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Third Hankel determinant for a subclass of analytic functions of reciprocal order defined by Srivastava-Attiya integral operator

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Abstract. *The aim of this paper is to investigate coefficient estimates, Fekete-Szegő inequality, and upper bound of third Hankel determinant for a subclass of analytic functions of reciprocal order defined by Srivastava-Attiya integral operator.*

Keywords: Analytic functions, Zeta function, Srivastava-Attiya integral operator, Fekete-Szegő inequality, reciprocal order, Third Hankel.
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1. Introduction

Let A be the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

in the open unit disc $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

The Lipschitz- Lerch zeta function is a series characterized as follows

$$R(a,x,s) \equiv \sum_{k=0}^{\infty} \frac{e^{2k\pi i x}}{(a+k)^s}, \qquad s,x,a \in \mathbb{C},$$

with conditions $1 - a \notin \mathbb{N}$ and $x \geq 0$. The series converges $\forall s \in \mathbb{C}$ if $x > 0$ and represents an entire function of s. The series converges absolutely for $\Re(s) > 1$ if $x = 0$. Lerch [23] and Lipschitz [26] studied this type of function with regard to Dirichlet's well known theorem on primes in arithmetic progression. The Lipschitz-Lerch zeta function reduces to the meromorphic Hurwitz zeta function $\zeta(s,a)$ if $x \in \mathbb{Z}$ with one single pole at $s = 1$ [37, Section 2, 3 Eq(2)].

By using a different notation for the Lipschitz-Lerch zeta function, Bateman gave the following function [3]:

$$\Phi(z,s,a) \equiv \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}\,,$$
$$\big(a \in \mathbb{C}\setminus \mathbb{Z}_0^-;\; s\in\mathbb{C} \text{ when } |z|<1;\; \Re(s)>1 \text{ when } |z|=1\big)\,. \quad (2)$$

The equation (2) is connected to Lipschitz-Lerch zeta function by the relation $\Phi(e^{2k\pi ix}, s, a) = R(a, x, s)$ and called later Hurwitz-Lerch zeta function.

Also, the Riemann zeta function $\zeta(s)$, the Hurwitz (or generalized) zeta function $\zeta(s,a)$ and the Lerch zeta function $\ell_s(\xi)$ are defined respectively as follows (see, for details, [3, Chapter 1] and [37, Chapter 2]):

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1,s,1) = \zeta(s,1)\,, \qquad (\Re(s)>1)\,,$$

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1,s,a)\,, \qquad (\Re(s)>1; a\in\mathbb{C}\setminus\mathbb{Z}_0^-)\,,$$

and
$$\ell_s(\xi) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+1)^s} = \Phi(e^{2\pi i\xi}, s, 1)\,, \qquad (\Re(s)>1; \xi\in\mathbb{R}).$$

In addition, an important function of Analytic Number Theory such as the Polylogarithmic function (or de Jonquière's function) $Li_s(z)$ is given by:

$$Li_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1)\,,$$
$$(s\in\mathbb{C} \text{ when } |z|<1;\; \Re(s)>1 \text{ when } |z|=1).$$

It is known that the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ in (2) can be written as

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt$$
$$(\Re(a)>0;\; \Re(s)>0 \text{ when } |z|\le 1 \;\; (z\neq 1);\; \Re(s)>1 \text{ when } z=1). \quad (3)$$

Besides, since

$$\sum_{n=0}^{\infty} f(n) = \sum_{j=0}^{k-1}\sum_{n=0}^{\infty} f(kn+j)\,, \qquad (k\in\mathbb{N}),$$

we have

$$\Phi(z,s,a)=k^{-s}\sum_{j=0}^{k-1}\Phi\left(z^k,s,\frac{a+j}{k}\right)z^j\,,\qquad (k\in\mathbb{N}). \tag{4}$$

By combining (3) and (4), immediately we have:

$$\Phi(z,s,a)=\sum_{j=0}^{k-1}\frac{z^j}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(a+j)t}}{1-z^ke^{-kt}}dt$$
$$(k\in\mathbb{N};\ \Re(a)>0;\ \Re(s)>0 \text{ when } |z|\leq 1 (z\neq 1);\ \Re(s)>1 \text{ when } z=1). \tag{5}$$

The above equation is mainly prompted by the sum-integral representation in which the authors introduce an analogous investigation of the following general family of the Hurwitz-Lerch zeta function by using $(\mu)_{\rho n}$ and $(\nu)_{\sigma n}$ (see [24]):

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a):=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{z^n}{(n+a)^s},$$
$$(\,\mu\in\mathbb{C};\ a,\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \rho,\sigma\in\mathbb{R}^+;\ \rho<\sigma \text{ when } s,z\in\mathbb{C};$$
$$\rho=\sigma \text{ and } s\in\mathbb{C} \text{ when } |z|<1:\rho=\sigma \text{ and } \Re(s-\mu+\nu)>1 \text{ when } |z|=1).$$

Here, and for the remainder of this paper, $(\gamma)_k$ denotes the Pochhammer symbol defined in terms of Gamma function, by

$$(\gamma)_k:=\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}=\begin{cases}\gamma(\gamma+1)\ldots(\gamma+n-1) & (k=n\in\mathbb{N};\ \gamma\in\mathbb{C})\\ 1 & (k=0;\ \gamma\in\mathbb{C}\setminus\{0\}).\end{cases}$$

We then have

$$\Phi_{\nu,\nu}^{(\sigma,\sigma)}(z,s,a)=\Phi_{\mu,\nu}^{(0,0)}(z,s,a)=\Phi(z,s,a)$$

and

$$\Phi_{\mu,1}^{(1,1)}(z,s,a)=\Phi_{\mu}^{*}(z,s,a):=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(n+a)^s}\,. \tag{6}$$

Recently, Goyal and Laddha ([14], p. 100, Eq. (1.5)) studied the generalized Hurwitz-Lerch zeta function $\Phi_{\mu}^{*}(z,s,a)$ given by (6).

For functions $f\in A$ given by (1) and $g\in A$ $(g(z)=z+\sum_{n=2}^{\infty}b_nz^n)$, the Hadamard product (or convolution) of f and g can be defined by

$$(f*g)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\qquad z\in U.$$

The Hurwitz-Lerch zeta function $\Phi(z,s,a)$ given in (3) was recently studied by Choi and Srivastava [8], Ferreira and Lopez [12], Garg *et al.* [13], Lin *et al.*[25], Srivastava and Attiya [36], Lin and Srivastava *et al* [38] and others (see [4, 5, 6, 7]).

Now,

$$J_{s,a}: A \to A,$$

$$J_{s,a}f(z) = G_{s,a} * f(z), \quad (z \in U;\ a \in \mathbb{C} \setminus \{Z_0^-\};\ s \in \mathbb{C};\ f \in A) \tag{7}$$

where, for convenience

$$G_{s,a}(z) := (1+a)^s[\Phi(z,s,a) - a^{-s}] \qquad (z \in U). \tag{8}$$

Successfully, by utilizing (1), (7) and (8), we can obtain

$$J_{s,a}f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right)^s a_n z^n.$$

Also let $S^*(\alpha)$ be classes of starlike functions and $K(\alpha)$ classes of convex functions of order α, $0 \le \alpha < 1$. In 1975, Silverman [33] proved that $f(z) \in S^*(\alpha)$ if the following condition is satisfied:

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha, \qquad (z \in U). \tag{9}$$

Geometrical importance of inequality (9) is that $zf'(z)/f(z)$ maps U onto the inside of the circle with radius $1-\alpha$ and center at 1.

We can define $S_*(\alpha)$ (classes of starlike functions of reciprocal order α) and $K_*(\alpha)$ (classes of convex functions of reciprocal order α), $0 \le \alpha < 1$, individually by

$$S_*(\alpha) = \left\{ f(z) \in A : \Re \frac{f(z)}{zf'(z)} > \alpha,\ (z \in U) \right\},$$

$$K_*(\alpha) = \left\{ f(z) \in A : \Re \frac{f'(z)}{zf''(z) + f'(z)} > \alpha,\ (z \in U) \right\}.$$

In 2008, Nunokawa and his coauthors [29] enhanced inequality (9) for the class $S_*(\alpha)$ and they showed that $f(z) \in S_*(\alpha)$, $0 < \alpha < \frac{1}{2}$, if and only if next inequality holds:

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \qquad (z \in U).$$

In perspective of these outcomes, we now characterize the accompanying subclass of analytic functions of reciprocal order and study its different properties.

Definition 1.1. *A function $f \in A$ is said to be in the class $L(a, s, \gamma)$ with $\gamma \in \mathbb{C} \setminus \{0, \frac{1}{2}\}$ and $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$, if it satisfies the following inequality*

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{J_{s,a}f(z)}{zJ'_{s,a}f(z)} - 1\right)\right) > 0,$$

where $J_{s,a}f(z) = G_{s,a}(z) * f(z)$.

Example 1.2: Let us define the function $J_{s,a}f(z)$ by

$$J_{s,a}f(z) = \frac{z}{(1 + (2\gamma - 1)z)^{2\gamma/(2\gamma-1)}}.$$

This implies that

$$\frac{zJ'_{s,a}f(z)}{J_{s,a}f(z)} = \frac{1 - z}{1 + (2\gamma - 1)z}.$$

Hence

$$1 + \frac{1}{\gamma}\left(\frac{J_{s,a}f(z)}{zJ'_{s,a}f(z)} - 1\right) = \frac{1 + z}{1 - z},$$

this further implies that

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{J_{s,a}f(z)}{zJ'_{s,a}f(z)} - 1\right)\right) = \Re\frac{1 + z}{1 - z} > 0, \qquad (z \in U).$$

Noonan and Thomas [28] considered the qth Hankel determinant $H_q(n)$, $q \geq 1$, $n \geq 1$ for a function $f \in A$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}, \qquad a_1 = 1.$$

In the literature, many authors have shed light on the determinant $H_q(n)$, where $H_2(2)$ refer to the second Hankel determinant. After that Janteng *et al.* ([16, 17]), Singh and Singh [35], and many authors have studied sharp upper bounds on $H_2(2)$. Yavuz [39] studied the analytic functions defined by Ruscheweyh derivative and got an upper bound for the second Hankel determinant $|a_2a_4 - a_3{}^2|$ for it in the unit disc. Mishra and Kund [22] studied a class of analytic functions related to the Carlson-Shaffer operator in the unit disc and estimated the second Hankel determinant for this class. Singh and Mehrok [34] investigated p-valent α-convex functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ in the unit disc and got the sharp upper bound of $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for $f(z)$.

Deniz *et al.* [10] researched bi-starlike and bi-convex functions of order β which are important subclasses of bi-univalent functions and obtained for the second Hankel determinant $H_2(2)$ of these subclasses. Deekonda and Thoutreddy in [9] by using Toeplitz determinants concentrated on the functions belonging to certain subclasses of analytic functions, and obtained an upper bound on the second Hankel determinant $|a_2a_4 - {a_3}^2|$ for this class. Krishna and Ramreddy [21] by using Toeplitz determinants, considered p-valent starlike and convex functions of order α and obtained an upper bound on the second Hankel determinant $|a_{p+1}a_{p+3} - a_{p+2}^2|$. We refer to $H_3(1)$ as the third Hankel determinant. In 2014 Arif *et al.* [2] studied some families of starlike and convex functions of reciprocal order defined by Al-Oboudi operator and obtained coefficient estimates, Fekete-Szegő inequality, and upper bound on third Hankel determinant for these families. Recently Mishra *et al.* [27] investigated upper bounds on the third Hankel determinants for the starlike and convex functions with respect to symmetric points in the open unit disc. Shanmugam *et al.* [32] investigated the third Hankel determinant, $H_3(1)$, for normalized univalent functions $f(z) = z + a_2z^2 + ...$ belonging to the class of α starlike functions. In 2015 Prajapat *et al.* [31] focused on the functions belonging to the class of close-to-convex functions and obtained upper bound on third Hankel determinant for this class. Other examples defined on various classes can be read in [1, 18, 19].

In this paper, the authors study the upper bound on $H_3(1)$ for a subclass of analytic functions of reciprocal order by using Toeplitz determinant. Some useful results include coefficient estimates, Fekete-Szegő inequality, and upper bound of third Hankel determinant for the functions belonging to the class $L(a, s, \gamma)$.

To achieve the results, we need the following lemmas:

Lemma 1.3 ([30]). *If $q(z)$ is a function with $\Re q(z) > 0$ and is of the form*

$$q(z) = 1 + c_1z + c_2z^2 + ..., \tag{10}$$

then

$$|c_n| \leq 2, \quad for \;\; n \geq 1.$$

Lemma 1.4 ([20]). *If $q(z)$ is of the form (10) with positive real part, then the following sharp estimate holds:*

$$\left|c_2 - \nu {c_1}^2\right| \leq 2 \;\; max\{1, |2\nu - 1|\}, for \;\; all \;\; \nu \in \mathbb{C}.$$

Lemma 1.5 ([15]). *If $q(z)$ is of the form (10) with positive real part, then*

$$2c_2 = {c_1}^2 + (4 - {c_1}^2)x$$

and

$$4c_3 = {c_1}^3 + 2c_1(4 - {c_1}^2)x - c_1(4 - {c_1}^2)x^2 + 2(4 - {c_1}^2)(1 - |x|^2)z,$$

for some x and z satisfy $|x| \leq 1$, $|z| \leq 1$ and $c_1 \in [0, 2]$.

2. Some properties of the class $L(a, s, \gamma)$

THEOREM 2.1. *Let $f(z) \in L(a, s, \gamma)$. Then*

$$|a_2| \leq \frac{2|\gamma|}{\left(\frac{1+a}{2+a}\right)^s}$$

and for all $n = 3, 4, 5, ...$

$$|a_n| \leq \frac{2|\gamma|}{(n-1)\left(\frac{1+a}{n+a}\right)^s} \prod_{k=2}^{n-1} \left(1 + \frac{2|\gamma|k}{k-1}\right).$$

Proof. The function $q(z)$ can be characterized as

$$q(z) = 1 + \frac{1}{\gamma}\left(\frac{J_{s,a}f(z)}{zJ'_{s,a}f(z)} - 1\right),$$

where $J_{s,a}f(z)$ is given by (7) with

$$J_{s,a}f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right)^s a_n z^n,$$

and $q(z)$ is analytic in U with $q(0) = 1$, $\Re q(z) > 0$.

Now, by using (1) and (10), we get

$$z + \sum_{k=2}^{\infty} A_k z^k = \left[1 + \gamma\left(\sum_{k=1}^{\infty} c_k z^k\right)\right]\left(z + \sum_{k=2}^{\infty} kA_k z^k\right),$$

where

$$A_k = \left(\frac{1+a}{k+a}\right)^s a_k. \tag{11}$$

Comparing coefficient of z^n, we get

$$(1-n)A_n = \gamma\big\{c_{n-1} + 2A_2c_{n-2} + ... + (n-1)A_{n-1}c_1\big\}. \tag{12}$$

Using triangle inequality and Lemma 1.3, we obtain

$$|(1-n)A_n| \leq 2|\gamma|\big\{1+2|A_2|+...+(n-1)|A_{n-1}|\big\}. \tag{13}$$

For $n=2$ and $n=3$ in (13), we can get the following easily

$$|a_2| \leq \frac{2|\gamma|}{\left(\frac{1+a}{2+a}\right)^s}, \qquad |a_3| \leq \frac{|\gamma|(1+4|\gamma|)}{\left(\frac{1+a}{3+a}\right)^s}.$$

Making $n=4$ in (13), we note that

$$|a_4| \leq \frac{2|\gamma|(1+4|\gamma|)(1+3|\gamma|)}{3\left(\frac{1+a}{4+a}\right)^s}.$$

In general, by using the principle of mathematical induction, we can obtain

$$|A_n| \leq \frac{2|\gamma|}{(n-1)}\prod_{k=2}^{n-1}\left(1+\frac{2|\gamma|k}{k-1}\right).$$

Presently, using relation (11), we get the required result:

$$|a_n| \leq \frac{2|\gamma|}{(n-1)\left(\frac{1+a}{n+a}\right)^s}\prod_{k=2}^{n-1}\left(1+\frac{2|\gamma|k}{k-1}\right).$$

□

With $\gamma = 1-\alpha$ and $s=0$, we obtain the following result.

Corollary 2.2 ([14]). *Let $f(z) \in S_*(\alpha)$. Then, for $n = 3, 4, 5, ...$, one has*

$$|a_n| \leq \frac{2(1-\alpha)}{(n-1)}\prod_{k=2}^{n-1}\left(1+\frac{2(1-\alpha)k}{k-1}\right)$$

with $|a_2| \leq 2(1-\alpha)$.

If we make $s=1$ and $\gamma = 1-\alpha$, we can get the following easily

Corollary 2.3 ([14]). *Let $f(z) \in K_*(\alpha)$. Then, for $n = 3, 4, 5, ...$, one has*

$$|a_n| \leq \frac{2(1-\alpha)}{(n-1)\left(\frac{1+a}{n+a}\right)}\prod_{k=2}^{n-1}\left(1+\frac{2(1-\alpha)k}{k-1}\right)$$

with $|a_2| \leq \frac{2(1-\alpha)}{\left(\frac{1+a}{2+a}\right)}$.

THEOREM 2.4. *If $f(z) \in L(a,s,\gamma)$ and is of the form (1). Then*

$$|a_3 - \mu a_2{}^2| \leq \frac{|\gamma|}{\left(\frac{1+a}{3+a}\right)^s} max\{1, |2\nu - 1|\},$$

where

$$\nu = 2\gamma \left(\frac{1+a}{3+a}\right)^s \left(\frac{1}{\left(\frac{1+a}{3+a}\right)^s} - \frac{\mu}{\left(\frac{1+a}{2+a}\right)^{2s}} \right). \tag{14}$$

Proof. Let $f(z) \in L(a,s,\gamma)$. Then from (12) we have

$$a_2 = \frac{-\gamma c_1}{\left(\frac{1+a}{2+a}\right)^s}, \quad a_3 = \frac{-\gamma}{2\left(\frac{1+a}{3+a}\right)^s}(c_2 - 2\gamma c_1{}^2).$$

We now consider

$$|a_3 - \mu a_2{}^2| = \frac{|\gamma|}{2\left(\frac{1+a}{3+a}\right)^s} \left| c_2 - 2\gamma \left(\frac{1+a}{3+a}\right)^s \left(\frac{1}{\left(\frac{1+a}{3+a}\right)^s} - \frac{\mu}{\left(\frac{1+a}{2+a}\right)^{2s}} \right) c_1{}^2 \right|.$$

Using Lemma 1.4, we obtain

$$|a_3 - \mu a_2{}^2| \leq \frac{|\gamma|}{\left(\frac{1+a}{3+a}\right)^s} max\{1, |2\nu - 1|\},$$

where ν is given by (14). □

Putting $\mu = 1$, we get

COROLLARY 2.5. *If $f(z) \in L(a,s,\gamma)$. Then*

$$|a_3 - a_2{}^2| \leq \frac{|\gamma|}{\left(\frac{1+a}{3+a}\right)^s}.$$

THEOREM 2.6. *Let $f(z) \in L(a,s,\gamma)$ and be of the form (1). Then*

$$|a_2 a_4 - a_3{}^2| \leq \left[\frac{4\left(\frac{1+a}{3+a}\right)^{2s} + |\gamma|\left(28\left(\frac{1+a}{3+a}\right)^{2s} + 24\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s\right)}{3\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{2s} \left(\frac{1+a}{4+a}\right)^s} \right.$$

$$\left. + \frac{48|\gamma|^2\left(\left(\frac{1+a}{3+a}\right)^{2s} + \left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s\right) + 3\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{4+a}\right)^s}{3\left(\frac{1+a}{2+a}\right)^s \left(\frac{1+a}{3+a}\right)^{2s} \left(\frac{1+a}{4+a}\right)^s} \right] \times |\gamma|^2.$$

Proof. Let $f(z) \in L(a,s,\gamma)$. Then, from (12), we have

$$a_2 = \frac{-\gamma c_1}{\left(\frac{1+a}{2+a}\right)^s}, \qquad a_3 = \frac{-\gamma}{2\left(\frac{1+a}{3+a}\right)^s}(c_2 - 2\gamma {c_1}^2)$$
$$\text{and} \qquad a_4 = \frac{-\gamma}{3\left(\frac{1+a}{4+a}\right)^s}\left[c_3 - \frac{7}{2}\gamma c_1 c_2 + 3\gamma^2 {c_1}^3\right]. \quad (15)$$

Consider

$$|a_2a_4 - {a_3}^2| = \left| \frac{-\gamma c_1}{\left(\frac{1+a}{2+a}\right)^s} \cdot \frac{-\gamma}{3\left(\frac{1+a}{4+a}\right)^s}\left[c_3 - \frac{7}{2}\gamma c_1 c_2 + 3\gamma^2 {c_1}^3\right] - \frac{\gamma^2}{4\left(\frac{1+a}{3+a}\right)^{2s}}(c_2 - 2\gamma {c_1}^2)^2 \right|$$

$$\begin{aligned}|a_2a_4 - {a_3}^2| = \Big|&\Big(\frac{\gamma^2}{12\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^{2s}\Big(\frac{1+a}{4+a}\Big)^s}\Big)\Big(4\Big(\frac{1+a}{3+a}\Big)^{2s} c_1c_3 \\ &- 2\gamma\Big(7\Big(\frac{1+a}{3+a}\Big)^{2s} - 6\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big)c_2{c_1}^2 + 12\gamma^2\Big(\Big(\frac{1+a}{3+a}\Big)^{2s} \\ &- \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big){c_1}^4 - 3\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s{c_2}^2\Big)\Big|.\end{aligned}$$

Now using values of c_2 and c_3 from Lemma 1.5, we obtain

$$\begin{aligned}|a_2a_4 - {a_3}^2| = &\frac{\gamma^2}{12\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^{2s}\Big(\frac{1+a}{4+a}\Big)^s} \times \Big|\Big\{\Big(\frac{1+a}{3+a}\Big)^{2s} - \gamma\Big(7\Big(\frac{1+a}{3+a}\Big)^{2s} \\ &- 6\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) + 12\gamma^2\Big(\Big(\frac{1+a}{3+a}\Big)^{2s} - \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) \\ &- \frac{3}{4}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big\}{c_1}^4 + \Big\{2\Big(\frac{1+a}{3+a}\Big)^{2s} - \gamma\Big(7\Big(\frac{1+a}{3+a}\Big)^{2s} \\ &- 6\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) - \frac{3}{2}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big\}(4-{c_1}^2){c_1}^2x \\ &- \Big\{\Big(\frac{1+a}{3+a}\Big)^{2s}{c_1}^2 + \frac{3}{4}\Big(\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big)(4-{c_1}^2)\Big\}(4-{c_1}^2)x^2 \\ &+ 2c_1\Big(\frac{1+a}{3+a}\Big)^{2s}(4-{c_1}^2)(1-|x|^2)z\Big|.\end{aligned}$$

Applying triangle inequality and replacing c_1 by c, $|x|$ by ρ, and $|z|$ by 1, we get

$$|a_2a_4 - {a_3}^2| \leq \frac{|\gamma|^2}{12\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{2s}\left(\frac{1+a}{4+a}\right)^s} \times \Bigg[\Big\{\Big(\frac{1+a}{3+a}\Big)^{2s} + |\gamma|\Big(7\Big(\frac{1+a}{3+a}\Big)^{2s}$$
$$+ 6\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) + 12|\gamma|^2\Big(\Big(\frac{1+a}{3+a}\Big)^{2s} + \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big)$$
$$+ \frac{3}{4}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big\}c^4 + \Big\{2\Big(\frac{1+a}{3+a}\Big)^{2s} + |\gamma|\Big(7\Big(\frac{1+a}{3+a}\Big)^{2s}$$
$$+ 6\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) + \frac{3}{2}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big\}(4-c^2)c^2\rho$$
$$+ \Big\{\Big(\frac{1+a}{3+a}\Big)^{2s}c^2 + \frac{3}{4}\Big(\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big)(4-c^2)\Big\}(4-c^2)\rho^2$$
$$+ 2c\Big(\frac{1+a}{3+a}\Big)^{2s}(4-c^2)(1-\rho^2)\Bigg] = F(c,\rho).$$

Differentiating with respect to ρ, we get

$$\frac{\partial F(c,\rho)}{\partial \rho} = \frac{|\gamma|^2}{12\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{2s}\left(\frac{1+a}{4+a}\right)^s} \times \Bigg[\Big\{2\Big(\frac{1+a}{3+a}\Big)^{2s} + |\gamma|\Big(7\Big(\frac{1+a}{3+a}\Big)^{2s}$$
$$+ 6\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) + \frac{3}{2}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big\}(4-c^2)c^2$$
$$+ \Big\{2\Big(\frac{1+a}{3+a}\Big)^{2s}c^2 + \frac{3}{2}\Big(\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big)(4-c^2)\Big\}(4-c^2)\rho$$
$$- 4c\Big(\frac{1+a}{3+a}\Big)^{2s}(4-c^2)\rho\Bigg].$$

Since $\frac{\partial F(c,\rho)}{\partial \rho} > 0$ for $\rho \in [0,1]$ and $c \in [0,2]$, the maximize of $F(c,\rho)$ will exist at $\rho = 1$. Let $F(c,1) = G(c)$, then

$$G(c) = \frac{|\gamma|^2}{12\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{2s}\left(\frac{1+a}{4+a}\right)^s} \times \Bigg[\Big\{\Big(\frac{1+a}{3+a}\Big)^{2s} + |\gamma|\Big(7\Big(\frac{1+a}{3+a}\Big)^{2s}$$
$$+ 6\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) + 12|\gamma|^2\Big(\Big(\frac{1+a}{3+a}\Big)^{2s} + \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big)$$
$$+ \frac{3}{4}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big\}c^4 + \Big\{2\Big(\frac{1+a}{3+a}\Big)^{2s}|\gamma|\Big(7\Big(\frac{1+a}{3+a}\Big)^{2s}$$
$$+ 6\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) + \frac{3}{2}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big\}(4-c^2)c^2$$
$$+ \Big\{\Big(\frac{1+a}{3+a}\Big)^{2s}c^2 + \frac{3}{4}\Big(\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big)(4-c^2)\Big\}(4-c^2)\Bigg].$$

Now by differentiating with respect to c, we obtain

$$G'(c) = \frac{|\gamma|^2}{12\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{2s}\left(\frac{1+a}{4+a}\right)^s} \times \Big[4\Big\{\Big(\frac{1+a}{3+a}\Big)^{2s} + |\gamma|\Big(7\Big(\frac{1+a}{3+a}\Big)^{2s}$$
$$+ 6\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) + 12|\gamma|^2\Big(\Big(\frac{1+a}{3+a}\Big)^{2s} + \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big)$$
$$+ \frac{3}{4}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big\}c^3 + \Big\{2\Big(\frac{1+a}{3+a}\Big)^{2s} + |\gamma|\Big(7\Big(\frac{1+a}{3+a}\Big)^{2s}$$
$$+ 6\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) + \frac{3}{2}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big\}(8c-4c^3)$$
$$+ \Big\{\Big(\frac{1+a}{3+a}\Big)^{2s}(8c-4c^3) - 3\Big(\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big)(4c-c^3)\Big\}\Big].$$

Since $\partial G(c)/\partial c > 0$ for $c \in [0,2]$, $G(c)$ has a maximum value at $c = 2$ and hence

$$|a_2a_4 - a_3{}^2| \leq \frac{|\gamma|^2}{3\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{2s}\left(\frac{1+a}{4+a}\right)^s}$$
$$\times\Big\{4\Big(\frac{1+a}{3+a}\Big)^{2s} + |\gamma|\Big(28\Big(\frac{1+a}{3+a}\Big)^{2s} + 24\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big)$$
$$+ 48|\gamma|^2\Big(\Big(\frac{1+a}{3+a}\Big)^{2s} + \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big) + 3\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{4+a}\Big)^s\Big\}.$$

□

THEOREM 2.7. *Let $f(z) \in L(a,s,\gamma)$ and be of the form (1). Then*

$$|a_2a_3 - a_4| \leq \frac{|\gamma|}{3\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s\left(\frac{1+a}{4+a}\right)^s}$$
$$\times\Big\{24|\gamma|^2\Big(\Big(\frac{1+a}{4+a}\Big)^s + \Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) + 2|\gamma|\Big(3\Big(\frac{1+a}{4+a}\Big)^s$$
$$+ 7\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) + 2\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big\}.$$

Proof. From (15), we can write

$$|a_2a_3 - a_4| = \frac{|\gamma|}{6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s\left(\frac{1+a}{4+a}\right)^s}$$
$$\times\Big| -6\gamma^2\Big(\Big(\frac{1+a}{4+a}\Big)^s - \Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big)c_1{}^3 + \gamma\Big(3\Big(\frac{1+a}{4+a}\Big)^s$$
$$- 7\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big)c_1c_2 + 2\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s c_3\Big|.$$

Using Lemma 1.5 for the values of c_2 and c_3, we have

$$\begin{aligned}
|a_2a_3 - a_4| = & \frac{|\gamma|}{6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s\left(\frac{1+a}{4+a}\right)^s} \\
\times\Big|\Big\{ & -6\gamma^2\Big(\Big(\frac{1+a}{4+a}\Big)^s - \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) + \frac{\gamma}{2}\Big(3\Big(\frac{1+a}{4+a}\Big)^s \\
& - 7\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) + \frac{1}{2}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big\}{c_1}^3 + \Big\{\frac{\gamma}{2}\Big(3\Big(\frac{1+a}{4+a}\Big)^s \\
& - 7\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) + \Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big\}(4-{c_1}^2)c_1x \\
& - \Big\{\frac{1}{2}\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big\}c_1(4-{c_1}^2)x^2 \\
& + \Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s(4-{c_1}^2)(1-|x|^2z)\Big|.
\end{aligned}$$

Applying triangle inequality and then putting $|z| = 1$, $|x| = \rho$, and $c_1 = c$, we have

$$\begin{aligned}
|a_2a_3 - a_4| \leq & \frac{|\gamma|}{6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s\left(\frac{1+a}{4+a}\right)^s} \\
& \times\Big[\Big\{6|\gamma|^2\Big(\Big(\frac{1+a}{4+a}\Big)^s + \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) \\
& \qquad + \frac{|\gamma|}{2}\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) \\
& \qquad\quad + \frac{1}{2}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big\}c^3 \\
& \quad + \Big\{\frac{|\gamma|}{2}\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) \\
& \qquad\quad + \Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big\}(4-c^2)c\rho \\
& \quad + \Big\{\frac{1}{2}\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big\}c(4-c^2)\rho^2 \\
& \quad + \Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s(4-c^2)(1-\rho^2)\Big] \\
& = F(c,\rho).
\end{aligned}$$

Differentiating with respect to ρ, we get

$$\begin{aligned}\frac{\partial F(c,\rho)}{\partial \rho} =& \frac{|\gamma|}{6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s\left(\frac{1+a}{4+a}\right)^s} \\ &\times \Big[\Big\{\frac{|\gamma|}{2}\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) \\ &\qquad + \Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big\}(4-c^2)c \\ &\quad + \Big\{\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big\}c(4-c^2)\rho \\ &\qquad - 2\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s(4-c^2)\rho\Big].\end{aligned}$$

Now since $\frac{\partial F(c,\rho)}{\partial \rho} > 0$ for $c \in [0,2]$ and $\rho \in [0,1]$, a maximum of $F(c,\rho)$ will exist at $\rho = 1$ and let $F(c,1) = G(c)$. Then

$$\begin{aligned}G(c) =& \frac{|\gamma|}{6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s\left(\frac{1+a}{4+a}\right)^s} \\ &\times \Big[\Big\{6|\gamma|^2\Big(\Big(\frac{1+a}{4+a}\Big)^s + \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) \\ &\qquad + \frac{|\gamma|}{2}\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) \\ &\qquad + \frac{1}{2}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big\}c^3 \\ &\quad + \Big\{\frac{|\gamma|}{2}\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) \\ &\qquad + \frac{3}{2}\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big\}(4-c^2)c\Big].\end{aligned}$$

Now by differentiating with respect to c, we obtain

$$\begin{aligned}G'(c) =& \frac{|\gamma|}{6\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s\left(\frac{1+a}{4+a}\right)^s} \\ &\times \Big[3\Big\{6|\gamma|^2\Big(\Big(\frac{1+a}{4+a}\Big)^s + \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) \\ &\quad + \frac{|\gamma|}{2}\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) + \frac{1}{2}\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big\}c^2 \\ &\quad + \Big\{\frac{|\gamma|}{2}\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) \\ &\qquad + \frac{3}{2}\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big\}(4-3c^2)\Big].\end{aligned}$$

Since $\partial G(c)/\partial c > 0$ for $c \in [0,2]$, $G(c)$ has a maximum value at $c = 2$, hence

$$
\begin{aligned}
|a_2a_3 - a_4| \leq & \frac{|\gamma|}{3\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s\left(\frac{1+a}{4+a}\right)^s} \\
& \times \Big\{24|\gamma|^2\Big(\Big(\frac{1+a}{4+a}\Big)^s + \Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) \\
& + 2|\gamma|\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) \\
& + 2\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big\}.
\end{aligned}
$$

□

THEOREM 2.8. *Let $f(z) \in L(a,s,\gamma)$ and be of the form (1). Then*

$$
\begin{aligned}
|H_3(1)| \leq & \frac{|\gamma|^3(1+4|\gamma|)}{3\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{3s}\left(\frac{1+a}{4+a}\right)^s} \\
& \times\Big[4\Big(\frac{1+a}{3+a}\Big)^{2s} + |\gamma|\Big(28\Big(\frac{1+a}{3+a}\Big)^{2s} + 24\Big(\frac{1+a}{4+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) \\
& + 48|\gamma|^2\Big(\Big(\frac{1+a}{3+a}\Big)^{2s} + \Big(\frac{1+a}{4+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) + 3\Big(\frac{1+a}{4+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big] \\
& + \frac{4|\gamma|^2(1+4|\gamma|)(1+3|\gamma|)}{9\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s\left(\frac{1+a}{4+a}\right)^{2s}} \\
& \times\Big[12|\gamma|^2\Big(\Big(\frac{1+a}{4+a}\Big)^s + \Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) \\
& + |\gamma|\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) + \Big(\frac{1+a}{3+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big] \\
& + \frac{|\gamma|^2(29|\gamma| + 92|\gamma|^2 + 96|\gamma|^3 + 3)}{6\left(\frac{1+a}{5+a}\right)^s\left(\frac{1+a}{3+a}\right)^s}.
\end{aligned}
$$

Proof. Since

$$|H_3(1)| \leq |a_3||a_2a_4 - {a_3}^2| + |a_4||a_2a_3 - a_1a_4| + |a_5||a_3 - {a_2}^2|,$$

using Corollary 2.5, Theorem 2.6, Theorems 2.7 and a_5, we have

$$
\begin{aligned}
|H_3(1)| \leq\ & \frac{|\gamma|(1+4|\gamma|)}{\left(\frac{1+a}{3+a}\right)^s} \times \frac{|\gamma|^2}{3\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{2s}\left(\frac{1+a}{4+a}\right)^s} \\
& \times \Bigg[4\Big(\frac{1+a}{3+a}\Big)^{2s} + |\gamma|\Big(28\Big(\frac{1+a}{3+a}\Big)^{2s} + 24\Big(\frac{1+a}{4+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) \\
& \quad + 48|\gamma|^2\Big(\Big(\frac{1+a}{3+a}\Big)^{2s} + \Big(\frac{1+a}{4+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) + 3\Big(\frac{1+a}{4+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Bigg] \\
& + \frac{2|\gamma|(1+4|\gamma|)(1+3|\gamma|)}{3\left(\frac{1+a}{4+a}\right)^s} \times \frac{|\gamma|}{3\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s\left(\frac{1+a}{4+a}\right)^s} \\
& \times \Bigg[24|\gamma|^2\Big(\Big(\frac{1+a}{4+a}\Big)^s + \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) \\
& \quad + 2|\gamma|\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) + 2\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Bigg] \\
& + \frac{2|\gamma|\Big(\frac{29}{3}|\gamma| + \frac{92}{3}|\gamma|^2 + 32|\gamma|^3 + 1\Big)}{4\left(\frac{1+a}{5+a}\right)^s} \times \frac{|\gamma|}{\left(\frac{1+a}{3+a}\right)^s}.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
|H_3(1)| \leq\ & \frac{(1+4|\gamma|)|\gamma|^3}{3\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^{3s}\left(\frac{1+a}{4+a}\right)^s} \\
& \times \Bigg[4\Big(\frac{1+a}{3+a}\Big)^{2s} + |\gamma|\Big(28\Big(\frac{1+a}{3+a}\Big)^{2s} + 24\Big(\frac{1+a}{4+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) \\
& \quad + 48|\gamma|^2\Big(\Big(\frac{1+a}{3+a}\Big)^{2s} + \Big(\frac{1+a}{4+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Big) + 3\Big(\frac{1+a}{4+a}\Big)^s\Big(\frac{1+a}{2+a}\Big)^s\Bigg] \\
& + \frac{4|\gamma|^2(1+4|\gamma|)(1+3|\gamma|)}{9\left(\frac{1+a}{4+a}\right)^{2s}\left(\frac{1+a}{2+a}\right)^s\left(\frac{1+a}{3+a}\right)^s} \\
& \times \Bigg[12|\gamma|^2\Big(\Big(\frac{1+a}{4+a}\Big)^s + \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) \\
& \quad + |\gamma|\Big(3\Big(\frac{1+a}{4+a}\Big)^s + 7\Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Big) + \Big(\frac{1+a}{2+a}\Big)^s\Big(\frac{1+a}{3+a}\Big)^s\Bigg] \\
& + \frac{|\gamma|^2\Big(29|\gamma| + 92|\gamma|^2 + 96|\gamma|^3 + 3\Big)}{6\left(\frac{1+a}{5+a}\right)^s\left(\frac{1+a}{3+a}\right)^s}.
\end{aligned}
$$

This completes the proof. □

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