i$.*

Proof. See the proof of Corollary 4 in Asci [4]. □

Henceforth, we will suppose that I, J are sets such that $\emptyset \neq I, J \subset \mathbf{N}^*$; moreover, for any $k \in \mathbf{N}^*$, we will indicate by I_k the set of the first k elements of I (with the natural order and with the convention $I_k = I$ if $|I| < k$); furthermore, for any $i \in I$, set $|i|_I = |I \cap (0, i]|$. Analogously, define J_k and $|j|_J$, for any $k \in \mathbf{N}^*$ and for any $j \in J$.

The following theorem generalizes a result proved in Rao [14] (Theorem 3, page 349), and can be considered a generalization of the Tonelli's theorem, in the integration of a function over an infinite-dimensional measure space. The integral of the function is the limit of a sequence of integrals of the same function, with respect to a finite subset of variables.

THEOREM 2.2. *Let (S_i, Σ_i, μ_i) be a measure space such that μ_i is finite, for any $i \in I$, and $\prod_{i \in I} \mu_i(S_i) \in [0, +\infty)$; moreover, let $(S, \Sigma, \mu) = \left(\prod_{i \in I} S_i, \bigotimes_{i \in I} \Sigma_i, \bigotimes_{i \in I} \mu_i\right)$, let $f \in L^1(S, \Sigma, \mu)$ and, for any $H \subset I$ such that $0 < |H| < +\infty$, let the measurable function $f_{H^c} : (S, \Sigma) \longrightarrow (\mathbf{R}, \mathcal{B})$ defined by*

$$f_{H^c}(x) = \int_{S_H} f(\cdot, x_{H^c}) d\mu_H,$$

where (S_H, Σ_H, μ_H) is the measure space $\left(\prod_{i \in H} S_i, \bigotimes_{i \in H} \Sigma_i, \bigotimes_{i \in H} \mu_i\right)$. Then, there exists $D \in \Sigma$ such that $\mu(D) = 0$ and such that, for any $x \in D^c$, one has $\lim_{n \to +\infty} f_{I_n^c}(x) = \int_S f d\mu$.

Proof. See the proof of Corollary 3 in Asci [5]. □

DEFINITION 2.3. *For any set $I \neq \emptyset$, define the function $\|\cdot\|_I : \mathbf{R}^I \longrightarrow [0, +\infty]$ by*

$$\|x\|_I = \sup_{i \in I} |x_i|, \ \forall\, x = (x_i : i \in I) \in \mathbf{R}^I,$$

and define the vector space

$$E_I = \{x \in \mathbf{R}^I : \|x\|_I < +\infty\}.$$

Moreover, indicate by $\mathcal{B}_I$ the σ-algebra $\mathcal{B}^{(I)}(E_I)$, by τ_I the topology $\tau^{(I)}(E_I)$, and by $\tau_{\|\cdot\|_I}$ the topology induced on E_I by the distance $d_I : E_I \times E_I \longrightarrow [0, +\infty)$ defined by $d_I(x, y) = \|x - y\|_I$, for any $x, y \in E_I$; furthermore, for any set

$A \subset E_I$, indicate by $\tau_{\|\cdot\|_I}(A)$ the topology induced on A by $\tau_{\|\cdot\|_I}$. Finally, for any $x_0 \in E_I$ and for any $\delta \in \mathbf{R}^+$, indicate by $B_I(x_0, \delta)$ the set $\{x \in E_I : \|x - x_0\|_I < \delta\}$.

PROPOSITION 2.4. *Let H, I be sets such that $\emptyset \neq H \subsetneq I$, and let $A \subset E_H$, $B \subset E_{I \setminus H}$; then:*

1. *E_I is a Banach space, with the norm $\|\cdot\|_I$.*

2. *$\tau_{\|\cdot\|_I}(A \times B)$ is the product of the topologies $\tau_{\|\cdot\|_H}(A)$ and $\tau_{\|\cdot\|_{I\setminus H}}(B)$.*

3. *Let $A = \left(\prod_{i \in I} A_i\right) \cap E_I \neq \emptyset$, where $A_i \in \tau$, for any $i \in I$; then, one has $A \in \tau_{\|\cdot\|_I}$ if and only if there exists $h \in I$ such that $A_i = \mathbf{R}$, for any $i \in I \setminus I_h$.*

4. *One has $\tau_I \subset \tau_{\|\cdot\|_I}$; moreover, if $|I| = +\infty$, then $\tau_I \subsetneq \tau_{\|\cdot\|_I}$.*

Proof. 1. See, for example, the proof of Remark 2 in [4].

2. Indicate by $\tau_{\|\cdot\|_H}(A) \bigotimes \tau_{\|\cdot\|_{I\setminus H}}(B)$ the product of the topologies $\tau_{\|\cdot\|_H}(A)$ and $\tau_{\|\cdot\|_{I\setminus H}}(B)$; $\forall\, D \in \tau_{\|\cdot\|_H}(A)$, let $D' \in \tau_{\|\cdot\|_H}$ such that $D = D' \cap A$; then, $\forall\, x = \left(x_H, x_{I\setminus H}\right) \in D' \times E_{I\setminus H}$, there exists $\delta \in \mathbf{R}^+$ such that $x_H \in B_H\left(x_H, \delta\right) \subset D'$, $x_{I\setminus H} \in B_{I\setminus H}\left(x_{I\setminus H}, \delta\right) \subset E_{I\setminus H}$, and so $x \in B_I\left(x, \delta\right) \subset D' \times E_{I\setminus H}$; then, we have $D' \times E_{I\setminus H} \in \tau_{\|\cdot\|_I}$, from which $D \times B = \left(D' \times E_{I\setminus H}\right) \cap (A \times B) \in \tau_{\|\cdot\|_I}(A \times B)$; analogously, $\forall\, E \in \tau_{\|\cdot\|_{I\setminus H}}(B)$, we have $A \times E \in \tau_{\|\cdot\|_I}(A \times B)$, and so $D \times E = (D \times B) \cap (A \times E) \in \tau_{\|\cdot\|_I}(A \times B)$; then, we obtain $\tau_{\|\cdot\|_H}(A) \bigotimes \tau_{\|\cdot\|_{I\setminus H}}(B) \subset \tau_{\|\cdot\|_I}(A \times B)$.

   Conversely, $\forall\, x = \left(x_H, x_{I\setminus H}\right) \in E_I$, $\forall\, \delta \in \mathbf{R}^+$, we have $B_I(x, \delta) \cap (A \times B) = (B_H(x_H, \delta) \cap A) \times \left(B_{I\setminus H}(x_{I\setminus H}, \delta) \cap B\right) \in \tau_{\|\cdot\|_H}(A) \bigotimes \tau_{\|\cdot\|_{I\setminus H}}(B)$, from which $\tau_{\|\cdot\|_I}(A \times B) \subset \tau_{\|\cdot\|_H}(A) \bigotimes \tau_{\|\cdot\|_{I\setminus H}}(B)$.

3. We can suppose $|I| = +\infty$. If there exists $h \in I$ such that $A_i = \mathbf{R}$, for any $i \in I \setminus I_h$, then $A = \left(\prod_{i \in I_h} A_i\right) \times E_{I\setminus I_h}$; thus, since $\prod_{i \in I_h} A_i \in \tau_{\|\cdot\|_{I_h}}$, $E_{I\setminus I_h} \in \tau_{\|\cdot\|_{I\setminus I_h}}$, from point 2 we have $A \in \tau_{\|\cdot\|_I}$.

   Conversely, suppose that there exists $J \subset I$ such that $|J| = +\infty$ and such that $A_j \neq \mathbf{R}$, $\forall\, j \in J$; then, since $A \neq \emptyset$, there exists $x \in A$ such that $d_I\left(x, E_I \setminus A\right) = 0$, and so $A \notin \tau_{\|\cdot\|_I}$.

4. Let

$$\mathcal{E} = \left\{ A = \left(\prod_{i \in I} A_i \right) \cap E_I : A_i \in \tau, \, \forall \, i \in I, \right.$$
$$\left. A_i = \mathbf{R}, \, \forall \, i \in I \backslash I_h, \text{ for some } h \in I \right\};$$

as we observed in the proof of point 3, we have $\mathcal{E} \subset \tau_{\|\cdot\|_I}$; moreover, by definition of τ_I, we have $\tau_I = \tau(\mathcal{E}) \subset \tau_{\|\cdot\|_I}$; furthermore, if $|I| = +\infty$, $\forall \, x \in E_I$, $\forall \, \delta \in \mathbf{R}^+$, we have $B_I(x, \delta) \in \tau_{\|\cdot\|_I}$, $B_I(x, \delta) \notin \tau_I$, and so $\tau_I \subsetneqq \tau_{\|\cdot\|_I}$.

□

PROPOSITION 2.5. *Let H, I be sets such that $\emptyset \neq H \subset I$, and let $\overline{\pi}_{I,H} : E_I \longrightarrow E_H$ be the function given by $\overline{\pi}_{I,H}(x) = \pi_{I,H}(x)$, for any $x \in E_I$; then:*

1. *$\overline{\pi}_{I,H} : \left(E_I, \tau_{\|\cdot\|_I}\right) \longrightarrow \left(E_H, \tau_{\|\cdot\|_H}\right)$ is continuous and open.*
2. *$\overline{\pi}_{I,H} : (E_I, \tau_I) \longrightarrow (E_H, \tau_H)$ is continuous and open.*
3. *$\overline{\pi}_{I,H} : (E_I, \mathcal{B}_I) \longrightarrow (E_H, \mathcal{B}_H)$ is measurable.*

Proof. Points 1 and 2 are proved, for example, in Proposition 6 in [5]; moreover, the proof of point 3 is analogous to the proof of the continuity of the function $\overline{\pi}_{I,H} : (E_I, \tau_I) \longrightarrow (E_H, \tau_H)$. □

REMARK 2.6: Let H, I, J be sets such that $\emptyset \neq H \subsetneqq J$, let $U = U_1 \times U_2 \in \tau_{\|\cdot\|_J}$, where $U_1 \in \tau_{\|\cdot\|_H}$, $U_2 \in \tau_{\|\cdot\|_{J \backslash H}}$, let $\psi : U_1 \subset E_H \longrightarrow E_I$ be a function and let $\varphi : U \subset E_J \longrightarrow E_I$ be the function given by $\varphi(x) = \psi(x_H)$, for any $x = \left(x_H, x_{J \backslash H}\right) \in U$; then:

1. ψ is $\left(\tau_{\|\cdot\|_H}(U_1), \tau_{\|\cdot\|_I}\right)$-continuous if and only if φ is $\left(\tau_{\|\cdot\|_J}(U), \tau_{\|\cdot\|_I}\right)$-continuous.
2. ψ is $\left(\tau^{(H)}(U_1), \tau_I\right)$-continuous if and only if φ is $\left(\tau^{(J)}(U), \tau_I\right)$-continuous.
3. If ψ is $\left(\mathcal{B}^{(H)}(U_1), \mathcal{B}_I\right)$-measurable, then φ is $\left(\mathcal{B}^{(J)}(U), \mathcal{B}_I\right)$-measurable.

Proof. $\forall \, A \subset E_I$, we have

$$\varphi^{-1}(A) = \left(\overline{\pi}_{J,H}^{-1} \circ \psi^{-1}\right)(A), \psi^{-1}(A) = \left(\overline{\pi}_{J,H} \circ \varphi^{-1}\right)(A);$$

then, from Proposition 2.5, we obtain the statement. □

DEFINITION 2.7. *Let $U \in \tau_{\|\cdot\|_J}$, let $x_0 \in U$, let $l \in E_I$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function; we say that $\lim\limits_{x \to x_0} \varphi(x) = l$ if, for any $\varepsilon \in \mathbf{R}^+$, there exists a neighbourhood $N \in \tau_{\|\cdot\|_J}(U)$ of x_0 such that, for any $x \in N \backslash \{x_0\}$, one has $\|\varphi(x) - l\|_I < \varepsilon$.*

DEFINITION 2.8. *Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function; we say that φ is continuous in $x_0 \in U$ if $\lim\limits_{x \to x_0} \varphi(x) = \varphi(x_0)$, and we say that φ is continuous in U if, for any $x \in U$, φ is continuous in x.*

REMARK 2.9: Let $U \in \tau_{\|\cdot\|_J}$, let $V \in \tau_{\|\cdot\|_I}$ and let $\varphi : U \subset E_J \longrightarrow V \subset E_I$ be a function; then, $\varphi : \big(U, \tau_{\|\cdot\|_J}(U)\big) \longrightarrow \big(V, \tau_{\|\cdot\|_I}(V)\big)$ is continuous if and only if φ is continuous in U.

DEFINITION 2.10. *Let $U \in \tau_{\|\cdot\|_J}$, let $V \in \tau_{\|\cdot\|_I}$ and let $\varphi : U \subset E_J \longrightarrow V \subset E_I$ be a function; we say that φ is a homeomorphism if φ is bijective and the functions $\varphi : \big(U, \tau_{\|\cdot\|_J}(U)\big) \longrightarrow \big(V, \tau_{\|\cdot\|_I}(V)\big)$ and $\varphi^{-1} : \big(V, \tau_{\|\cdot\|_I}(V)\big) \longrightarrow \big(U, \tau_{\|\cdot\|_J}(U)\big)$ are continuous.*

DEFINITION 2.11. *Let $U \in \tau_{\|\cdot\|_J}$, let $A \subset U$, let $\varphi : U \subset E_J \longrightarrow E_I$ be a functions and let $\{\varphi_n\}_{n \in \mathbf{N}}$ be a sequence of functions such that $\varphi_n : U \longrightarrow E_I$, for any $n \in \mathbf{N}$; we say that:*

1. *The sequence $\{\varphi_n\}_{n \in \mathbf{N}}$ converges to φ over A if, for any $\varepsilon \in \mathbf{R}^+$ and for any $x \in A$, there exists $n_0 \in \mathbf{N}$ such that, for any $n \in \mathbf{N}$, $n \geq n_0$, one has $\|\varphi_n(x) - \varphi(x)\|_I < \varepsilon$.*
2. *The sequence $\{\varphi_n\}_{n \in \mathbf{N}}$ converges uniformly to φ over A if, for any $\varepsilon \in \mathbf{R}^+$, there exists $n_0 \in \mathbf{N}$ such that, for any $n \in \mathbf{N}$, $n \geq n_0$, and for any $x \in A$, one has $\|\varphi_n(x) - \varphi(x)\|_I < \varepsilon$.*

The following concept generalizes Definition 6 in [4] (see also the theory in the Lang's book [12] and that in the Weidmann's book [20]).

DEFINITION 2.12. *Let $A = (a_{ij})_{i \in I, j \in J}$ be a real matrix $I \times J$ (eventually infinite); then, define the linear function $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow \mathbf{R}^I$, and write $x \longrightarrow Ax$, in the following manner:*

$$(Ax)_i = \sum_{j \in J} a_{ij} x_j, \ \forall\, x \in E_J, \ \forall\, i \in I, \tag{1}$$

on condition that, for any $i \in I$, the sum in (1) converges to a real number. In *particular, if $|I| = |J|$, indicate by $\mathbf{I}_{I,J} = \left(\overline{\delta}_{ij}\right)_{i\in I, j\in J}$ the real matrix defined by*

$$\overline{\delta}_{ij} = \begin{cases} 1 & \text{if } |i|_I = |j|_J \\ 0 & \text{otherwise} \end{cases},$$

and call $\overline{\delta}_{ij}$ generalized Kronecker symbol. Moreover, indicate by $A^{(L,N)}$ the real matrix $(a_{ij})_{i\in L, j\in N}$, for any $\emptyset \neq L \subset I$, for any $\emptyset \neq N \subset J$, and indicate by ${}^tA = (b_{ji})_{j\in J, i\in I} : E_I \longrightarrow \mathbf{R}^J$ the linear function defined by $b_{ji} = a_{ij}$, for any $j \in J$ and for any $i \in I$. Furthermore, if $I = J$ and $A = {}^tA$, we say that A is a symmetric function. Finally, if $B = (b_{jk})_{j\in J, k\in K}$ is a real matrix $J \times K$, define the $I \times K$ real matrix $AB = ((AB)_{ik})_{i\in I, k\in K}$ by

$$(AB)_{ik} = \sum_{j\in J} a_{ij} b_{jk}, \tag{2}$$

on condition that, for any $i \in I$ and for any $k \in K$, the sum in (2) converges to a real number.

Proposition 2.13. *Let $A = (a_{ij})_{i\in I, j\in J}$ be a real matrix $I \times J$; then:*

1. *The linear function $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow \mathbf{R}^I$ given by (1) is defined if and only if, for any $i \in I$, $\sum_{j\in J} |a_{ij}| < +\infty$.*
2. *One has $\sup_{i\in I} \sum_{j\in J} |a_{ij}| < +\infty$ if and only if $A(E_J) \subset E_I$ and A is continuous; moreover, if $A(E_J) \subset E_I$, then $\|A\| = \sup_{i\in I} \sum_{j\in J} |a_{ij}|$.*
3. *If $B = (b_{jk})_{j\in J, k\in K} : E_K \longrightarrow E_J$ is a linear function, then the linear function $A \circ B : E_K \longrightarrow \mathbf{R}^I$ is defined by the real matrix AB.*
4. *If $A(E_J) \subset E_I$, then, for any $\emptyset \neq L \subset I$, for any $\emptyset \neq N \subset J$, one has $A^{(L,N)}(E_N) \subset E_L$.*

Proof. The proofs of points 1 and 2 are analogous to the proof of Proposition 7 in [4]. Moreover, the proof of point 3 is analogous to that one true in the particular case $|I|, |J|, |K| < +\infty$ (see, e.g., the Lang's book [12]). Finally, suppose that $A(E_J) \subset E_I$; let $\emptyset \neq L \subset I$, let $\emptyset \neq N \subset J$, let $x = (x_n : n \in N) \in E_N$ and let $y = (y_j : j \in J) \in E_J$ such that $y_j = x_j$, $\forall\, j \in N$, and $y_j = 0$,

$\forall\, j \in J \backslash N$; we have

$$\begin{aligned}\sup_{i\in L}\left|\left(A^{(L,N)}x\right)_i\right| &= \sup_{i\in L}\left|\sum_{j\in N} a_{ij}(x_j)\right| = \sup_{i\in L}\left|\sum_{j\in J} a_{ij}(y_j)\right| \\ &\le \sup_{i\in I}\left|\sum_{j\in J} a_{ij}(y_j)\right| = \sup_{i\in I}|(Ay)_i| < +\infty \quad \Rightarrow \quad A^{(L,N)}x \in E_L;\end{aligned}$$

then, point 4 follows. □

The following definitions (from Definition 2.14 to Definition 2.18) can be found in [5] and generalize the differentiation theory in the finite case (see, e.g., the Lang's book [11]).

DEFINITION 2.14. *Let $U \in \tau_{\|\cdot\|_J}$; a function $\varphi : U \subset E_J \longrightarrow E_I$ is called differentiable in $x_0 \in U$ if there exists a linear and continuous function $A : E_J \longrightarrow E_I$ defined by a real matrix $A = (a_{ij})_{i\in I, j\in J}$, and one has*

$$\lim_{h\to 0} \frac{\|\varphi(x_0+h) - \varphi(x_0) - Ah\|_I}{\|h\|_J} = 0. \tag{3}$$

If φ is differentiable in x_0 for any $x_0 \in U$, φ is called differentiable in U. The function A is called differential of the function φ in x_0, and it is indicated by the symbol $d\varphi(x_0)$.

DEFINITION 2.15. *Let $U \in \tau_{\|\cdot\|_J}$, let $v \in E_J$ such that $\|v\|_J = 1$ and let a function $\varphi : U \subset E_J \longrightarrow \mathbf{R}^I$; for any $i \in I$, the function φ_i is called differentiable in $x_0 \in U$ in the direction v if there exists the limit*

$$\lim_{t\to 0} \frac{\varphi_i(x_0+tv) - \varphi_i(x_0)}{t}.$$

This limit is indicated by $\frac{\partial\varphi_i}{\partial v}(x_0)$, and it is called derivative of φ_i in x_0 in the direction v. If, for some $j \in J$, one has $v = e_j$, where $(e_j)_k = \delta_{jk}$, for any $k \in J$, indicate $\frac{\partial\varphi_i}{\partial v}(x_0)$ by $\frac{\partial\varphi_i}{\partial x_j}(x_0)$, and call it partial derivative of φ_i in x_0, with respect to x_j. Moreover, if there exists the linear function defined by the matrix $J_\varphi(x_0) = \left((J_\varphi(x_0))_{ij}\right)_{i\in I, j\in J} : E_J \longrightarrow \mathbf{R}^I$, where $(J_\varphi(x_0))_{ij} = \frac{\partial\varphi_i}{\partial x_j}(x_0)$, for any $i \in I$, $j \in J$, then $J_\varphi(x_0)$ is called Jacobian matrix of the function φ in x_0. Finally, if, for any $x \in U$, there exists $J_\varphi(x)$, then the function $x \longrightarrow J_\varphi(x)$ is indicated by J_φ.

Definition 2.16. *Let $U \in \tau_{\|\cdot\|_J}$, let $i, j \in J$ and let $\varphi : U \subset E_J \longrightarrow \mathbf{R}$ be a function differentiable in $x_0 \in U$ with respect to x_i, such that the function $\frac{\partial \varphi}{\partial x_i}$ is differentiable in x_0 with respect to x_j. Indicate $\frac{\partial}{\partial x_j}\left(\frac{\partial \varphi}{\partial x_i}\right)(x_0)$ by $\frac{\partial^2 \varphi}{\partial x_j \partial x_i}(x_0)$ and call it second partial derivative of φ in x_0 with respect to x_i and x_j. If $i = j$, it is indicated by $\frac{\partial^2 \varphi}{\partial x_i^2}(x_0)$. Analogously, for any $k \in \mathbf{N}^*$ and for any $j_1, ..., j_k \in J$, define $\frac{\partial^k \varphi}{\partial x_{j_k} ... \partial x_{j_1}}(x_0)$ and call it k-th partial derivative of φ in x_0 with respect to $x_{j_1}, ...x_{j_k}$.*

Definition 2.17. *Let $U \in \tau_{\|\cdot\|_J}$ and let $k \in \mathbf{N}^*$; a function $\varphi : U \subset E_J \longrightarrow \mathbf{R}^I$ is called C^k in $x_0 \in U$ if, in a neighbourhood $V \in \tau_{\|\cdot\|_J}(U)$ of x_0, for any $i \in I$ and for any $j_1, ..., j_k \in J$, there exists the function defined by $x \longrightarrow \frac{\partial^k \varphi_i}{\partial x_{j_k} ... \partial x_{j_1}}(x)$, and this function is continuous in x_0; φ is called C^k in U if, for any $x_0 \in U$, φ is C^k in x_0.*

Definition 2.18. *Let $U \in \tau_{\|\cdot\|_J}$ and let $V \in \tau_{\|\cdot\|_I}$; a function $\varphi : U \subset E_J \longrightarrow V \subset E_I$ is called diffeomorphism if φ is bijective and C^1 in U, and the function $\varphi^{-1} : V \subset E_I \longrightarrow U \subset E_J$ is C^1 in V.*

3. Theory of the (m, σ)-general functions

The following definition introduces a class of functions, called m-general, that generalize the linear functions $(a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ (see the next Remark 3.15). For example, the equation corresponding to a 1-general function is obtained by formula 1, by substituting the functions $x_j \longrightarrow a_{ij} x_j$ for some functions φ_{ij}.

Definition 3.1. *Let $m \in \mathbf{N}^*$ and let $\emptyset \neq U = \left(U^{(m)} \times \prod_{j \in J \setminus J_m} A_j\right) \cap E_J \in \tau_{\|\cdot\|_J}$, where $U^{(m)} \in \tau^{(m)}$, $A_j \in \tau$, for any $j \in J \setminus J_m$. A function $\varphi : U \subset E_J \longrightarrow E_I$ is called m-general if, for any $i \in I$ and for any $j \in J \setminus J_m$, there exist some functions $\varphi_i^{(I,m)} : U^{(m)} \longrightarrow \mathbf{R}$ and $\varphi_{ij} : A_j \longrightarrow \mathbf{R}$ such that*

$$\varphi_i(x) = \varphi_i^{(I,m)}(x_{J_m}) + \sum_{j \in J \setminus J_m} \varphi_{ij}(x_j), \; \forall\, x \in U.$$

Moreover, for any $\emptyset \neq L \subset I$ and for any $J_m \subset N \subset J$, indicate by $\varphi^{(L,N)}$ the function $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ defined by

$$\varphi_i^{(L,N)}(x_N) = \varphi_i^{(I,m)}(x_{J_m}) + \sum_{j \in N \setminus J_m} \varphi_{ij}(x_j), \; \forall\, x_N \in \pi_{J,N}(U), \; \forall\, i \in L. \quad (4)$$

Furthermore, for any $\emptyset \neq L \subset I$ *and for any* $\emptyset \neq N \subset J \backslash J_m$*, indicate by* $\varphi^{(L,N)}$ *the function* $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ *given by*

$$\varphi_i^{(L,N)}(x_N) = \sum_{j \in N} \varphi_{ij}(x_j), \, \forall \, x_N \in \pi_{J,N}(U), \, \forall \, i \in L. \tag{5}$$

In particular, suppose that $m = 1$*; then, let* $j \in J$ *such that* $\{j\} = J_1$ *and indicate* $U^{(1)}$ *by* A_j *and* $\varphi_i^{(I,1)}$ *by* φ_{ij}*, for any* $i \in I$*; moreover, for any* $\emptyset \neq L \subset I$ *and for any* $\emptyset \neq N \subset J$*, indicate by* $\varphi^{(L,N)}$ *the function* $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ *defined by formula (5).*

Furthermore, for any $l, n \in \mathbf{N}^*$*, indicate* $\varphi^{(I_l,N)}$ *by* $\varphi^{(l,N)}$*,* $\varphi^{(L,J_n)}$ *by* $\varphi^{(L,n)}$*, and* $\varphi^{(I_l,J_n)}$ *by* $\varphi^{(l,n)}$*.*

The following definition introduces a class of m-general functions $\varphi : U \subset E_J \longrightarrow E_I$, called (m,σ)-general, that will be used to provide a change of variables' formula for the integration of the measurable real functions over $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$. In fact, the properties of some (m,σ)-general functions generalize the analogous ones of the standard finite-dimensional diffeomorphisms. In particular, if A is a linear (m,σ)-general function, we can define the determinant of A (see the next Theorem 3.18 and Definition 3.19): a concept without sense, if A is an arbitrary matrix $I \times J$.

Definition 3.2. *Let* $m \in \mathbf{N}^*$*, let* $\emptyset \neq U = \left(U^{(m)} \times \prod_{j \in J \backslash J_m} A_j\right) \cap E_J \in \tau_{\|\cdot\|_J}$*, where* $U^{(m)} \in \tau^{(m)}$*,* $A_j \in \tau$*, for any* $j \in J \backslash J_m$*, and let* $\sigma : I \backslash I_m \longrightarrow J \backslash J_m$ *be an increasing function; a function* $\varphi : U \subset E_J \longrightarrow E_I$ *m-general and such that* $|J| = |I|$ *is called* (m,σ)*-general if:*

1. $\forall \, i \in I \backslash I_m$*,* $\forall \, j \in J \backslash (J_m \cup \{\sigma(i)\})$*,* $\forall \, t \in A_j$*, one has* $\varphi_{ij}(t) = 0$*; moreover*
$$\varphi^{(I \backslash I_m, J \backslash J_m)}\left(\pi_{J, J \backslash J_m}(U)\right) \subset E_{I \backslash I_m}.$$
2. $\forall \, i \in I \backslash I_m$*,* $\forall \, x \in U$*, there exists* $J_{\varphi_i}(x) : E_J \longrightarrow \mathbf{R}$*; moreover,* $\forall \, x_{J_m} \in U^{(m)}$*, one has* $\displaystyle\sum_{i \in I \backslash I_m} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| < +\infty$*.*
3. $\forall \, i \in I \backslash I_m$*, the function* $\varphi_{i,\sigma(i)} : A_{\sigma(i)} \longrightarrow \mathbf{R}$ *is constant or injective; moreover,* $\forall \, x_{\sigma(I \backslash I_m)} \in \displaystyle\prod_{j \in \sigma(I \backslash I_m)} A_j$*, one has* $\displaystyle\sup_{i \in I \backslash I_m} \left|\varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right)\right| < +\infty$ *and* $\displaystyle\inf_{i \in \mathcal{I}_\varphi} \left|\varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right)\right| > 0$*, where* $\mathcal{I}_\varphi = \{i \in I \backslash I_m : \varphi_{i,\sigma(i)}$ *is injective*$\}$*.*

4. If, for some $h \in \mathbf{N}$, $h \geq m$, one has $|\sigma(i)|_{J \backslash J_m} = |i|_{I \backslash I_m}$, $\forall\, i \in I \backslash I_h$, then, $\forall\, x_{\sigma(I \backslash I_m)} \in \prod_{j \in \sigma(I \backslash I_m)} A_j$, there exists $\prod_{i \in \mathcal{I}_\varphi} \varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right) \in \mathbf{R}^*$.

Moreover, set

$$\mathcal{A} = \mathcal{A}(\varphi) = \left\{h \in \mathbf{N},\, h \geq m : |\sigma(i)|_{J \backslash J_m} = |i|_{I \backslash I_m}\,,\, \forall\, i \in I \backslash I_h\right\}.$$

If the sequence $\left\{J_{\varphi_i^{(I,m)}}(x_{J_m})\right\}_{i \in I \backslash I_m}$ converges uniformly on $U^{(m)}$ to the matrix $(0 \ldots 0)$ and there exists $a \in \mathbf{R}$ such that, for any $\varepsilon > 0$, there exists $i_0 \in \mathbf{N}$, $i_0 \geq m$, such that, for any $i \in \mathcal{I}_\varphi \cap (I \backslash I_{i_0})$ and for any $t \in A_{\sigma(i)}$, one has $\left|\varphi'_{i,\sigma(i)}(t) - a\right| < \varepsilon$, then φ is called strongly (m,σ)-general.

Furthermore, for any $I_m \subset L \subset I$ and for any $J_m \subset N \subset J$, define the function $\overline{\varphi}^{(L,N)} : U \subset E_J \longrightarrow \mathbf{R}^I$ in the following manner:

$$\overline{\varphi}_i^{(L,N)}(x) = \begin{cases} \varphi_i^{(L,N)}(x_N) & \forall\, i \in I_m,\, \forall\, x \in U \\ \varphi_i(x) & \forall\, i \in L \backslash I_m,\, \forall\, x \in U \\ \varphi_{i,\sigma(i)}(x_{\sigma(i)}) & \forall\, i \in I \backslash L,\, \forall\, x \in U \end{cases}.$$

Finally, for any $l, n \in \mathbf{N}$, $l, n \geq m$, indicate $\overline{\varphi}^{(I_l,N)}$ by $\overline{\varphi}^{(l,N)}$, $\overline{\varphi}^{(L,J_n)}$ by $\overline{\varphi}^{(L,n)}$, $\overline{\varphi}^{(I_l,J_n)}$ by $\overline{\varphi}^{(l,n)}$, and $\overline{\varphi}^{(m,m)}$ by $\overline{\varphi}$.

Definition 3.3. *A function $\varphi : U \subset E_J \longrightarrow E_I$ (m,σ)-general is called (m,σ)-standard (or (m,σ) of the first type) if, for any $i \in I \backslash I_m$ and for any $x_{J_m} \in U^{(m)}$, one has $\varphi_i^{(I,m)}(x_{J_m}) = 0$. Moreover, a function $\varphi : U \subset E_J \longrightarrow E_I$ (m,σ)-standard and strongly (m,σ)-general is called strongly (m,σ)-standard (see also Definition 28 in [5]).*

Remark 3.4: Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function; then:

1. σ is injective if and only if, for any $i_1, i_2 \in I \backslash I_m$ such that $i_1 < i_2$, one has $\sigma(i_1) < \sigma(i_2)$.

2. σ is bijective if and only if, for any $i \in I \backslash I_m$, one has $|\sigma(i)|_{J \backslash J_m} = |i|_{I \backslash I_m}$.

3. There exists $m_0 \in \mathbf{N}$, $m_0 \geq m$, such that $A_j = \mathbf{R}$, for any $j \in J \backslash J_{m_0}$.

Proof. The statement follows from Definition 3.2 and point 3 of Proposition 2.4. $\square$

Proposition 3.5. *Let $I_m \subset L \subset I$, let $J_m \subset N \subset J$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function; then, one has $\overline{\varphi}^{(L,N)}(U) \subset E_I$, and the function $\overline{\varphi}^{(L,N)} : U \subset E_J \longrightarrow E_I$ is (m,σ)-general. Moreover, suppose that, for any $j \in J\backslash J_m$, for any $t \in A_j$, one has $\sum\limits_{i \in I\backslash I_m} \left|\varphi'_{i,j}(t)\right| < +\infty$; then, for any $n \in \mathbf{N}$, $n \geq m$, $\overline{\varphi}^{(L,N)}$ is (n,τ)-general, where the function $\tau : I\backslash I_n \longrightarrow J\backslash J_n$ is defined by*

$$\tau(i) = \begin{cases} \sigma(i) & \text{if } \sigma(i) \in J\backslash J_n \\ \min\left(J\backslash J_n\right) & \text{if } \sigma(i) \notin J\backslash J_n \end{cases}, \forall\, i \in I\backslash I_n. \tag{6}$$

Proof. Since $I_m \subset L \subset I$ and $J_m \subset N \subset J$, $\forall\, i \in I\backslash I_m$, $\forall\, x \in U$, we have

$$\left|\overline{\varphi}_i^{(L,N)}(x)\right| \leq \left|\varphi_i^{(I,m)}(x_{J_m})\right| + \left|\varphi_{i,\sigma(i)}\left(x_{\sigma(i)}\right)\right|,$$

and so $\sup\limits_{i \in I\backslash I_m} \left|\overline{\varphi}_i^{(L,N)}(x)\right| < +\infty$; then, $\overline{\varphi}^{(L,N)}(U) \subset E_I$. Moreover, from the definition of $\overline{\varphi}^{(L,N)}$, the function $\overline{\varphi}^{(L,N)} : U \subset E_J \longrightarrow E_I$ is (m,σ)-general. Furthermore, suppose that, for any $j \in J\backslash J_m$, for any $t \in A_j$, one has $\sum\limits_{i \in I\backslash I_m} \left|\varphi'_{i,j}(t)\right| < +\infty$; $\forall\, n \in \mathbf{N}$, $n \geq m$, and $\forall\, x_{J_n} \in \pi_{J,J_n}(U)$, we have

$$\begin{aligned} \sum_{i \in I\backslash I_n} \left\| J_{\left(\overline{\varphi}^{(L,N)}\right)_i^{(I,J_n)}}(x_{J_n}) \right\| &\leq \sum_{i \in I\backslash I_n} \left\| J_{\varphi_i^{(I,J_n)}}(x_{J_n}) \right\| \\ &= \sum_{i \in I\backslash I_n} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| + \sum_{j \in J_n\backslash J_m} \left(\sum_{i \in I\backslash I_n} \left|\varphi'_{i,j}(x_j)\right| \right) < +\infty; \end{aligned}$$

then, $\overline{\varphi}^{(L,N)}$ is (n,τ)-general, where the function $\tau : I\backslash I_n \longrightarrow J\backslash J_n$ is defined by formula (6). □

Proposition 3.6. *Let $\emptyset \neq L \subset I$, let $\emptyset \neq N \subset J$ such that $J_m \subset N$ or $N \subset J\backslash J_m$, and let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function; then:*

1. *For any $x \in U$, there exists the function $J_{\varphi^{(L,N)}}(x) : E_N \longrightarrow \mathbf{R}^L$ if and only if, for any $i \in L \cap I_m$ and for any $j \in N$, there exists the partial derivative $\frac{\partial \varphi_i}{\partial x_j}(x)$, and for any $i \in L \cap I_m$ one has $\sum\limits_{j \in N} \left|\frac{\partial \varphi_i}{\partial x_j}(x)\right| < +\infty$; moreover, in this case one has $J_{\varphi^{(L,N)}}(x)\left(E_N\right) \subset E_L$, and $J_{\varphi^{(L,N)}}(x)$ is continuous.*
2. *For any $x \in U$, there exists the function $J_{\varphi^{(I\backslash I_m,J)}}(x) : E_J \longrightarrow E_{I\backslash I_m}$, and it is continuous.*

3. *Suppose that $I_m \subset L$ and $J_m \subset N$, and let $x \in U$; then, there exists the function $J_{\overline{\varphi}^{(L,N)}}(x) : E_J \longrightarrow \mathbf{R}^I$ if and only if, for any $i \in I_m$ and for any $j \in N$, there exists the partial derivative $\frac{\partial \varphi_i}{\partial x_j}(x)$, and for any $i \in I_m$ one has $\sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| < +\infty$; moreover, in this case one has $J_{\overline{\varphi}^{(L,N)}}(x)(E_J) \subset E_I$, and $J_{\overline{\varphi}^{(L,N)}}(x)$ is continuous and (m,σ)-general.*

Proof. 1. From Definition 3.2, $\forall\, i \in L \cap (I \backslash I_m)$ and $\forall\, j \in N$, there exists the partial derivative $\frac{\partial \varphi_i^{(L,N)}}{\partial x_j}(x) = \frac{\partial \varphi_i}{\partial x_j}(x)$, and one has

$$\sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| \leq \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| + \left| \varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right) \right| < +\infty,$$

$$\forall\, i \in L \cap (I \backslash I_m); \quad (7)$$

then, from Proposition 2.13, there exists the function $J_{\varphi^{(L,N)}}(x) : E_N \longrightarrow \mathbf{R}^L$ if and only if, $\forall\, i \in L \cap I_m$ and $\forall\, j \in N$,there exists the partial derivative $\frac{\partial \varphi_i^{(L,N)}}{\partial x_j}(x) = \frac{\partial \varphi_i}{\partial x_j}(x)$, and $\forall\, i \in L \cap I_m$ one has $\sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| < +\infty$.

Furthermore, since $\sum_{i \in I \backslash I_m} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| < +\infty$, we have

$$\sup_{i \in L \cap (I \backslash I_m)} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| < +\infty,$$

and so formula (7) implies

$$\sup_{i \in L \cap (I \backslash I_m)} \sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| \leq \sup_{i \in L \cap (I \backslash I_m)} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| + \sup_{i \in L \cap (I \backslash I_m)} \left| \varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right) \right| < +\infty;$$

thus, if there exists the function $J_{\varphi^{(L,N)}}(x)$, we obtain $\sup_{i \in L} \sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| < +\infty$; then, from Proposition 2.13, we have $J_{\varphi^{(L,N)}}(x)(E_N) \subset E_L$, and $J_{\varphi^{(L,N)}}(x)$ is continuous.

2. The statement follows from point 1.

3. By Definition 3.2, $\forall\, i \in I \backslash I_m$ and $\forall\, j \in J$, there exists the partial derivative $\frac{\partial \overline{\varphi}_i^{(L,N)}}{\partial x_j}(x)$, and one has

$$\sum_{j \in J} \left| \frac{\partial \overline{\varphi}_i^{(L,N)}}{\partial x_j}(x) \right| \leq \sum_{j \in J} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right|$$
$$\leq \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| + \left| \varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right) \right| < +\infty, \forall\, i \in I \backslash I_m; \quad (8)$$

then, from Proposition 2.13, there exists the function $J_{\overline{\varphi}^{(L,N)}}(x) : E_J \longrightarrow \mathbf{R}^I$ if and only if, $\forall\, i \in I_m$ and $\forall\, j \in J$,there exists the partial derivative $\frac{\partial \overline{\varphi}_i^{(L,N)}}{\partial x_j}(x)$, and $\forall\, i \in I_m$ one has $\sum_{j \in J} \left| \frac{\partial \overline{\varphi}_i^{(L,N)}}{\partial x_j}(x) \right| < +\infty$; thus, this happens if and only if, $\forall\, i \in I_m$ and $\forall\, j \in N$,there exists the partial derivative $\frac{\partial \varphi_i}{\partial x_j}(x)$, and $\forall\, i \in I_m$ one has $\sum_{j \in N} \left| \frac{\partial \varphi_i}{\partial x_j}(x) \right| < +\infty$.

Moreover, from formula (8), we have

$$\sup_{i \in I \backslash I_m} \sum_{j \in J} \left| \frac{\partial \overline{\varphi}_i^{(L,N)}}{\partial x_j}(x) \right|$$
$$\leq \sup_{i \in I \backslash I_m} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| + \sup_{i \in I \backslash I_m} \left| \varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right) \right| < +\infty;$$

then, if there exists the function $J_{\overline{\varphi}^{(L,N)}}(x)$, we obtain

$$\sup_{i \in I} \sum_{j \in J} \left| \frac{\partial \overline{\varphi}_i^{(L,N)}}{\partial x_j}(x) \right| < +\infty;$$

thus, from Proposition 2.13, we have $J_{\overline{\varphi}^{(L,N)}}(x)(E_J) \subset E_I$, and $J_{\overline{\varphi}^{(L,N)}}(x)$ is continuous; furthermore, by Definition 3.2, $J_{\overline{\varphi}^{(L,N)}}(x)$ is (m, σ)-general.

□

PROPOSITION 3.7. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m, σ)-standard function; then:*

1. *Suppose that φ is injective, $\pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, for any $H \subset I \backslash I_m$ such that $0 < |H| \leq 2$, the function $\varphi_i : U \longrightarrow \mathbf{R}$ is C^1, for any $i \in I_m$, and* $\det J_{\varphi^{(m,m)}}(\mathbf{x}) \neq 0$*, for any $\mathbf{x} \in U^{(m)}$; then the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \backslash I_m$, and $\varphi^{(m,m)}$ are injective, and σ is bijective.*
2. *Suppose that φ is bijective, the function $\varphi_i : U \longrightarrow \mathbf{R}$ is C^1, for any $i \in I_m$, and* $\det J_{\varphi^{(m,m)}}(\mathbf{x}) \neq 0$*, for any $\mathbf{x} \in U^{(m)}$; then the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \backslash I_m$, $\varphi^{(m,m)}$ and σ are bijective.*

3. *Suppose that* $\varphi_{ij}(x_j) = 0$, *for any* $i \in I_m$, *for any* $j \in J \backslash J_m$, *for any* $x_j \in A_j$, φ *is injective, and* $\pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, *for any* $H \subset I \backslash I_m$ *such that* $0 < |H| \leq 2$; *then the functions* $\varphi_{i,\sigma(i)}$, *for any* $i \in I \backslash I_m$, *and* $\varphi^{(m,m)}$ *are injective, and* σ *is bijective.*

4. *Suppose that* $\varphi_{ij}(x_j) = 0$, *for any* $i \in I_m$, *for any* $j \in J \backslash J_m$, *for any* $x_j \in A_j$, *and* φ *is bijective; then the functions* $\varphi_{i,\sigma(i)}$, *for any* $i \in I \backslash I_m$, $\varphi^{(m,m)}$ *and* σ *are bijective.*

5. *If the functions* $\varphi_{i,\sigma(i)}$, *for any* $i \in I \backslash I_m$, *and* $\varphi^{(m,m)}$ *are injective, and* σ *is bijective, then* φ *is injective.*

6. *If the functions* $\varphi_{i,\sigma(i)}$, *for any* $i \in I \backslash I_m$, $\varphi^{(m,m)}$ *and* σ *are bijective, then* φ *is bijective.*

Proof. The statement follows from Proposition 31, Proposition 32 and Remark 33 in [5]. $\square$

COROLLARY 3.8. *Let* $\varphi : U \subset E_J \longrightarrow E_I$ *be a* (m, σ)*-general function; then:*

1. If $\overline{\varphi}$ is injective and $\pi_{I,H}(\overline{\varphi}(U)) \in \tau^{(H)}$, for any $H \subset I \backslash I_m$ such that $0 < |H| \leq 2$, then the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \backslash I_m$, and $\varphi^{(m,m)}$ are injective, and σ is bijective.

2. If $\overline{\varphi}$ is bijective, then the functions $\varphi_{i,\sigma(i)}$, for any $i \in I \backslash I_m$, $\varphi^{(m,m)}$ and σ are bijective.

Proof. Observe that $\overline{\varphi}$ is (m, σ)-standard, and $\overline{\varphi}_{ij}(x_j) = 0$, for any $i \in I_m$, for any $j \in J \backslash J_m$, for any $x_j \in A_j$; then, from points 3 and 4 of Proposition 3.7, we obtain the statements 1 and 2. $\square$

PROPOSITION 3.9. *Let* $m \in \mathbf{N}^*$, *let* $\emptyset \neq L \subset I$, *let* $\emptyset \neq N \subset J$ *such that* $J_m \subset N$ *or* $N \subset J \backslash J_m$, *and let* $\varphi : U \subset E_J \longrightarrow E_I$ *be a function* m*-general and such that, for any* $i \in L$ *and for any* $j \in N \backslash J_m$, *the functions* $\varphi_i^{(I,m)} : \left(U^{(m)}, \mathcal{B}^{(m)}\left(U^{(m)}\right)\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ *and* $\varphi_{ij} : (A_j, \mathcal{B}(A_j)) \longrightarrow (\mathbf{R}, \mathcal{B})$ *are measurable; then:*

1. *The function*

$$\varphi^{(L,N)} : (\pi_{J,N}(U), \mathcal{B}^{(N)}(\pi_{J,N}(U))) \longrightarrow \left(\mathbf{R}^L, \mathcal{B}^{(L)}\right)$$

is measurable; in particular, suppose that, for any $i \in I$ *and for any* $j \in J \backslash J_m$, $\varphi_i^{(I,m)}$ *and* φ_{ij} *are measurable functions; then,* $\varphi : (U, \mathcal{B}^{(J)}(U)) \longrightarrow (E_I, \mathcal{B}_I)$ *is measurable.*

2. *If φ is (m,σ)-general, $I_m \subset L$ and $J_m \subset N$, then the function $\overline{\varphi}^{(L,N)}$: $(U, \mathcal{B}^{(J)}(U)) \longrightarrow (E_I, \mathcal{B}_I)$ is measurable.*

Proof. 1. $\forall\, i \in L$ and $\forall\, M \subset N$ such that $J_m \subset M$ or $M \subset J \backslash J_m$, consider the function $\widehat{\varphi}^{(i,M,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}$ defined by

$$\widehat{\varphi}^{(i,M,N)}(x) = \begin{cases} \varphi^{(\{i\},M)}(x_M) & \text{if } M \neq \emptyset \\ 0 & \text{if } M = \emptyset \end{cases}, \ \forall\, x \in \pi_{J,N}(U);$$

observe that, $\forall\, n \in \mathbf{N}$, $n \geq m$, we have

$$\widehat{\varphi}^{(i,N\cap J_n,N)}(x) = \widehat{\varphi}^{(i,N\cap J_m,N)}(x) + \sum_{j\in N\cap(J_n\backslash J_m)} \widehat{\varphi}^{(i,\{j\},N)}(x), \quad \forall\, x \in \pi_{J,N}(U); \quad (9)$$

moreover, from Remark 2.6, the functions $\widehat{\varphi}^{(i,N\cap J_m,N)}$ and $\widehat{\varphi}^{(i,\{j\},N)}$, $\forall\, j \in N \cap (J_n \backslash J_m)$, are $\left(\mathcal{B}^{(N)}(\pi_{J,N}(U)), \mathcal{B}\right)$-measurable; thus, from formula (9), $\widehat{\varphi}^{(i,N\cap J_n,N)}$ is $\left(\mathcal{B}^{(N)}(\pi_{J,N}(U)), \mathcal{B}\right)$-measurable; then, since

$$\lim_{n\longrightarrow+\infty} \widehat{\varphi}^{(i,N\cap J_n,N)} = \varphi_i^{(L,N)},$$

$\varphi_i^{(L,N)}$ is $\left(\mathcal{B}^{(N)}(\pi_{J,N}(U)), \mathcal{B}\right)$-measurable too. Furthermore, let

$$\Sigma(L) = \left\{ B = \prod_{i\in L} B_i : B_i \in \mathcal{B},\, \forall\, i \in L \right\};$$

$\forall\, B = \prod_{i\in L} B_i \in \Sigma(L)$, we have

$$\left(\varphi^{(L,N)}\right)^{-1}(B) = \bigcap_{i\in L} \left(\varphi_i^{(L,N)}\right)^{-1}(B_i) \in \mathcal{B}^{(N)}(\pi_{J,N}(U)).$$

Finally, since $\sigma\left(\Sigma(L)\right) = \mathcal{B}^{(L)}$, $\forall\, B \in \mathcal{B}^{(L)}$, we obtain $\left(\varphi^{(L,N)}\right)^{-1}(B) \in \mathcal{B}^{(N)}(\pi_{J,N}(U))$, and so $\varphi^{(L,N)} : (\pi_{J,N}(U), \mathcal{B}^{(N)}(\pi_{J,N}(U))) \longrightarrow \left(\mathbf{R}^L, \mathcal{B}^{(L)}\right)$ is measurable. In particular, suppose that, $\forall\, i \in I$ and $\forall\, j \in J\backslash J_m$, the functions $\varphi_i^{(I,m)}$ and φ_{ij} are measurable; then, $\varphi : (U, \mathcal{B}^{(J)}(U)) \longrightarrow \left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$ is measurable; thus, since $\varphi(U) \subset E_I$, we obtain that φ is $\left(\mathcal{B}^{(J)}(U), \mathcal{B}_I\right)$-measurable.

2. If φ is (m,σ)-general, $I_m \subset L$ and $J_m \subset N$, from Proposition 3.5, the function $\overline{\varphi}^{(L,N)} : U \subset E_J \longrightarrow E_I$ is (m,σ)-general, and so m-general. Moreover, we have

$$\overline{\varphi}_i^{(L,N)}(x) = \psi_i^{(I,m)}(x_{J_m}) + \sum_{j\in J\backslash J_m} \psi_{ij}(x_j),\ \forall\, x \in U,\ \forall\, i \in I,$$

where

$$\psi_i^{(I,m)} = \begin{cases} \varphi_i^{(I,m)} & \text{if } i \in L \\ 0 & \text{if } i \in I \backslash L \end{cases},$$

$$\psi_{ij} = \begin{cases} \varphi_{ij} & \text{if } (i,j) \in (I_m \times (N \backslash J_m)) \cup ((I \backslash I_m) \times (J \backslash J_m)) \\ 0 & \text{if } (i,j) \in I_m \times (J \backslash N) \end{cases};$$

furthermore, $\forall\, i \in I$, $\forall\, j \in J \backslash J_m$, $\psi_i^{(I,m)} : \left(U^{(m)}, \mathcal{B}^{(m)}\left(U^{(m)}\right)\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ and $\psi_{ij} : (A_j, \mathcal{B}(A_j)) \longrightarrow (\mathbf{R}, \mathcal{B})$ are measurable functions, and so, from point 1, $\overline{\varphi}^{(L,N)} : (U, \mathcal{B}^{(J)}(U)) \longrightarrow \left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$ is measurable; finally, since $\overline{\varphi}^{(L,N)}(U) \subset E_I$, we obtain that $\overline{\varphi}^{(L,N)}$ is $\left(\mathcal{B}^{(J)}(U), \mathcal{B}_I\right)$-measurable.

□

Proposition 3.10. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function such that σ is bijective and $\pi_{I,I\backslash I_m} \circ \overline{\varphi} : \left(U, \tau_{\|\cdot\|_J}(U)\right) \longrightarrow \left(E_{I\backslash I_m}, \tau_{\|\cdot\|_{I\backslash I_m}}\right)$ is continuous; then, for any $n \in \mathbf{N}$, $n \geq m$, $\varphi^{(n,n)} : \left(\pi_{J,J_n}(U), \tau^{(n)}(\pi_{J,J_n}(U))\right) \longrightarrow \left(\mathbf{R}^n, \tau^{(n)}\right)$ is continuous if and only if $\overline{\varphi}^{(n,n)} : \left(U, \tau_{\|\cdot\|_J}(U)\right) \longrightarrow \left(E_I, \tau_{\|\cdot\|_I}\right)$ is continuous.*

Proof. Let $n \in \mathbf{N}$, $n \geq m$, and suppose that $\varphi^{(n,n)}$ is continuous; moreover, let $B = B_1 \times B_2 \in \tau_{\|\cdot\|_I}$, where $B_1 \in \tau^{(n)}$, $B_2 \in \tau_{\|\cdot\|_{I\backslash I_n}}$; since σ is bijective, we have

$$\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B) = \left(\varphi^{(n,n)}\right)^{-1}(B_1) \times \pi_{J,J\backslash J_n}\left(\left(\pi_{I,I\backslash I_n} \circ \overline{\varphi}\right)^{-1}(B_2)\right);$$

moreover, since $\varphi^{(n,n)}$ and $\pi_{I,I\backslash I_m} \circ \overline{\varphi}$ are continuous, and $\mathbf{R}^{n-m} \times B_2 \in \tau_{\|\cdot\|_{I\backslash I_m}}$, we have

$$\left(\varphi^{(n,n)}\right)^{-1}(B_1) \in \tau^{(n)}(\pi_{J,J_n}(U)),$$

$$\begin{aligned} \left(\pi_{I,I\backslash I_n} \circ \overline{\varphi}\right)^{-1}(B_2) &= \left(\pi_{I\backslash I_m, I\backslash I_n} \circ \left(\pi_{I,I\backslash I_m} \circ \overline{\varphi}\right)\right)^{-1}(B_2) \\ &= \left(\pi_{I,I\backslash I_m} \circ \overline{\varphi}\right)^{-1}\left(\mathbf{R}^{n-m} \times B_2\right) \in \tau_{\|\cdot\|_J}(U), \end{aligned}$$

and so $\pi_{J,J\backslash J_n}\left(\left(\pi_{I,I\backslash I_n} \circ \overline{\varphi}\right)^{-1}(B_2)\right) \in \tau_{\|\cdot\|_{J\backslash J_n}}\left(\pi_{J,J\backslash J_n}(U)\right)$, from Proposition 2.5; then, we obtain $\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B) \in \tau_{\|\cdot\|_J}(U)$; finally, from Proposition 2.4, $\forall\, B \in \tau_{\|\cdot\|_I}$, we have $\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B) \in \tau_{\|\cdot\|_J}(U)$, and so $\overline{\varphi}^{(n,n)}$ is continuous.

Conversely, suppose that $\overline{\varphi}^{(n,n)}$ is continuous; $\forall\, B \in \tau^{(n)}$, we have $B \times E_{I\backslash I_n} \in \tau_{\|\cdot\|_I}$, and so $\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B \times E_{I\backslash I_n}) \in \tau_{\|\cdot\|_J}(U)$; then, $\left(\varphi^{(n,n)}\right)^{-1}(B) = \pi_{J,J_n}\left(\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B \times E_{I\backslash I_n})\right) \in \tau^{(n)}(\pi_{J,J_n}(U))$. □

PROPOSITION 3.11. *Let $m \in \mathbf{N}^*$, let $\emptyset \neq L \subset I$, let $\emptyset \neq N \subset J$ such that $J_m \subset N$ or $N \subset J \backslash J_m$, and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function m-general and C^1 in $x_0 = (x_{0,j} : j \in J) \in U$; then:*

1. *The function $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ is C^1 in $(x_{0,j} : j \in N)$.*
2. *If φ is (m,σ)-general, $I_m \subset L$ and $J_m \subset N$, then the function $\overline{\varphi}^{(L,N)}$: $U \subset E_J \longrightarrow E_I$ is C^1 in x_0.*

Proof. See the proof of Proposition 2.28 in [6]. □

PROPOSITION 3.12. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function such that $\overline{\varphi} : U \longrightarrow \overline{\varphi}(U)$ is a homeomorphism. Then, the functions $\varphi^{(m,m)}$: $U^{(m)} \longrightarrow \varphi^{(m,m)}\left(U^{(m)}\right)$ and $\varphi_{i,\sigma(i)} : A_i \longrightarrow \varphi_{i,\sigma(i)}(A_i)$, for any $i \in I \backslash I_m$, are homeomorphisms, and σ is bijective.*

Proof. From Proposition 37 in [5], the statement is true if φ is (m,σ)-standard; moreover, observe that $\overline{\varphi}$ is (m,σ)-standard, $\overline{\varphi} = \overline{(\overline{\varphi})}$, $\varphi^{(m,m)} = (\overline{\varphi})^{(m,m)}$, $\varphi_{i,\sigma(i)} = \overline{\varphi}_{i,\sigma(i)}$, $\forall\, i \in I \backslash I_m$; then, the statement is true if φ is (m,σ)-general too. □

PROPOSITION 3.13. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function. Then, $\overline{\varphi} : U \longrightarrow \overline{\varphi}(U)$ is a diffeomorphism if and only if the functions $\varphi^{(m,m)}$: $U^{(m)} \longrightarrow \varphi^{(m,m)}\left(U^{(m)}\right)$ and $\varphi_{i,\sigma(i)} : A_i \longrightarrow \varphi_{i,\sigma(i)}(A_i)$, for any $i \in I \backslash I_m$, are diffeomorphisms, and σ is bijective.*

Proof. From Proposition 38 in [5], the statement is true if φ is (m,σ)-standard; then, as we observed in the proof of Proposition 3.12, the statement is true if φ is (m,σ)-general too. □

DEFINITION 3.14. *Let $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ be a linear (m,σ)-general function; $\forall\, i \in I \backslash I_m$, set $\lambda_i = \lambda_i(A) = a_{i,\sigma(i)}$.*

REMARK 3.15: For any $m \in \mathbf{N}^*$, a linear function $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ is m-general; moreover, if $|J| = |I|$ and $\sigma : I \backslash I_m \longrightarrow J \backslash J_m$ is an increasing function, A is (m,σ)-general if and only if:

1. $\forall\, i \in I \backslash I_m$, $\forall\, j \in J \backslash (J_m \cup \{\sigma(i)\})$, one has $a_{ij} = 0$.
2. $\forall\, j \in J_m$, $\sum_{i \in I \backslash I_m} |a_{ij}| < +\infty$; moreover, one has $\sup_{i \in I \backslash I_m} |\lambda_i| < +\infty$ and $\inf_{i \in I \backslash I_m : \lambda_i \neq 0} |\lambda_i| > 0$.

3. If $\mathcal{A} \neq \emptyset$, there exists $\prod_{i \in I \setminus I_m : \lambda_i \neq 0} \lambda_i \in \mathbf{R}^*$.

Furthermore, A is strongly (m, σ)-general if and only if A is (m, σ)-general and there exists $a \in \mathbf{R}$ such that the sequence $\{\lambda_i\}_{i \in I \setminus I_m : \lambda_i \neq 0}$ converges to a.

Finally, A is (m, σ)-standard if and only if A is (m, σ)-general and $a_{ij} = 0$, for any $i \in I \setminus I_m$, for any $j \in J_m$.

Corollary 3.16. *Let $m \in \mathbf{N}^*$, let $\emptyset \neq L \subset I$, let $\emptyset \neq N \subset J$ and let $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ be a linear function; then:*

1. The function $A^{(L,N)} : (E_N, \mathcal{B}_N) \longrightarrow (E_L, \mathcal{B}_L)$ is measurable; in particular, $A : (E_J, \mathcal{B}_J) \longrightarrow (E_I, \mathcal{B}_I)$ is measurable.
2. If A is (m, σ)-general, $I_m \subset L$ and $J_m \subset N$, then the function $\overline{A}^{(L,N)}$: $(E_J, \mathcal{B}_J) \longrightarrow (E_I, \mathcal{B}_I)$ is measurable.

Proof. 1. From Proposition 2.13, we have $A^{(L,N)}(E_N) \subset E_L$; furthermore, from Remark 3.15, A is 1-general; moreover, we have $J_1 \subset N$ or $N \subset J \setminus J_1$; then, from Proposition 3.9, $A^{(L,N)} : (E_N, \mathcal{B}_N) \longrightarrow \left(\mathbf{R}^L, \mathcal{B}^{(L)}\right)$ is measurable, and so $A^{(L,N)} : (E_N, \mathcal{B}_N) \longrightarrow (E_L, \mathcal{B}_L)$ is measurable; in particular, $A : (E_J, \mathcal{B}_J) \longrightarrow (E_I, \mathcal{B}_I)$ is measurable.

2. The statement follows from Proposition 3.9.

□

Henceforth, we will suppose that $|I| = +\infty$. The following definitions and results (from Proposition 3.17 to Proposition 3.21) can be found in [6] and generalize the standard theory of the $m \times m$ matrices.

Proposition 3.17. *Let $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ be a linear (m, σ)-general function; then, A is continuous.*

Theorem 3.18. *Let $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ be a linear (m, σ)-general function; then, the sequence $\left\{\det A^{(n,n)}\right\}_{n \geq m}$ converges to a real number. Moreover, if $\mathcal{A} \neq \emptyset$, by setting $\overline{m} = \min \mathcal{A}$, we have*

$$\lim_{n \longrightarrow +\infty} \det A^{(n,n)} = \sum_{p \in I \setminus I_{\overline{m}}} \left(\prod_{q \in I \setminus I_{|p|}} \lambda_q \right) \sum_{j \in J_m} a_{p,j} \left(cof A^{(|p|,|p|)} \right)_{p,j} + \det A^{(\overline{m},\overline{m})} \left(\prod_{q \in I \setminus I_{\overline{m}}} \lambda_q \right). \quad (10)$$

Conversely, if $\mathcal{A} = \emptyset$*, we have* $\lim_{n \longrightarrow +\infty} \det A^{(n,n)} = 0$.

Definition 3.19. *Let* $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ *be a linear* (m, σ)*-general function; define the determinant of* A*, and call it* $\det A$*, the real number*

$$\det A = \lim_{n \longrightarrow +\infty} \det A^{(n,n)}.$$

Corollary 3.20. *Let* $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ *be a linear* (m, σ)*-general function such that* $a_{ij} = 0$, $\forall i \in I_m$, $\forall j \in J \backslash J_m$*, or* A *is* (m, σ)*-standard. Then, if* σ *is bijective, we have*

$$\det A = \det A^{(m,m)} \prod_{i \in I \backslash I_m} \lambda_i.$$

Conversely, if σ *is not bijective, we have* $\det A = 0$*. In particular, if* $A = \mathbf{I}_{I,J}$*, we have* $\det A = 1$.

Proposition 3.21. *Let* $\varphi : U \subset E_J \longrightarrow E_I$ *be a* (m, σ)*-general function and let* $x_0 = (x_{0,j} : j \in J) \in U$ *such that there exists the function* $J_\varphi(x_0) : E_J \longrightarrow E_I$*; then,* $J_\varphi(x_0)$ *is* (m, σ)*-general; moreover, for any* $n \in \mathbf{N}$, $n \geq m$*, there exists the linear* (m, σ)*-general function* $J_{\overline{\varphi}^{(n,n)}}(x_0) : E_J \longrightarrow E_I$*, and one has*

$$\det J_\varphi(x_0) = \lim_{n \to +\infty} \det J_{\overline{\varphi}^{(n,n)}}(x_0).$$

Proposition 3.22. *Let* $m \in \mathbf{N}^*$*, let* $n \in \mathbf{N}$, $n \geq m$*, and let* $\varphi : U \subset E_J \longrightarrow E_I$ *be a function* m*-general such that, for any* $i \in I_n$*, for any* $j_1 \in J_m$ *and for any* $j_2 \in J_n \backslash J_m$*, there exist the functions* $\frac{\partial \varphi_i^{(I,m)}}{\partial x_{j_1}} : \left(U^{(m)}, \mathcal{B}^{(m)}\left(U^{(m)}\right)\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ *and* $\frac{\partial \varphi_{ij_2}}{\partial x_{j_2}} : (A_{j_2}, \mathcal{B}(A_{j_2})) \longrightarrow (\mathbf{R}, \mathcal{B})$*, and they are measurable; then:*

1. *The function* $\det J_{\varphi^{(n,n)}} : \left(\pi_{J,J_n}(U), \mathcal{B}^{(n)}(\pi_{J,J_n}(U))\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ *is measurable.*
2. *Suppose that* φ *is* (m, σ)*-general and, for any* $i \in I \backslash I_m$*, the function*

$$\varphi'_{i,\sigma(i)} : (A_{\sigma(i)}, \mathcal{B}(A_{\sigma(i)})) \longrightarrow (\mathbf{R}, \mathcal{B})$$

is measurable; then, for any $x \in U$*, there exists the function* $J_{\overline{\varphi}^{(n,n)}}(x) : E_J \longrightarrow E_I$*, and it is* (m, σ)*-general; moreover, the function* $\det J_{\overline{\varphi}^{(n,n)}} : \left(U, \mathcal{B}^{(J)}(U)\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ *is measurable.*

3. *Suppose that φ is (m,σ)-general and, for any $x \in U$, there exists the function $J_\varphi(x) : E_J \longrightarrow E_I$; moreover, suppose that, for any $i \in I$, for any $j_1 \in J_m$ and for any $j_2 \in J \backslash J_m$, the functions*

$$\frac{\partial \varphi_i^{(I,m)}}{\partial x_{j_1}} : \left(U^{(m)}, \mathcal{B}^{(m)}\left(U^{(m)}\right)\right) \longrightarrow (\mathbf{R}, \mathcal{B})$$

and $\frac{\partial \varphi_{ij_2}}{\partial x_{j_2}} : (A_{j_2}, \mathcal{B}(A_{j_2})) \longrightarrow (\mathbf{R}, \mathcal{B})$ *are measurable; then the function* $\det J_\varphi : \left(U, \mathcal{B}^{(J)}(U)\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ *is measurable.*

Proof. 1. From Remark 2.6, $\forall\, i \in I_n$, $\forall\, j \in J_n$, the function

$$\frac{\partial \varphi_i^{(I,n)}}{\partial x_j} : \left(\pi_{J,J_n}(U), \mathcal{B}^{(n)}(\pi_{J,J_n}(U))\right) \longrightarrow (\mathbf{R}, \mathcal{B})$$

is measurable; moreover, we have

$$\left(J_{\varphi^{(n,n)}}(x)\right)_{ij} = \frac{\partial \varphi_i^{(I,n)}}{\partial x_j}(x),\ \forall\, x \in \pi_{J,J_n}(U);$$

then, by definition of determinant, the function

$$\det J_{\varphi^{(n,n)}} : \left(\pi_{J,J_n}(U), \mathcal{B}^{(n)}(\pi_{J,J_n}(U))\right) \longrightarrow (\mathbf{R}, \mathcal{B})$$

is measurable too.

2. If φ is (m,σ)-general, from Proposition 3.5, $\overline{\varphi}^{(n,n)}$ is (m,σ)-general too; then, from Proposition 2.13, $\forall\, x \in U$, there exists the function $J_{\overline{\varphi}^{(n,n)}}(x) : E_J \longrightarrow E_I$, and it is (m,σ)-general, from Remark 3.15.

If $\mathcal{A}(\varphi) = \emptyset$, $\forall\, x \in U$, we have $\mathcal{A}\left(J_{\overline{\varphi}^{(n,n)}}(x)\right) = \emptyset$, and so $\det J_{\overline{\varphi}^{(n,n)}}(x) = 0$; then, the function $\det J_{\overline{\varphi}^{(n,n)}} : \left(U, \mathcal{B}^{(J)}(U)\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable.

Conversely, if $\mathcal{A}(\varphi) \neq \emptyset$, set $\overline{m} = \min \mathcal{A}(\varphi)$, $\widehat{m} = \max\{n, \overline{m}\}$; observe that $\overline{\varphi}^{(n,n)}$ is $(\widehat{m}, \rho)$-standard, where the bijective increasing function $\rho : I \backslash I_{\widehat{m}} \longrightarrow J \backslash J_{\widehat{m}}$ is defined by $\rho(i) = \sigma(i)$, $\forall\, i \in I \backslash I_{\widehat{m}}$; thus, $\forall\, x \in U$, $J_{\overline{\varphi}^{(n,n)}}(x)$ is $(\widehat{m}, \rho)$-standard too, and so Corollary 3.20 implies

$$\det J_{\overline{\varphi}^{(n,n)}}(x) = \det \left(J_{\overline{\varphi}^{(n,n)}}\right)^{(\widehat{m},\widehat{m})}(x_{J_{\widehat{m}}}) \prod_{i \in I \backslash I_{\widehat{m}}} \varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right),\ \forall\, x \in U. \tag{11}$$

If $\widehat{m} > n$, we have $\det \left(J_{\overline{\varphi}^{(n,n)}}\right)^{(\widehat{m},\widehat{m})}(x_{J_{\widehat{m}}}) = 0$, and so $\det J_{\overline{\varphi}^{(n,n)}}(x) = 0$, $\forall\, x \in U$; then, $\det J_{\overline{\varphi}^{(n,n)}} : \left(U, \mathcal{B}^{(J)}(U)\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable. Finally, if $\widehat{m} = n$, from formula (11), we have

$$\det J_{\overline{\varphi}^{(n,n)}}(x) = \det J_{\varphi^{(n,n)}}(x_{J_n}) \prod_{i \in I \backslash I_n} \varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right),\ \forall\, x \in U;$$

moreover, from point 1, the function

$$\det J_{\varphi^{(n,n)}} : \Big(\pi_{J,J_n}(U), \mathcal{B}^{(n)}(\pi_{J,J_n}(U))\Big) \longrightarrow (\mathbf{R}, \mathcal{B})$$

is measurable, and so it is $\big(\mathcal{B}^{(J)}(U), \mathcal{B}\big)$-measurable, from Remark 2.6; analogously, $\forall\, i \in I \backslash I_n$, $\varphi'_{i,\sigma(i)} : \big(U, \mathcal{B}^{(J)}(U)\big) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable; then, $\forall\, h \in \mathbf{N}$, $h \geq n$, the function $f_h : \big(U, \mathcal{B}^{(J)}(U)\big) \longrightarrow (\mathbf{R}, \mathcal{B})$ defined by

$$f_h(x) = \det J_{\varphi^{(n,n)}}\left(x_{J_n}\right) \prod_{i \in I_h \backslash I_n} \varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right), \ \forall\, x \in U,$$

is measurable; furthermore, we have $\det J_{\overline{\varphi}^{(n,n)}}(x) = \lim_{h \to +\infty} f_h(x)$, $\forall\, x \in U$, and so $\det J_{\overline{\varphi}^{(n,n)}} : \big(U, \mathcal{B}^{(J)}(U)\big) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable too.

3. By assumption and from point 2, $\forall\, n \in \mathbf{N}$, $n \geq m$, there exists the function $\det J_{\overline{\varphi}^{(n,n)}} : \big(U, \mathcal{B}^{(J)}(U)\big) \longrightarrow (\mathbf{R}, \mathcal{B})$, and it is measurable; moreover, from Proposition 3.21, we have $\det J_{\varphi}(x) = \lim_{n \to +\infty} \det J_{\overline{\varphi}^{(n,n)}}(x)$, $\forall\, x \in U$, and so $\det J_{\varphi} : \big(U, \mathcal{B}^{(J)}(U)\big) \longrightarrow (\mathbf{R}, \mathcal{B})$ is measurable.

□

PROPOSITION 3.23. *Let $m \in \mathbf{N}^*$, let $n \in \mathbf{N}$, $n \geq m$, and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function m-general such that $\varphi^{(n,n)}$ is C^1; then, the function $\det J_{\varphi^{(n,n)}} : \big(\pi_{J,J_n}(U), \tau^{(n)}(\pi_{J,J_n}(U))\big) \longrightarrow (\mathbf{R}, \tau)$ is continuous.*

Proof. Since $\varphi^{(n,n)}$ is C^1, $\forall\, i \in I_n$, $\forall\, j \in J_n$, the function

$$\frac{\partial \varphi_i^{(n,n)}}{\partial x_j} : \Big(\pi_{J,J_n}(U), \tau^{(n)}(\pi_{J,J_n}(U))\Big) \longrightarrow (\mathbf{R}, \tau)$$

is continuous; then, by definition of determinant, the function $\det J_{\varphi^{(n,n)}} : \big(\pi_{J,J_n}(U), \tau^{(n)}(\pi_{J,J_n}(U))\big) \longrightarrow (\mathbf{R}, \tau)$ is continuous too. □

4. Change of variables' formula

DEFINITION 4.1. *Let $k \in \mathbf{N}^*$, let $M, N \in \mathbf{R}^+$, let $a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod_{i \in I : a_i \neq 0} a_i \in \mathbf{R}^+$, and let $v = (v_i : i \in I) \in E_I$; define the following sets in $\mathcal{B}_I$:*

$$E_{N,a,v}^{(k,I)} = \mathbf{R}^k \times \prod_{i \in I \backslash I_k} \left[v_i - \frac{N}{2} a_i, v_i + \frac{N}{2} a_i\right];$$

$$E_{M,N,a,v}^{(k,I)} = \prod_{h \in I_k} [v_h - M, v_h + M] \times \prod_{i \in I \backslash I_k} \left[v_i - \frac{N}{2} a_i, v_i + \frac{N}{2} a_i\right].$$

Moreover, define the σ-finite measure $\lambda_{N,a,v}^{(k,I)}$ over $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$ in the following manner:

$$\lambda_{N,a,v}^{(k,I)} = Leb^{(k)} \otimes \left(\bigotimes_{i \in I \setminus I_k} \frac{1}{N} Leb \left(\cdot \cap \left[v_i - \frac{N}{2} a_i, v_i + \frac{N}{2} a_i \right] \right) \right).$$

Lemma 4.2. *Let $k \in \mathbf{N}^*$, let $N \in \mathbf{R}^+$, let $a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod_{i \in I : a_i \neq 0} a_i \in \mathbf{R}^+$, and let $v = (v_i : i \in I) \in E_I$; then, for any measurable function $f : \left(\mathbf{R}^I, \mathcal{B}^{(I)}\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ such that f^+ (or f^-) is $\lambda_{N,a,v}^{(k,I)}$-integrable, one has*

$$\int_{\mathbf{R}^I} f d\lambda_{N,a,v}^{(k,I)} = \int_{E_{N,a,v}^{(k,I)}} f d\lambda_{N,a,v}^{(k,I)}.$$

Proof. See the proof of Lemma 46 in [5]. □

Proposition 4.3. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m, σ)-general function such that the function $\overline{\varphi}$ is bijective, and suppose that there exists $\varepsilon = (\varepsilon_i : i \in I \setminus I_m) \in [0, +\infty)^{I \setminus I_m}$ such that $\left| \varphi_i^{(I,m)} (x_{J_m}) \right| \leq \varepsilon_i$, for any $i \in I \setminus I_m$, for any $x_{J_m} \in U^{(m)}$, and such that $\prod_{i \in I \setminus I_m} (1 + 2\varepsilon_i) \in \mathbf{R}^+$; moreover, let $N \in [1, +\infty)$, let $a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod_{i \in I : a_i \neq 0} a_i \in \mathbf{R}^+$, and let $v \in E_I$; then:*

1. *There exist $b = (b_j : j \in J) \in [0, +\infty)^J$ and $z \in E_J$ such that $\prod_{j \in J : b_j \neq 0} b_j \in \mathbf{R}^+$ and such that, for any $l, n, k \in \mathbf{N}$, $l, n, k \geq m$, one has*

$$\varphi^{-1} \left(E_{N,a,v}^{(k,I)} \right) \subset E_{N,b,z}^{(k,J)},$$
$$\left(\overline{\varphi}^{(l,n)} \right)^{-1} \left(E_{N,a,v}^{(k,I)} \right) \subset E_{N,b,z}^{(k,J)}.$$

In particular, if φ is (m, σ)-standard, the statement is true for any $N \in \mathbf{R}^+$, and one has

$$\varphi^{-1} \left(E_{N,a,v}^{(k,I)} \right) = \left(\overline{\varphi}^{(l,n)} \right)^{-1} \left(E_{N,a,v}^{(k,I)} \right) = E_{N,b,z}^{(k,J)},$$
$$\varphi^{-1} \left(\overset{\circ}{E}{}_{N,a,v}^{(k,I)} \right) = \left(\overline{\varphi}^{(l,n)} \right)^{-1} \left(\overset{\circ}{E}{}_{N,a,v}^{(k,I)} \right) = \overset{\circ}{E}{}_{N,b,z}^{(k,J)}.$$

2. *Suppose that the function φ_{ij} is continuous, for any $i \in I_m$, for any $j \in J \backslash J_m$, and the function $\varphi^{(m,m)} : \left(U^{(m)}, \tau^{(m)}(U^{(m)})\right) \longrightarrow \left(\mathbf{R}^m, \tau^{(m)}\right)$ is open; then, for any $M \in \mathbf{R}^+$, there exists $O \in \mathbf{R}^+$ such that, for any $l, n, k \in \mathbf{N}$, $l, n, k \geq m$, one has*

$$\varphi^{-1}\left(E^{(k,I)}_{M,N,a,v}\right) \subset E^{(k,J)}_{O,N,b,z},$$
$$\left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(E^{(k,I)}_{M,N,a,v}\right) \subset E^{(k,J)}_{O,N,b,z}.$$

In particular, if φ is (m, σ)-standard, the statement is true for any $N \in \mathbf{R}^+$.

Proof. 1. Since $\overline{\varphi}$ is bijective, from Corollary 3.8, the functions $\varphi_{i,\sigma(i)}$, $\forall\, i \in I \backslash I_m$, and σ are bijective.

Let $N \in [1, +\infty)$, let $a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod_{i \in I : a_i \neq 0} a_i \in \mathbf{R}^+$, let $v \in E_I$, and let $\overline{a} = (\overline{a}_i : i \in I \backslash I_m) \in [0, +\infty)^{I \backslash I_m}$, where

$$\overline{a}_i = \begin{cases} \max\{1, a_i\} & \text{if } \varepsilon_i > 0 \\ a_i & \text{if } \varepsilon_i = 0 \end{cases}, \forall\, i \in I \backslash I_m;$$

define $b = (b_j : j \in J) \in [0, +\infty)^J$ and $z = (z_j : j \in J) \in [0, +\infty)^J$ such that $b_j = z_j = 1$, $\forall\, j \in J_m$; moreover, $\forall\, i \in I \backslash I_m$, set

$$\begin{aligned} b_{\sigma(i)} &= \frac{\left|\varphi^{-1}_{i,\sigma(i)}\left(v_i + \frac{N}{2}\overline{a}_i\left(1 + 2\varepsilon_i\right)\right) - \varphi^{-1}_{i,\sigma(i)}\left(v_i - \frac{N}{2}\overline{a}_i\left(1 + 2\varepsilon_i\right)\right)\right|}{N}, \\ z_{\sigma(i)} &= \frac{\varphi^{-1}_{i,\sigma(i)}\left(v_i - \frac{N}{2}\overline{a}_i\left(1 + 2\varepsilon_i\right)\right) + \varphi^{-1}_{i,\sigma(i)}\left(v_i + \frac{N}{2}\overline{a}_i\left(1 + 2\varepsilon_i\right)\right)}{2}. \end{aligned} \tag{12}$$

Observe that, $\forall\, i \in I \backslash I_m$, we have $b_{\sigma(i)} \neq 0$ if and only if $\overline{a}_i \neq 0$; then, since $\sigma\left(I \backslash I_m\right) = J \backslash J_m$, we have

$$\begin{aligned} &\prod_{j \in J : b_j \neq 0} b_j = \prod_{j \in J \backslash J_m : b_j \neq 0} b_j = \prod_{i \in I \backslash I_m : \overline{a}_i \neq 0} b_{\sigma(i)} \\ &= \left(\prod_{i \in I \backslash I_m : \overline{a}_i \neq 0} \frac{\left|\varphi^{-1}_{i,\sigma(i)}\left(v_i + \frac{N}{2}\overline{a}_i\left(1+2\varepsilon_i\right)\right) - \varphi^{-1}_{i,\sigma(i)}\left(v_i - \frac{N}{2}\overline{a}_i\left(1+2\varepsilon_i\right)\right)\right|}{N\overline{a}_i\left(1 + 2\varepsilon_i\right)}\right) \\ &\qquad \cdot \left(\prod_{i \in I \backslash I_m : \overline{a}_i \neq 0} \overline{a}_i\right)\left(\prod_{i \in I \backslash I_m : \overline{a}_i \neq 0} \left(1 + 2\varepsilon_i\right)\right). \end{aligned} \tag{13}$$

Moreover, $\forall\, i \in I \backslash I_m$ the function $\varphi_{i,\sigma(i)}^{-1}$ is derivable on $\mathbf{R}$; then, if $\overline{a}_i \neq 0$, the Lagrange theorem implies that, for some $\xi_i \in (v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\,, v_i + \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i))$, we have

$$\begin{aligned}
&\frac{\left|\varphi_{i,\sigma(i)}^{-1}\left(v_i + \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\right) - \varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\right)\right|}{N\overline{a}_i\,(1+2\varepsilon_i)} \\
&\qquad = \left|\left(\varphi_{i,\sigma(i)}^{-1}\right)'(\xi_i)\right| = \frac{1}{\left|\varphi_{i,\sigma(i)}'(\varphi_{i,\sigma(i)}^{-1}(\xi_i))\right|}; \qquad (14)
\end{aligned}$$

furthermore, $\forall\, i \in I \backslash I_m$, $\varphi_{i,\sigma(i)}$ is injective, and so $\mathcal{I}_\varphi = I \backslash I_m$; then

$$\prod_{i \in I \backslash I_m : \overline{a}_i \neq 0} \left|\varphi_{i,\sigma(i)}'(\varphi_{i,\sigma(i)}^{-1}(\xi_i))\right| = \prod_{i \in \mathcal{I}_\varphi : \overline{a}_i \neq 0} \left|\varphi_{i,\sigma(i)}'(\varphi_{i,\sigma(i)}^{-1}(\xi_i))\right| \in \mathbf{R}^+, \qquad (15)$$

from Definition 3.2. Moreover, we have

$$\prod_{i \in I \backslash I_m : \overline{a}_i \neq 0} \overline{a}_i = \left(\prod_{i \in I \backslash I_m : a_i > 1, \varepsilon_i > 0} a_i\right)\left(\prod_{i \in I \backslash I_m : a_i \neq 0, \varepsilon_i = 0} a_i\right) \in \mathbf{R}^+,$$

$$\prod_{i \in I \backslash I_m : \overline{a}_i \neq 0} (1+2\varepsilon_i) \in \mathbf{R}^+;$$

then, from formulas (13), (14) and (15), we obtain $\prod_{j \in J : b_j \neq 0} b_j \in \mathbf{R}^+$.

Moreover, let $x_0 = (x_{0,j} : j \in J) \in U$; $\forall\, i \in I \backslash I_m$, we have

$$\begin{aligned}
&\left|\varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\right)\right| \\
&\qquad = \left|\varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\right) - x_{0,\sigma(i)} + x_{0,\sigma(i)}\right| \\
&\leq \left|\varphi_{i,\sigma(i)}^{-1}(v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)) - \varphi_{i,\sigma(i)}^{-1}(\varphi_{i,\sigma(i)}(x_{0,\sigma(i)}))\right| + \left|x_{0,\sigma(i)}\right|; \qquad (16)
\end{aligned}$$

furthermore, from the Lagrange theorem, there exists $\zeta_i \in (\rho_i, \tau_i)$, where

$$\rho_i = \min\{v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\,, \varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\},$$
$$\tau_i = \max\{v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\,, \varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\},$$

such that

$$\begin{aligned}\left|\varphi_{i,\sigma(i)}^{-1}(v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)) - \varphi_{i,\sigma(i)}^{-1}(\varphi_{i,\sigma(i)}(x_{0,\sigma(i)}))\right| \\ = \left|\left(\varphi_{i,\sigma(i)}^{-1}\right)'(\zeta_i)\right|\left|v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i) - \varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\right| \\ = \frac{\left|v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i) - \varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\right|}{\left|\varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\zeta_i))\right|};\end{aligned}$$

thus, from (16), we obtain

$$\begin{aligned}\left|\varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\right)\right| \\ \leq \frac{\left|v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i) - \varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\right|}{\left|\varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\zeta_i))\right|} + \left|x_{0,\sigma(i)}\right|. \quad (17)\end{aligned}$$

We have $\sup_{i\in I\setminus I_m}\left|v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\right| \leq \|v\|_I + \frac{N}{2}\,\|\overline{a}\|_I\,(1+2\,\|\varepsilon\|_I) < +\infty$; moreover, from Definition 3.2, we have

$$\sup_{i\in I\setminus I_m}\left|\varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\right| = \sup_{i\in I\setminus I_m}\left|\varphi_i^{(I\setminus I_m, J\setminus J_m)}\left((x_0)_{J\setminus J_m}\right)\right| < +\infty,$$

$$\inf_{i\in I\setminus I_m}\left|\varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\zeta_i)\right| = \inf_{i\in \mathcal{I}_\varphi}\left|\varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\zeta_i)\right| > 0;$$

then, there exists $c \in \mathbf{R}^+$ such that $\sup_{i\in I\setminus I_m}\left|\varphi'_{i,\sigma(i)}(\varphi_{i,\sigma(i)}^{-1}(\zeta_i))\right|^{-1} \leq c$, and so formula (17) implies

$$\begin{aligned}\sup_{i\in I\setminus I_m}\left|\varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\right)\right| \\ \leq c\left(\sup_{i\in I\setminus I_m}\left|v_i - \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\right| + \sup_{i\in I\setminus I_m}\left|\varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\right|\right) \\ + \|x_0\|_J < +\infty.\end{aligned}$$

Analogously, we have

$$\sup_{i\in I\setminus I_m}\left|\varphi_{i,\sigma(i)}^{-1}\left(v_i + \frac{N}{2}\overline{a}_i\,(1+2\varepsilon_i)\right)\right| < +\infty;$$

then, from formula (12), we obtain that $\sup_{i\in I\setminus I_m}\left|z_{\sigma(i)}\right| < +\infty$, and so $z \in E_J$.

Moreover, let $k \in \mathbf{N}$, $k \geq m$, and let $x = (x_j : j \in J) \in \varphi^{-1}\left(E_{N,a,v}^{(k,I)}\right)$; $\forall\, i \in I \backslash I_k$, we have

$$\begin{aligned}
&\varphi_i^{(I,m)}\left(x_{J_m}\right) + \varphi_{i,\sigma(i)}(x_{\sigma(i)}) = \varphi_i(x) \in \left[v_i - \frac{N}{2}a_i, v_i + \frac{N}{2}a_i\right] \\
&\Rightarrow \varphi_{i,\sigma(i)}(x_{\sigma(i)}) \in \left[v_i - \frac{N}{2}a_i - \varphi_i^{(I,m)}\left(x_{J_m}\right), v_i + \frac{N}{2}a_i - \varphi_i^{(I,m)}\left(x_{J_m}\right)\right] \\
&\qquad \subset \left[v_i - \frac{N}{2}\overline{a}_i - \varepsilon_i, v_i + \frac{N}{2}\overline{a}_i + \varepsilon_i\right];
\end{aligned}$$

moreover, since $N \geq 1$, we have $\frac{N}{2}\overline{a}_i + \varepsilon_i \leq \frac{N}{2}\overline{a}_i\left(1 + 2\varepsilon_i\right)$, and so $x_{\sigma(i)} \in [\alpha_i, \beta_i]$, where

$$\begin{aligned}
\alpha_i &= \min\left\{\varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{N}{2}\overline{a}_i\left(1 + 2\varepsilon_i\right)\right), \varphi_{i,\sigma(i)}^{-1}\left(v_i + \frac{N}{2}\overline{a}_i\left(1 + 2\varepsilon_i\right)\right)\right\}, \\
\beta_i &= \max\left\{\varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{N}{2}\overline{a}_i\left(1 + 2\varepsilon_i\right)\right), \varphi_{i,\sigma(i)}^{-1}\left(v_i + \frac{N}{2}\overline{a}_i\left(1 + 2\varepsilon_i\right)\right)\right\};
\end{aligned}$$

thus, formula (12) implies

$$x_{\sigma(i)} \in \left[z_{\sigma(i)} - \frac{N}{2}b_{\sigma(i)}, z_{\sigma(i)} + \frac{N}{2}b_{\sigma(i)}\right]; \tag{18}$$

finally, since $\sigma\left(I \backslash I_k\right) = J \backslash J_k$, we obtain $\varphi^{-1}\left(E_{N,a,v}^{(k,I)}\right) \subset E_{N,b,z}^{(k,J)}$.

Furthermore, let $l, n \in \mathbf{N}$, $l, n \geq m$, and let

$$x = (x_j : j \in J) \in \left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(E_{N,a,v}^{(k,I)}\right);$$

$\forall\, i \in I_l \backslash I_k$, since $\varphi_i(x) = \overline{\varphi}_i^{(l,n)}(x)$, by repeating the previous arguments, we have formula (18); conversely, $\forall\, i \in I \backslash I_l$, we have

$$\varphi_{i,\sigma(i)}(x_{\sigma(i)}) = \varphi_i(x) \in \left[v_i - \frac{N}{2}a_i, v_i + \frac{N}{2}a_i\right],$$

and so $x_{\sigma(i)} \in [\gamma_i, \delta_i]$, where

$$\begin{aligned}
\gamma_i &= \min\left\{\varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{N}{2}a_i\right), \varphi_{i,\sigma(i)}^{-1}\left(v_i + \frac{N}{2}a_i\right)\right\}, \\
\delta_i &= \max\left\{\varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{N}{2}a_i\right), \varphi_{i,\sigma(i)}^{-1}\left(v_i + \frac{N}{2}a_i\right)\right\};
\end{aligned} \tag{19}$$

then, since $[\gamma_i, \delta_i] \subset [\alpha_i, \beta_i]$, we obtain formula (18) again; thus, we have $\left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(E^{(k,I)}_{N,a,v}\right) \subset E^{(k,J)}_{N,b,z}$.

In particular, if φ is (m,σ)-standard, $\forall\, i \in I\backslash I_m$, we have $\varepsilon_i = 0$, and so $\overline{a}_i = a_i$; then, $\forall\, N \in \mathbf{R}^+$, we have

$$\varphi^{-1}_{i,\sigma(i)}\left(\left[v_i - \frac{N}{2}a_i, v_i + \frac{N}{2}a_i\right]\right) = [\gamma_i, \delta_i]$$
$$= \left[z_{\sigma(i)} - \frac{N}{2}b_{\sigma(i)}, z_{\sigma(i)} + \frac{N}{2}b_{\sigma(i)}\right]; \quad (20)$$

thus, $\forall\, k \in \mathbf{N}$, $k \geq m$, we obtain $\varphi^{-1}\left(E^{(k,I)}_{N,a,v}\right) = E^{(k,J)}_{N,b,z}$, $\varphi^{-1}\left(\overset{\circ}{E}{}^{(k,I)}_{N,a,v}\right) = \overset{\circ}{E}{}^{(k,J)}_{N,b,z}$; analogously, $\forall\, l, n \in \mathbf{N}$, $l, n \geq m$, from formula (20), we have $\left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(E^{(k,I)}_{N,a,v}\right) = E^{(k,J)}_{N,b,z}$, $\left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(\overset{\circ}{E}{}^{(k,I)}_{N,a,v}\right) = \overset{\circ}{E}{}^{(k,J)}_{N,b,z}$.

2. Suppose that the function φ_{ij} is continuous, $\forall\, i \in I_m$, $\forall\, j \in J\backslash J_m$, and the function $\varphi^{(m,m)} : \left(U^{(m)}, \tau^{(m)}(U^{(m)})\right) \longrightarrow \left(\mathbf{R}^m, \tau^{(m)}\right)$ is open; since $\overline{\varphi}$ is bijective, from Corollary 3.8, $\varphi^{(m,m)}$ is bijective too; moreover, $\forall\, M \in \mathbf{R}^+$, consider the set

$$\overline{E}^{(I)}_{M,N,a,v} = \prod_{i\in I}\left[v_i - \frac{\overline{N}}{2}\overline{\overline{a}}_i, v_i + \frac{\overline{N}}{2}\overline{\overline{a}}_i\right],$$

where $\overline{N} = \max\{2M, N\} \in [1, +\infty)$, $\overline{\overline{a}}_i = \max\{1, a_i\}$, $\forall\, i \in I$. We have $\overline{E}^{(I)}_{M,N,a,v} \subset E^{(m,I)}_{\overline{N},\overline{\overline{a}},v}$, where $\overline{\overline{a}} = \left(\overline{\overline{a}}_i : i \in I\right) \in [1, +\infty)^I$; moreover, we have

$$\prod_{i\in I\backslash I_m : \overline{\overline{a}}_i \neq 0} \overline{\overline{a}}_i = \prod_{i\in I\backslash I_m : a_i > 1} a_i \in \mathbf{R}^+;$$

then, from point 1, there exist $\overline{b} = \left(\overline{b}_j : j \in J\right) \in [0, +\infty)^J$ and $\overline{z} \in E_J$ such that $\displaystyle\prod_{j\in J:\overline{b}_j\neq 0} \overline{b}_j \in \mathbf{R}^+$ and such that

$$\varphi^{-1}\left(\overline{E}^{(I)}_{M,N,a,v}\right) \subset \varphi^{-1}\left(E^{(m,I)}_{\overline{N},\overline{\overline{a}},v}\right) \subset E^{(m,J)}_{\overline{N},\overline{b},\overline{z}};$$

then, $\forall\, x = (x_j : j \in J) \in \varphi^{-1}\left(\overline{E}^{(I)}_{M,N,a,v}\right)$, we have $\left\|x_{J\backslash J_m}\right\|_{J\backslash J_m} \leq \|\overline{z}\|_{J\backslash J_m} + \frac{\overline{N}}{2}\left\|\overline{b}\right\|_{J\backslash J_m} \equiv O_1 \in \mathbf{R}^+$. Moreover, $\forall\, i \in I_m$, we have

$$\varphi_i(x) = \varphi^{(m,m)}_i(x_{J_m}) + \sum_{j\in J\backslash J_m} \varphi_{ij}(x_j),$$

and so

$$x_{J_m} = \left(\varphi^{(m,m)}\right)^{-1} w_{I_m}, \tag{21}$$

where

$$w_i = \varphi_i(x) - \sum_{j \in J \setminus J_m} \varphi_{ij}\left(x_j\right), \forall\, i \in I_m; \tag{22}$$

furthermore, $\forall\, i \in I \setminus I_m$, we have

$$\varphi_i^{(I,m)}\left(x_{J_m}\right) + \varphi_{i,\sigma(i)}(x_{\sigma(i)}) = \varphi_i(x) \in \left[v_i - \frac{\overline{N}}{2}\overline{\overline{a}}_i, v_i + \frac{\overline{N}}{2}\overline{\overline{a}}_i\right]$$

$$\Rightarrow \varphi_{i,\sigma(i)}(x_{\sigma(i)}) \in \left[v_i - \frac{\overline{N}}{2}\overline{\overline{a}}_i - \varphi_i^{(I,m)}\left(x_{J_m}\right), v_i + \frac{\overline{N}}{2}\overline{\overline{a}}_i - \varphi_i^{(I,m)}\left(x_{J_m}\right)\right]$$

$$\subset \left[v_i - \frac{\overline{N}}{2}\overline{\overline{a}}_i - \varepsilon_i, v_i + \frac{\overline{N}}{2}\overline{\overline{a}}_i + \varepsilon_i\right],$$

and so

$$x_{\sigma(i)} \in \left[\overline{\alpha}_i, \overline{\beta}_i\right] \subset A_{\sigma(i)}, \tag{23}$$

where

$$\overline{\alpha}_i = \min\left\{\varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{\overline{N}}{2}\overline{\overline{a}}_i - \varepsilon_i\right), \varphi_{i,\sigma(i)}^{-1}\left(v_i + \frac{\overline{N}}{2}\overline{\overline{a}}_i + \varepsilon_i\right)\right\},$$

$$\overline{\beta}_i = \max\left\{\varphi_{i,\sigma(i)}^{-1}\left(v_i - \frac{\overline{N}}{2}\overline{\overline{a}}_i - \varepsilon_i\right), \varphi_{i,\sigma(i)}^{-1}\left(v_i + \frac{\overline{N}}{2}\overline{\overline{a}}_i + \varepsilon_i\right)\right\};$$

then, since $\forall\, i \in I_m$, $\forall\, j \in J \setminus J_m$, the function φ_{ij} is continuous, there exists $O_2 = O_2\left(\varphi, M, N, a, v\right) \in \mathbf{R}^+$ such that

$$\sup_{i \in I_m} \sum_{j \in J \setminus J_m} \left|\varphi_{ij}\left(x_j\right)\right| \leq O_2,$$

and so $\left\|w_{I_m}\right\|_{I_m} \leq \left\|v\right\|_{I_m} + \frac{\overline{N}}{2}\left\|\overline{\overline{a}}\right\|_{I_m} + O_2 \equiv O_3 \in \mathbf{R}^+$, from (22); then, since the function $\left(\varphi^{(m,m)}\right)^{-1}$ is continuous, from (21), we have $\left\|x_{J_m}\right\|_{J_m} \leq O_4$, for some $O_4 = O_4\left(\varphi, M, N, a, v\right) \in \mathbf{R}^+$ such that

$$\left(\varphi^{(m,m)}\right)^{-1}\left(\left[-O_3, O_3\right]^m\right) \subset \left[-O_4, O_4\right]^m,$$

and so $\left\|x\right\|_J \leq \max\left\{O_1, O_4\right\}$. Thus, if b, z are the sequences defined by point 1, we have

$$\varphi^{-1}\left(\overline{E}_{M,N,a,v}^{(I)}\right) \subset \prod_{j \in J}\left[-\max\left\{O_1, O_4\right\}, \max\left\{O_1, O_4\right\}\right]$$

$$\subset \prod_{j \in J}\left[z_j - O, z_j + O\right], \tag{24}$$

where $O \equiv \max\{O_1, O_4\} + \|z\|_J \in \mathbf{R}^+$; moreover, $\forall\, k \in \mathbf{N}$, $k \geq m$, we have $E^{(k,I)}_{M,N,a,v} \subset E^{(k,I)}_{N,a,v} \cap \overline{E}^{(I)}_{M,N,a,v}$; then, from formula (24), we obtain

$$\varphi^{-1}\left(E^{(k,I)}_{M,N,a,v}\right) \subset \varphi^{-1}\left(E^{(k,I)}_{N,a,v}\right) \cap \varphi^{-1}\left(\overline{E}^{(I)}_{M,N,a,v}\right) \\ \subset E^{(k,J)}_{N,b,z} \cap \prod_{j\in J}\left[z_j - O, z_j + O\right] \subset E^{(k,J)}_{O,N,b,z}.$$

Furthermore, let $l, n \in \mathbf{N}$, $l, n \geq m$; from point 1, we have

$$\left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(\overline{E}^{(I)}_{M,N,a,v}\right) \subset \left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(E^{(m,I)}_{\overline{N},\overline{\overline{a}},v}\right) \subset E^{(m,J)}_{\overline{N},\overline{b},\overline{z}};$$

then, $\forall\, x = (x_j : j \in J) \in \left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(\overline{E}^{(I)}_{M,N,a,v}\right)$, we have $\left\|x_{J\setminus J_m}\right\|_{J\setminus J_m} \leq O_1$. Moreover, $\forall\, i \in I_m$, we have

$$\overline{\varphi}^{(l,n)}_i(x) = \varphi^{(m,m)}_i(x_{J_m}) + \sum_{j\in J_n\setminus J_m} \varphi_{ij}(x_j),$$

and so

$$x_{J_m} = \left(\varphi^{(m,m)}\right)^{-1} \overline{w}_{I_m}, \tag{25}$$

where

$$\overline{w}_i = \overline{\varphi}^{(l,n)}_i(x) - \sum_{j\in J_n\setminus J_m} \varphi_{ij}\left(x_j\right), \forall\, i \in I_m; \tag{26}$$

furthermore, $\forall\, i \in I_l\setminus I_m$, since $\varphi_i(x) = \overline{\varphi}^{(l,n)}_i(x)$, we have formula (23). Finally, $\forall\, i \in I\setminus I_l$, we have

$$\varphi_{i,\sigma(i)}(x_{\sigma(i)}) = \overline{\varphi}^{(l,n)}_i(x) \in \left[v_i - \frac{\overline{N}}{2}\overline{\overline{a}}_i, v_i + \frac{\overline{N}}{2}\overline{\overline{a}}_i\right] \\ \Rightarrow x_{\sigma(i)} \in \left[\overline{\gamma}_i, \overline{\delta}_i\right],$$

where

$$\begin{aligned} \overline{\gamma}_i &= \min\left\{\varphi^{-1}_{i,\sigma(i)}\left(v_i - \frac{\overline{N}}{2}\overline{\overline{a}}_i\right), \varphi^{-1}_{i,\sigma(i)}\left(v_i + \frac{\overline{N}}{2}\overline{\overline{a}}_i\right)\right\}, \\ \overline{\delta}_i &= \max\left\{\varphi^{-1}_{i,\sigma(i)}\left(v_i - \frac{\overline{N}}{2}\overline{\overline{a}}_i\right), \varphi^{-1}_{i,\sigma(i)}\left(v_i + \frac{\overline{N}}{2}\overline{\overline{a}}_i\right)\right\}; \end{aligned}$$

then, since $\left[\overline{\gamma}_i, \overline{\delta}_i\right] \subset \left[\overline{\alpha}_i, \overline{\beta}_i\right]$, we obtain formula (23) again, from which

$$\sup_{i\in I_m} \sum_{j\in J_n\setminus J_m} \left|\varphi_{ij}\left(x_j\right)\right| \leq \sup_{i\in I_m} \sum_{j\in J\setminus J_m} \left|\varphi_{ij}\left(x_j\right)\right| \leq O_2,$$

and so $\|\overline{w}_{I_m}\|_{I_m} \leq O_3$, from (26).

Then, since the function $\left(\varphi^{(m,m)}\right)^{-1}$ is continuous, from (25), we have $\|x_{J_m}\|_{J_m} \leq O_4$, and so $\|x\|_J \leq \max\{O_1, O_4\}$. Thus, we have

$$\left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(\overline{E}^{(I)}_{M,N,a,v}\right) \subset \prod_{j\in J}\left[-\max\{O_1,O_4\}, \max\{O_1,O_4\}\right] \subset \prod_{j\in J}\left[z_j - O, z_j + O\right]; \quad (27)$$

finally, $\forall\, k \in \mathbf{N}$, $k \geq m$, from point 1 and formula (27), we obtain

$$\left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(E^{(k,I)}_{M,N,a,v}\right) \subset \left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(E^{(k,I)}_{N,a,v}\right) \cap \left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(\overline{E}^{(I)}_{M,N,a,v}\right) \subset E^{(k,J)}_{N,b,z} \cap \prod_{j\in J}\left[z_j - O, z_j + O\right] \subset E^{(k,J)}_{O,N,b,z}.$$

In particular, if φ is (m,σ)-standard, $\forall\, N \in \mathbf{R}^+$, $\forall\, l,n,k \in \mathbf{N}$, $l,n,k \geq m$, from point 1, we have

$$\varphi^{-1}\left(E^{(k,I)}_{N,a,v}\right) = \left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(E^{(k,I)}_{N,a,v}\right) = E^{(k,J)}_{N,b,z};$$

moreover, we have formulas (24) and (27) again, from which

$$\varphi^{-1}\left(E^{(k,I)}_{M,N,a,v}\right) \subset E^{(k,J)}_{O,N,b,z},$$
$$\left(\overline{\varphi}^{(l,n)}\right)^{-1}\left(E^{(k,I)}_{M,N,a,v}\right) \subset E^{(k,J)}_{O,N,b,z}.$$

□

PROPOSITION 4.4. *Let (S,Σ) be a measurable space, let $\mathcal{I}$ be a π-system on S, and let μ_1 and μ_2 be two measures on (S,Σ), σ- finite on $\mathcal{I}$; if $\sigma(\mathcal{I}) = \Sigma$ and μ_1 and μ_2 coincide on $\mathcal{I}$, then μ_1 and μ_2 coincide on Σ.*

Proof. See, for example, Theorem 10.3 in Billingsley [8]. □

Now, we can prove the main result of our paper, that improves Theorem 47 in [5], and generalizes the change of variables' formula for the integration of a measurable function on $\mathbf{R}^m$ with values in $\mathbf{R}$ (see, for example, the Lang's book [11]).

THEOREM 4.5. *(Change of variables' formula). Let* $\varphi : U \subset E_J \longrightarrow E_I$ *be a bijective, continuous and* (m, σ)*-general function, such that* $\pi_{I,I\setminus I_m} \circ \overline{\varphi}$ *is continuous and such that, for any* $n \in \mathbf{N}$, $n \geq m$, *the function* $\overline{\varphi}^{(n,n)} : U \longrightarrow E_I$ *is a diffeomorphism; moreover, suppose that there exists* $\varepsilon = (\varepsilon_i : i \in I\setminus I_m) \in (\mathbf{R}^+)^{I\setminus I_m}$ *such that* $\left|\varphi_i^{(I,m)}(x_{J_m})\right| \leq \varepsilon_i$, *for any* $i \in I\setminus I_m$, *for any* $x_{J_m} \in U^{(m)}$, *and such that* $\prod\limits_{i \in I\setminus I_m} (1 + 2\varepsilon_i) \in \mathbf{R}^+$*; furthermore, suppose that the sequence* $\left\{\left(\overline{\varphi}^{(n,n)}\right)^{-1}\right\}_{n\geq m}$ *converges uniformly to* φ^{-1} *over the closed and bounded subsets of* E_I, *and the sequence* $\left\{\det J_{\overline{\varphi}^{(n,n)}}\right\}_{n\geq m}$ *converges uniformly over the closed and bounded subsets of* U*; finally, let* $N \in [1, +\infty)$, *let* $a = (a_i : i \in I) \in [0, +\infty)^I$ *such that* $\prod\limits_{i \in I: a_i \neq 0} a_i \in \mathbf{R}^+$, *let* $v \in E_I$, *and let* $b \in [0, +\infty)^J$ *and* $z \in E_J$ *defined by Proposition 4.3. Then, for any* $k \in \mathbf{N}$, $k \geq m$, *for any* $B \in \mathcal{B}^{(I)}\left(\overset{\circ}{E}{}^{(k,I)}_{N,a,v}\right)$ *and for any measurable function* $f : \left(\mathbf{R}^I, \mathcal{B}^{(I)}\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ *such that* f^+ *(or* f^-*) is* $\lambda^{(k,I)}_{N,a,v}$*-integrable, one has*

$$\int\limits_B f d\lambda^{(k,I)}_{N,a,v} = \int\limits_{\varphi^{-1}(B)} f(\varphi) \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda^{(k,J)}_{N,b,z}.$$

In particular, assume that, for any $x \in U$, *there exists the function* $J_\varphi(x) : E_J \longrightarrow E_I$*; then, one has*

$$\int\limits_B f d\lambda^{(k,I)}_{N,a,v} = \int\limits_{\varphi^{-1}(B)} f(\varphi) \left|\det J_\varphi\right| d\lambda^{(k,J)}_{N,b,z}.$$

Proof. The previous assumptions imply that $\overline{\varphi}$ is bijective, φ_{ij} is continuous, $\forall\, i \in I_m$, $\forall\, j \in J\setminus J_m$, and $\varphi^{(m,m)} : \left(U^{(m)}, \tau^{(m)}(U^{(m)})\right) \longrightarrow \left(\mathbf{R}^m, \tau^{(m)}\right)$ is open; thus, $\forall\, M \in \mathbf{R}^+$, $\forall\, N \in [1, +\infty)$, $\forall\, a = (a_i : i \in I) \in [0, +\infty)^I$ such that $\prod\limits_{i \in I: a_i \neq 0} a_i \in \mathbf{R}^+$, and $\forall\, v \in E_I$, let $O \in \mathbf{R}^+$ and let b, z be the sequences defined by Proposition 4.3. Then, $\forall\, n, k \in \mathbf{N}$, $n \geq k \geq m$, $\forall\, B = \prod\limits_{i\in I} B_i \in \mathcal{B}^{(I)}\left(E^{(k,I)}_{M,N,a,v}\right)$ and $\forall\, i \in I\setminus I_n$, we have $B_i \in \mathcal{B}\left(\left[v_i - \frac{N}{2}a_i, v_i + \frac{N}{2}a_i\right]\right)$; moreover, since $\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B) \subset E^{(k,J)}_{N,b,z}$, we have

$$\varphi^{-1}_{i,\sigma(i)}(B_i) \in \mathcal{B}\left(\left[z_{\sigma(i)} - \frac{N}{2} b_{\sigma(i)}, z_{\sigma(i)} + \frac{N}{2} b_{\sigma(i)}\right]\right),$$

from which

$$\int\limits_B d\lambda_{N,a,v}^{(k,I)} = \int\limits_{\prod\limits_{p\in I} B_p} d\left(Leb^{(k)} \otimes \left(\bigotimes_{q\in I\setminus I_k} \frac{1}{N} Leb\Big|_{\mathcal{B}\left(\left[v_q-\frac{N}{2}a_q, v_q+\frac{N}{2}a_q\right]\right)}\right)\right)$$

$$= \frac{1}{N^{n-k}} \int\limits_{\prod\limits_{p\in I_n} B_p \times \prod\limits_{q\in I\setminus I_n} B_q} d\left(Leb^{(n)} \otimes \left(\bigotimes_{q\in I\setminus I_n} \frac{1}{N} Leb\Big|_{\mathcal{B}\left(\left[v_q-\frac{N}{2}a_q, v_q+\frac{N}{2}a_q\right]\right)}\right)\right)$$

$$= \frac{1}{N^{n-k}} \int\limits_{\prod\limits_{p\in I_n} B_p} dLeb^{(n)} \cdot \int\limits_{\prod\limits_{q\in I\setminus I_n} B_q} d\left(\bigotimes_{q\in I\setminus I_n} \frac{1}{N} Leb\Big|_{\mathcal{B}\left(\left[v_q-\frac{N}{2}a_q, v_q+\frac{N}{2}a_q\right]\right)}\right). \quad (28)$$

Moreover, we have

$$\int\limits_{\prod\limits_{q\in I\setminus I_n} B_q} d\left(\bigotimes_{q\in I\setminus I_n} \frac{1}{N} Leb\Big|_{\mathcal{B}\left(\left[v_q-\frac{N}{2}a_q, v_q+\frac{N}{2}a_q\right]\right)}\right) = \int\limits_{\prod\limits_{q\in I\setminus I_n} B_q} d\left(\bigotimes_{q\in I\setminus I_n} \frac{1}{N} Leb\Big|_{\mathcal{B}(B_q)}\right)$$

$$= \lim_{p\to+\infty} \int\limits_{\prod\limits_{q\in I_p\setminus I_n} B_q} d\left(\bigotimes_{q\in I_p\setminus I_n} \frac{1}{N} Leb\Big|_{\mathcal{B}(B_q)}\right) \qquad \text{(by Theorem 2.1)}$$

$$= \lim_{p\to+\infty} \int\limits_{\prod\limits_{q\in I_p\setminus I_n} \varphi_{q,\sigma(q)}^{-1}(B_q)} \prod_{q\in I_p\setminus I_n} \left|\varphi'_{q,\sigma(q)}\right| \cdot d\left(\bigotimes_{q\in I_p\setminus I_n} \frac{1}{N} Leb\Big|_{\mathcal{B}(\varphi_{q,\sigma(q)}^{-1}(B_q))}\right)$$

(since, $\forall\, q \in I_p\setminus I_n$, $\varphi_{q,\sigma(q)}$ is a diffeomorphism, by Proposition 3.13)

$$= \int\limits_{\prod\limits_{q\in I\setminus I_n} \varphi_{q,\sigma(q)}^{-1}(B_q)} \prod_{q\in I\setminus I_n} \left|\varphi'_{q,\sigma(q)}\right| \cdot d\left(\bigotimes_{q\in I\setminus I_n} \frac{1}{N} Leb\Big|_{\mathcal{B}(\varphi_{q,\sigma(q)}^{-1}(B_q))}\right)$$

(by Theorem 2.2)

$$= \int\limits_{\prod\limits_{q\in I\setminus I_n} \varphi_{q,\sigma(q)}^{-1}(B_q)} \prod_{q\in I\setminus I_n} \left|\varphi'_{q,\sigma(q)}\right| \cdot d\left(\bigotimes_{q\in I\setminus I_n} \frac{1}{N} Leb\Big|_{\mathcal{B}\left(\left[z_{\sigma(q)}-\frac{N}{2}b_{\sigma(q)}, z_{\sigma(q)}+\frac{N}{2}b_{\sigma(q)}\right]\right)}\right).$$

Moreover, from Proposition 3.13, $\varphi^{(n,n)}$ is a diffeomorphism, and so formula

(28) implies

$$\int_B d\lambda_{N,a,v}^{(k,I)} = \frac{1}{N^{n-k}} \int_{\left(\varphi^{(n,n)}\right)^{-1}\left(\prod_{p\in I_n} B_p\right)} |\det J_{\varphi^{(n,n)}}| dLeb^{(n)}$$

$$\cdot \int_{\prod_{q\in I\setminus I_n} \varphi_{q,\sigma(q)}^{-1}(B_q)} \prod_{q\in I\setminus I_n} \left|\varphi'_{q,\sigma(q)}\right| \cdot d\left(\bigotimes_{q\in I\setminus I_n} \frac{1}{N} Leb\Big|_{\mathcal{B}\left(\left[z_{\sigma(q)}-\frac{N}{2}b_{\sigma(q)}, z_{\sigma(q)}+\frac{N}{2}b_{\sigma(q)}\right]\right)}\right)$$

$$= \frac{1}{N^{n-k}} \int_{\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B)} |\det J_{\overline{\varphi}^{(n,n)}}| d\left(Leb^{(n)}\right.$$

$$\left.\otimes\left(\bigotimes_{q\in I\setminus I_n} \frac{1}{N} Leb\Big|_{\mathcal{B}\left(\left[z_{\sigma(q)}-\frac{N}{2}b_{\sigma(q)}, z_{\sigma(q)}+\frac{N}{2}b_{\sigma(q)}\right]\right)}\right)\right)$$

$$= \int_{\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B)} |\det J_{\overline{\varphi}^{(n,n)}}| d\left(Leb^{(k)}\right.$$

$$\left.\otimes\left(\bigotimes_{q\in I\setminus I_k} \frac{1}{N} Leb\Big|_{\mathcal{B}\left(\left[z_{\sigma(q)}-\frac{N}{2}b_{\sigma(q)}, z_{\sigma(q)}+\frac{N}{2}b_{\sigma(q)}\right]\right)}\right)\right)$$

$$\left(\text{since } \left(\overline{\varphi}^{(n,n)}\right)^{-1}(B) \subset E_{N,b,z}^{(k,J)}\right)$$

$$= \int_{\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B)} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)}. \tag{29}$$

Consider the measures μ_1 and μ_2 on $\Sigma \equiv \mathcal{B}^{(I)}\left(E_{M,N,a,v}^{(k,I)}\right)$ defined by

$$\mu_1(B) = \int_B d\lambda_{N,a,v}^{(k,I)},$$

$$\mu_2(B) = \int_{\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B)} |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)};$$

from (29), μ_1 and μ_2 coincide on the set

$$\mathcal{I} = \left\{B \in \Sigma : B = \prod_{i\in I} B_i\right\};$$

moreover, we have $\mu_1\left(E^{(k,I)}_{M,N,a,v}\right)=\mu_2\left(E^{(k,I)}_{M,N,a,v}\right)<+\infty$, $E^{(k,I)}_{M,N,a,v}\in\mathcal{I}$, and so μ_1 and μ_2 are σ- finite on $\mathcal{I}$. Then, since $\mathcal{I}$ is a π-system on $E^{(k,I)}_{M,N,a,v}$ such that $\sigma(\mathcal{I})=\Sigma$, from Proposition 4.4, $\forall\, B\in\mathcal{B}^{(I)}\left(E^{(k,I)}_{M,N,a,v}\right)$, we have

$$\int\limits_B d\lambda^{(k,I)}_{N,a,v}=\int\limits_{\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B)}|\det J_{\overline{\varphi}^{(n,n)}}|d\lambda^{(k,J)}_{N,b,z}. \tag{30}$$

Moreover, since $E^{(k,I)}_{M,N,a,v}$ is closed and bounded, the sequence $\left\{\left(\overline{\varphi}^{(n,n)}\right)^{-1}\right\}_{n\geq k}$ converges uniformly to φ^{-1} over $E^{(k,I)}_{M,N,a,v}$; furthermore, since φ is continuous, $\varphi^{-1}\left(E^{(k,I)}_{M,N,a,v}\right)$ is closed; then, there exist $\overline{n}\in\mathbf{N}$, $\overline{n}\geq k$, and $\delta\in\mathbf{R}^+$ such that, $\forall\, i>\overline{n}$, $\left(\overline{\varphi}^{(i,i)}\right)^{-1}\left(E^{(k,I)}_{M,N,a,v}\right)\subset\varphi^{-1}\left(E^{(k,I)}_{M,N,a,v}\right)+\overline{B_J(0,\delta)}\subset U$, from which

$$\begin{aligned}\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B)&\subset\bigcup_{h\geq k}\left(\overline{\varphi}^{(h,h)}\right)^{-1}\left(E^{(k,I)}_{M,N,a,v}\right)\\&\subset\left(\bigcup_{h=k}^{\overline{n}}\left(\overline{\varphi}^{(h,h)}\right)^{-1}\left(E^{(k,I)}_{M,N,a,v}\right)\right)\bigcup\left(\varphi^{-1}\left(E^{(k,I)}_{M,N,a,v}\right)+\overline{B_J(0,\delta)}\right),\\&\qquad\forall\, n\geq k;\end{aligned}$$

then, from Proposition 4.3, $\forall\, n\geq k$, we have

$$\begin{aligned}\left(\overline{\varphi}^{(n,n)}\right)^{-1}(B)&\subset E^{(k,J)}_{O,N,b,z}\\&\cap\left(\left(\bigcup_{h=k}^{\overline{n}}\left(\overline{\varphi}^{(h,h)}\right)^{-1}\left(E^{(k,I)}_{M,N,a,v}\right)\right)\bigcup\left(\varphi^{-1}\left(E^{(k,I)}_{M,N,a,v}\right)+\overline{B_J(0,\delta)}\right)\right)\\&\equiv E^{(k,I,\varphi,\delta)}_{M,N,a,v}\subset U,\end{aligned}\tag{31}$$

and so

$$\int\limits_{E^{(k,I)}_{M,N,a,v}}1_B d\lambda^{(k,I)}_{N,a,v}=\int\limits_{E^{(k,I,\varphi,\delta)}_{M,N,a,v}}1_B(\overline{\varphi}^{(n,n)})|\det J_{\overline{\varphi}^{(n,n)}}|d\lambda^{(k,J)}_{N,b,z}. \tag{32}$$

Moreover, $\forall\, h\in\{k,...,\overline{n}\}$, $\varphi^{(h,h)}$ is continuous, since from Proposition 3.13 it is a diffeomorphism; then, since $\pi_{I,I\setminus I_m}\circ\overline{\varphi}$ is continuous, from Proposition 3.10, $\overline{\varphi}^{(h,h)}$ is continuous too, and so formula (31) implies that $E^{(k,I,\varphi,\delta)}_{M,N,a,v}$ is a closed subset of U; furthermore, we have $E^{(k,I,\varphi,\delta)}_{M,N,a,v} \subset E^{(k,J)}_{O,N,b,z}$, and so $E^{(k,I,\varphi,\delta)}_{M,N,a,v}$ is bounded.

From formula (32), if $\psi : \left(\mathbf{R}^I, \mathcal{B}^{(\mathbf{I})}\right) \longrightarrow ([0,+\infty), \mathcal{B}([0,+\infty)))$ is a simple function such that $\psi(x) = 0$, $\forall\, x \notin E^{(k,I)}_{M,N,a,v}$, we have

$$\int\limits_{E^{(k,I)}_{M,N,a,v}} \psi d\lambda^{(k,I)}_{N,a,v} = \int\limits_{E^{(k,I,\varphi,\delta)}_{M,N,a,v}} \psi(\overline{\varphi}^{(n,n)}) |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda^{(k,J)}_{N,b,z}.$$

Then, if $l : \left(\mathbf{R}^I, \mathcal{B}^{(\mathbf{I})}\right) \longrightarrow ([0,+\infty), \mathcal{B}([0,+\infty)))$ is a measurable function such that $l(x) = 0$, $\forall\, x \notin E^{(k,I)}_{M,N,a,v}$, and $\{\psi_i\}_{i\in\mathbf{N}}$ is a sequence of increasing positive simple functions over $\left(\mathbf{R}^I, \mathcal{B}^{(\mathbf{I})}\right)$ such that $\lim\limits_{i\longrightarrow+\infty} \psi_i = l$, $\psi_i(x) = 0$, $\forall\, x \notin E^{(k,I)}_{M,N,a,v}$, $\forall\, i \in \mathbf{N}$, from Beppo Levi theorem we have

$$\begin{aligned}
\int\limits_{E^{(k,I)}_{M,N,a,v}} l d\lambda^{(k,I)}_{N,a,v} &= \lim_{i\longrightarrow+\infty} \int\limits_{E^{(k,I)}_{M,N,a,v}} \psi_i d\lambda^{(k,I)}_{N,a,v} \\
&= \lim_{i\longrightarrow+\infty} \int\limits_{E^{(k,I,\varphi,\delta)}_{M,N,a,v}} \psi_i(\overline{\varphi}^{(n,n)}) |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda^{(k,J)}_{N,b,z} \\
&= \int\limits_{E^{(k,I,\varphi,\delta)}_{M,N,a,v}} l(\overline{\varphi}^{(n,n)}) |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda^{(k,J)}_{N,b,z}, \qquad (33)
\end{aligned}$$

from which

$$\int\limits_{E^{(k,I)}_{M,N,a,v}} l d\lambda^{(k,I)}_{N,a,v} = \lim_{n\longrightarrow+\infty} \int\limits_{E^{(k,I,\varphi,\delta)}_{M,N,a,v}} l(\overline{\varphi}^{(n,n)}) |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda^{(k,J)}_{N,b,z}. \qquad (34)$$

In particular, formula (34) is true if $l : \mathbf{R}^I \longrightarrow [0,+\infty)$ is $\left(\mathcal{B}^{(\mathbf{I})}, \mathcal{B}([0,+\infty))\right)$-measurable, $\left(\tau^{(I)}, \tau([0,+\infty))\right)$-continuous and such that $l\left(\mathbf{R}^I\right) \subset [0,1]$, $l(x) = 0$, $\forall\, x \notin E^{(k,I)}_{M,N,a,v}$. In this case, let $\{f_n\}_{n\geq k}$ be the sequence of the measurable functions

$$f_n : \left(E^{(k,I,\varphi,\delta)}_{M,N,a,v}, \mathcal{B}^{(J)}\left(E^{(k,I,\varphi,\delta)}_{M,N,a,v}\right)\right) \longrightarrow ([0,+\infty), \mathcal{B}([0,+\infty)))$$

given by

$$f_n(x) = l(\overline{\varphi}^{(n,n)}(x)) |\det J_{\overline{\varphi}^{(n,n)}}(x)|,\ \forall\, x \in E^{(k,I,\varphi,\delta)}_{M,N,a,v},\ \forall\, n \geq k;$$

since $E^{(k,I,\varphi,\delta)}_{M,N,a,v}$ is closed and bounded, the sequence $\left\{\det J_{\overline{\varphi}^{(n,n)}}\right\}_{n\geq k}$ converges uniformly over $E^{(k,I,\varphi,\delta)}_{M,N,a,v}$; then, there exists $\widehat{n} \in \mathbf{N}$, $\widehat{n} \geq k$, such that, $\forall\, x \in$

$E_{M,N,a,v}^{(k,I,\varphi,\delta)}$, $\forall\, n > \widehat{n}$, we have $|\det J_{\overline{\varphi}^{(n,n)}}(x)| \leq |\det J_{\overline{\varphi}^{(\hat{n},\hat{n})}}(x)| + 1$; thus, since $l\left(\mathbf{R}^I\right) \subset [0,1]$, $\forall\, n \geq k$, we have $|f_n| \leq |\det J_{\overline{\varphi}^{(n,n)}}| \leq g$, where

$$g : \left(E_{M,N,a,v}^{(k,I,\varphi,\delta)}, \mathcal{B}^{(J)}\left(E_{M,N,a,v}^{(k,I,\varphi,\delta)}\right)\right) \longrightarrow ([0,+\infty), \mathcal{B}\left([0,+\infty)\right))$$

is the measurable function defined by

$$g(x) = \sum_{h=k}^{\widehat{n}} |\det J_{\overline{\varphi}^{(h,h)}}(x)| + |\det J_{\overline{\varphi}^{(\hat{n},\hat{n})}}(x)| + 1,\ \forall\, x \in E_{M,N,a,v}^{(k,I,\varphi,\delta)}. \tag{35}$$

Moreover, $\forall\, h \in \{k, ..., \widehat{n}\}$, we have

$$|\det J_{\overline{\varphi}^{(h,h)}}(x)| = |\det J_{\varphi^{(h,h)}}(x_{J_h})| \prod_{i \in I \setminus I_h} \left|\varphi'_{i,\sigma(i)}(x_{\sigma(i)})\right|,\ \forall\, x \in E_{M,N,a,v}^{(k,I,\varphi,\delta)}; \tag{36}$$

furthermore, from Proposition 3.23 and Proposition 3.13, $\forall\, h \in \{k, ..., \widehat{n}\}$, $\forall\, i \in I \backslash I_h$, the functions $\det J_{\varphi^{(h,h)}}$ and $\varphi'_{i,\sigma(i)}$ are continuous; then, since the sets $\pi_{J,J_h}\left(E_{M,N,a,v}^{(k,I,\varphi,\delta)}\right)$ and $\pi_{J,\{\sigma(i)\}}\left(E_{M,N,a,v}^{(k,I,\varphi,\delta)}\right)$ are closed and bounded, from formulas (35) and (36), there exists $\beta \in \mathbf{R}^+$ such that $g(x) \leq \beta$, $\forall\, x \in E_{M,N,a,v}^{(k,I,\varphi,\delta)}$; thus, by definition of $E_{M,N,a,v}^{(k,I,\varphi,\delta)}$, we have

$$\begin{aligned}
\int\limits_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} g d\lambda_{N,b,z}^{(k,J)} &\leq \beta \lambda_{N,b,z}^{(k,J)}\left(E_{M,N,a,v}^{(k,I,\varphi,\delta)}\right) \leq \beta \lambda_{N,b,z}^{(k,J)}\left(E_{O,N,b,z}^{(k,J)}\right) \\
&= \beta \prod_{p \in J_k} Leb\left([z_p - O, z_p + O]\right) \prod_{q \in J \setminus J_k} \frac{1}{N} Leb\left(\left[z_q - \frac{N}{2} b_q, z_q + \frac{N}{2} b_q\right]\right) \\
&= \beta\,(2O)^k \prod_{q \in J \setminus J_k} b_q < +\infty.
\end{aligned}$$

Moreover, since $\lim\limits_{i \in I, i \longrightarrow +\infty} \varepsilon_i = 0$, we have $\lim\limits_{n \longrightarrow +\infty} \overline{\varphi}^{(n,n)} = \varphi$, and so

$$\lim_{n \longrightarrow +\infty} f_n(x) = l(\varphi(x)) \lim_{n \rightarrow +\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}(x)\right|, \forall\, x \in E_{M,N,a,v}^{(k,I,\varphi,\delta)};$$

then, from the dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n \longrightarrow +\infty} \int\limits_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} & l(\overline{\varphi}^{(n,n)}) |\det J_{\overline{\varphi}^{(n,n)}}| d\lambda_{N,b,z}^{(k,J)} \\
&= \int\limits_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} l(\varphi) \lim_{n \rightarrow +\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda_{N,b,z}^{(k,J)};
\end{aligned}$$

consequently, from (34), we have

$$\int\limits_{E_{M,N,a,v}^{(k,I)}} l d\lambda_{N,a,v}^{(k,I)} = \int\limits_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} l(\varphi) \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda_{N,b,z}^{(k,J)}. \tag{37}$$

Let $B = \prod_{i\in I} B_i \in \mathcal{B}^{(I)}\left(E_{M,N,a,v}^{(k,I)}\right)$, where $B_i = (\alpha_i, \beta_i)$, $\forall\, i \in I$, and let $\delta_i = \frac{\beta_i - \alpha_i}{2}$, $\forall\, i \in I$; moreover, $\forall\, h \in \mathbf{N}^*$, $\forall\, t \in [0,1]$, consider the set

$$A_{h,t} = \prod_{i\in I}\left(\alpha_i + \frac{t\delta_i}{h}, \beta_i - \frac{t\delta_i}{h}\right),$$

and consider the function $l_h : \mathbf{R}^I \longrightarrow [0,+\infty)$ defined by

$$l_h(x) = \begin{cases} 1 & \text{if } x \in \overset{\circ}{A}_{h,1} \\ t & \text{if } x \in \partial A_{h,t} \\ 0 & \text{if } x \in \mathbf{R}^I \backslash \overline{A_{h,0}} \end{cases}.$$

Observe that, $\forall\, h \in \mathbf{N}^*$, $l_h : \mathbf{R}^I \longrightarrow [0,+\infty)$ is a function such that $l_h\left(\mathbf{R}^I\right) \subset [0,1]$, $l_h(x) = 0$, $\forall\, x \notin E_{M,N,a,v}^{(k,I)}$; moreover, $\forall\, t_1, t_2 \in [0,+\infty)$ such that $t_1 < t_2$, we have

$$l_h^{-1}((t_1,t_2)) = \begin{cases} \emptyset & \text{if } t_1 \geq 1 \\ \overset{\circ}{A}_{h,t_1} & \text{if } t_1 < 1 < t_2 \\ \overset{\circ}{A}_{h,t_1} \backslash \overline{A_{h,t_2}} & \text{if } t_1 < t_2 \leq 1 \end{cases},$$

$$l_h^{-1}([0,t_2)) = \begin{cases} \mathbf{R}^I & \text{if } t_2 > 1 \\ \mathbf{R}^I \backslash \overline{A_{h,t_2}} & \text{if } t_2 \leq 1 \end{cases};$$

thus, l_h is $\left(\mathcal{B}^{(I)}, \mathcal{B}([0,+\infty))\right)$-measurable and $\left(\tau^{(I)}, \tau([0,+\infty))\right)$-continuous. Then, since $\{l_h\}_{h\in\mathbf{N}^*}$ is an increasing positive sequence such that $\lim\limits_{h\longrightarrow+\infty} l_h =$

1_B, from Beppo Levi theorem and (37), we have

$$\int_B d\lambda_{N,a,v}^{(k,I)} = \int_{E_{M,N,a,v}^{(k,I)}} 1_B d\lambda_{N,a,v}^{(k,I)} = \lim_{h\longrightarrow+\infty} \int_{E_{M,N,a,v}^{(k,I)}} l_h d\lambda_{N,a,v}^{(k,I)}$$
$$= \lim_{h\longrightarrow+\infty} \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} l_h(\varphi) \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda_{N,b,z}^{(k,J)}$$
$$= \int_{E_{M,N,a,v}^{(k,I,\varphi,\delta)}} 1_B(\varphi) \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda_{N,b,z}^{(k,J)}$$
$$= \int_{\varphi^{-1}(B)} \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda_{N,b,z}^{(k,J)}. \quad (38)$$

Moreover, Proposition 4.4 implies that the previous formula (38) is true $\forall\, B \in \mathcal{B}^{(I)}\left(\overset{\circ}{E}{}_{M,N,a,v}^{(k,I)}\right)$. Consider the measures μ and υ on $\left(\overset{\circ}{E}{}_{N,a,v}^{(k,I)}, \mathcal{B}^{(I)}\left(\overset{\circ}{E}{}_{N,a,v}^{(k,I)}\right)\right)$ defined by

$$\mu(B) = \int_B d\lambda_{N,a,v}^{(k,I)},$$
$$\upsilon(B) = \int_{\varphi^{-1}(B)} \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda_{N,b,z}^{(k,J)},$$

and set $B_l = B \cap \overset{\circ}{E}{}_{l,N,a,v}^{(k,I)}$, $\forall\, l \in \mathbf{N}^*$, $\forall\, B \in \mathcal{B}^{(I)}\left(\overset{\circ}{E}{}_{N,a,v}^{(k,I)}\right)$. Since $B_l \subset B_{l+1}$, $\varphi^{-1}(B_l) \subset \varphi^{-1}(B_{l+1})$, $\bigcup_{l\in\mathbf{N}^*} B_l = B$ and $\bigcup_{l\in\mathbf{N}^*} \varphi^{-1}(B_l) = \varphi^{-1}(B)$, from the continuity property of μ and υ and (38), we have

$$\int_B d\lambda_{N,a,v}^{(k,I)} = \lim_{l\longrightarrow+\infty} \int_{B_l} d\lambda_{N,a,v}^{(k,I)}$$
$$= \lim_{l\longrightarrow+\infty} \int_{\varphi^{-1}(B_l)} \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda_{N,b,z}^{(k,J)}$$
$$= \int_{\varphi^{-1}(B)} \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda_{N,b,z}^{(k,J)}. \quad (39)$$

Then, let $B \in \mathcal{B}^{(I)}\left(\overset{\circ}{E}{}^{(k,I)}_{N,a,v}\right)$ and let $g : \left(\mathbf{R}^I, \mathcal{B}^{(I)}\right) \longrightarrow ([0,+\infty), \mathcal{B}([0,+\infty)))$ be a measurable function; $\forall\, x \notin \overset{\circ}{E}{}^{(k,I)}_{N,a,v}$, we have $(g1_B)(x) = 0$; thus, by proceeding as in the proof of formula (33), formula (39) implies

$$\int_B g d\lambda^{(k,I)}_{N,a,v} = \int_{\mathbf{R}^I} 1_B g d\lambda^{(k,I)}_{N,a,v} = \int_{\mathbf{R}^J} (1_B g)(\varphi) \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda^{(k,J)}_{N,b,z}$$
$$= \int_{\varphi^{-1}(B)} g(\varphi) \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda^{(k,J)}_{N,b,z}.$$

Then, for any measurable function $f : \left(\mathbf{R}^I, \mathcal{B}^{(I)}\right) \longrightarrow (\mathbf{R}, \mathcal{B})$ such that f^+ (or f^-) is $\lambda^{(k,I)}_{N,a,v}$-integrable, we have

$$\int_B f d\lambda^{(k,I)}_{N,a,v} = \int_B f^+ d\lambda^{(k,I)}_{N,a,v} - \int_B f^- d\lambda^{(k,I)}_{N,a,v}$$
$$= \int_{\varphi^{-1}(B)} f^+(\varphi) \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda^{(k,J)}_{N,b,z}$$
$$- \int_{\varphi^{-1}(B)} f^-(\varphi) \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda^{(k,J)}_{N,b,z}$$
$$= \int_{\varphi^{-1}(B)} f(\varphi) \lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}\right| d\lambda^{(k,J)}_{N,b,z}. \quad (40)$$

In particular, assume that, $\forall\, x \in U$, there exists the function $J_\varphi(x) : E_J \longrightarrow E_I$; from Proposition 3.21, we have

$$\lim_{n\to+\infty} \left|\det J_{\overline{\varphi}^{(n,n)}}(x)\right| = \left|\lim_{n\to+\infty} \det J_{\overline{\varphi}^{(n,n)}}(x)\right| = \left|\det J_\varphi(x)\right|, \ \forall\, x \in U,$$

and so formula (40) implies

$$\int_B f d\lambda^{(k,I)}_{N,a,v} = \int_{\varphi^{-1}(B)} f(\varphi) \left|\det J_\varphi\right| d\lambda^{(k,J)}_{N,b,z}.$$

□

5. Problems for further study

A natural application of this paper, in the probabilistic framework, is the development of the theory of the infinite-dimensional continuous random variables, defined in the paper [4]. In particular, we can prove the formula of the density of such random variables composed with the (m, σ)-general functions, with further properties. Consequently, it is possible to introduce many random variables that generalize the well known continuous random vectors in $\mathbf{R}^m$ (for example, the Beta random variables in E_I defined by the (m, σ)-general matrices), and to develop some theoretical results and some applications in the statistical inference. Moreover, we can define a convolution between the laws of two independent and infinite-dimensional continuous random variables, as in the finite case.

Furthermore, in the statistical mechanics, it is possible to describe the systems of smooth hard particles, by using the Boltzmann equation (see, for example, the paper [18]), or the more general Master kinetic equation, described in the papers [17] and [16]. In order to study the evolution of these systems, we can consider the model of countable particles, such that their joint infinite-dimensional density can be determined by composing a particular random variable with a (m, σ)-general function.

Finally, we can generalize the papers [2] and [3] (where we estimate the rate of convergence of some Markov chains on $[0, p)^k$ to a uniform random vector) by considering the recursion $\{X_n\}_{n\in\mathbf{N}}$ on $[0, p)^{\mathbf{N}^*}$ defined by

$$X_{n+1} = AX_n + B_n \pmod{p},$$

where $X_0 = x_0 \in E_I$, A is a bijective, linear, integer and (m, σ)-general function, $p \in \mathbf{R}^+$, and $\{B_n\}_{n\in\mathbf{N}}$ is a sequence of independent and identically distributed random variables with values on E_I. As noted above, it is possible to determine the density of the random variable AX_n, for any $n \in \mathbf{N}^*$; consequently, we expect to prove that, with some assumptions on the law of B_n, the sequence $\{X_n\}_{n\in\mathbf{N}}$ converges with geometric rate to a random variable with law $\bigotimes_{i\in\mathbf{N}^*}\left(\frac{1}{p}Leb\Big|_{\mathcal{B}([0,p))}\right)$, that is the uniform random variable on $[0, p)^{\mathbf{N}^*}$. Moreover, we wish to quantify the rate of convergence in terms of A, p, m, and the law of B_n.

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