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An effective criterion for the additive decompositions of forms

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Abstract. *We give an effective criterion for the identifiability of additive decompositions of homogeneous forms of degree d in a fixed number of variables. Asymptotically for large d it has the same order of the Kruskal's criterion adapted to symmetric tensors given by L. Chiantini, G. Ottaviani and N. Vannieuwenhoven. We give a new case of identifiability for $d = 4$.*

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1. Introduction

Let $\mathbb{C}[z_0, \dots, z_n]_d$ denote the complex vector space of all homogeneous degree d polynomials in the variables $z_0, \dots, z_n$. An *additive decomposition* (or a Waring decomposition) of a form $f \in \mathbb{C}[z_0, \dots, z_n]_d \setminus \{0\}$ is a finite sum

$$f = \sum \ell_i^d \tag{1}$$

with each $\ell_i \in \mathbb{C}[z_0, \dots, z_n]_1$. The minimal number $R(f)$ of summands in an additive decomposition of f is called the *rank* of f. The form f is said to be *identifiable* if it has a unique decomposition (1), up to a permutation of the summands. Often it is called an additive decomposition of f a finite sum

$$f = \sum c_i \mu_i^d \tag{2}$$

with $c_i \in \mathbb{C}$ and $\mu_i \in \mathbb{C}[z_0, \dots, z_n]_1$. Taking $b_i \in \mathbb{C}$ such that $b_i^d = c_i$ and setting $\ell_i := b_i \mu_i$ we see that the two definitions coincide and that (1) and (2) have the same number of non-zero summands. Degree d forms in the variables $z_0, \dots, z_n$ correspond to symmetric tensors of format $(n+1) \times \cdots \times (n+1)$ (d times), i.e. to symmetric elements of $(\mathbb{C}^{n+1})^{\otimes d}$. An additive decomposition (2) of f is said to be *non redundant* or *irredundant* if there are no index i such that $c_i \ell_i^d$ is a linear combination of the other $c_j \ell_j^d$'s, $j \neq i$. See [23] for a long list of possible applications and the language needed. Obviously it is interesting to know when a non redundant decomposition of f has only $R(f)$ summands, because just from knowing the non redundant decomposition we would know that f has no shorter additive decompositions. More important (as stressed in [17, 18]) is to know if f is identifiable.

In [18] L. Chiantini, G. Ottaviani and N. Vannieuwenhoven stressed the importance (even for arbitrary tensors) of effective criteria for the identifiability and gave a long list of practical applications (with explicit examples even in Chemistry). We add to the list of potential applications the tensor networks ([13, 14, 25], at least for tensors without symmetries. For the case of bivariate forms, see [11]; for bivariate forms the identifiability of a form only depends on its rank and, for generic bivariate forms, on the parity of d by a theorem of Sylvester ([21, Theorem 1.5.3 (ii)]).

L. Chiantini, G. Ottaviani and N. Vannieuwenhoven stressed the importance of the true effectivity of the criterion to be tested as it happens in the case of the famous Kruskal's criterion for the tensor decomposition ([22]). They reshaped the Kruskal's criterion to the case of additive decompositions ([18, Theorem 4.6 and Proposition 4.8]) and proved that it is effective (for $d \geq 5$) for ranks at most $\sim n^{\lfloor (d-1)/2 \rfloor}$. The upper bound to which our criterion applies has the same asymptotic order when $d \gg 0$, but we hope that it is easy and efficient. Then in [3] E. Angelini, L. Chiantini and N. Vannieuwenhoven considered the case $d = 4$ and added the analysis of one more rank. Among the huge number of papers considering mostly "generic" identifiability we also mention [1, 2, 4, 15, 16, 17, 19]. An effective criterion should be something machine-testable in a reasonable time and that to be applied to the form f only requires data from the additive decomposition (1). In our case we need the forms ℓ_i's in the right hand side of (1) (we only need them up to a scalar multiple, but we need them exactly, not approximately) and the computation of the rank of a matrix with ρ rows and $\binom{n+t}{n}$ columns, where ρ is the number of summands in (1) and $t \leq \lfloor d/2 \rfloor$ (but t may be lower for lower ρ); see Remark 2.1 for more details.

To state our results we need the following geometric language for instance fully explained in [18, 23].

Set $\mathbb{P}^n := \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_1$. Thus points of the n-dimensional complex space p correspond to non-zero linear forms, up to a non-zero multiplicative constant. Set $r := \binom{n+d}{n} - 1$. Thus $\mathbb{P}^r := \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d$ is an r-dimensional projective space. Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^r$ denote the order d Veronese embedding, i.e. the map defined by the formula $[\ell] \mapsto [\ell^d]$. An additive decomposition (1) or (2) with k non-proportional non-zero terms corresponds to a subset $S \subset \mathbb{P}^n$ such that $|S| = k$ and $[f] \in \langle \nu_d(S) \rangle$, where $\langle\ \rangle$ denote the linear span. This decomposition is called *non redundant* and we say that the set $\nu_d(S)$ *irredundantly spans* $[f]$ if $[f] \in \langle \nu_d(S) \rangle$ and $[f] \notin \langle \nu_d(S') \rangle$ for each $S' \subsetneq S$. For any integer $t \geq 0$ each $p \in \mathbb{P}^n$ gives a linear condition to the vector space $\mathbb{C}[z_0, \dots, z_n]_t$ by taking

$p_1 = (a_0, \dots, a_n) \in \mathbb{C}^{n+1}$ with $[p_1] = p$ and evaluating each $f \in \mathbb{C}[z_0, \dots, z_n]_t$ at p_1. When we perform this evaluation for all points of a finite set $S \subset \mathbb{P}^n$ we get $|S|$ linear equations and the rank of the corresponding matrix does not depend on the choice of the representatives of the points of S.

We prove the following result.

THEOREM 1.1. *Fix $q \in \mathbb{P}^r$ and take a finite set $S \subset \mathbb{P}^n$ such that $\nu_d(S)$ irredundantly spans q.*

(a) If $|S| \le \binom{n+\lfloor d/2\rfloor}{n}$ and S gives $|S|$ independent conditions to the complex vector space $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2\rfloor}$, then q has rank $|S|$.

(b) If $|S| \le \binom{n+\lfloor d/2\rfloor-1}{n}$ and S gives $|S|$ independent conditions to the complex vector space $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2\rfloor-1}$, then S is the unique set evincing the rank of q.

In Remark 2.1 we explain why Theorem 1.1 effectively determines the rank of q (and in the set-up of (b) the identifability of f, i.e. the uniqueness statement often called “uniqueness of additive decomposition ” for homogeneous polynomials or for symmetric tensors). Indeed, to check that S satisfies the assumptions of part (a) (resp. part (b)) of Theorem 1.1 it is sufficient to check that a certain matrix with $|S|$ rows and $\binom{n+\lfloor d/2\rfloor}{n}$ (resp $\binom{n+\lfloor d/2\rfloor-1}{n}$) columns has rank $|S|$. This matrix has rank $|S|$ if S is sufficiently general, but the test is effective for a specific set S.

See [7] and [8] for results similar to Theorem 1.1 for tensors; roughly speaking [8, Corollary 3.10, Remark 3.11 and their proof] is equivalent to part (a) of Theorem 1.1. Part (a) of Theorem 1.1 is good, but one could hope to get part (b) when $|S| < \binom{n+\lfloor d/2\rfloor}{n}$, adding some other easily testable assumptions on S. We prove the following strong result (an essential step for the proof of part (b) of Theorem 1.1). To state it we recall the following notation: for any finite set $E \subset \mathbb{P}^n$ and any $t \in \mathbb{N}$ let $H^0(\mathcal{I}_E(t))$ denote the set of all $f \in \mathbb{C}[z_0, \dots, z_n]_t$ such that $f(p) = 0$ for all $p \in E$. The set $H^0(\mathcal{I}_E(t))$ is a vector space of dimension at least $\binom{n+t}{n} - |E|$. Set $|\mathcal{I}_E(t)| := \mathbb{P}H^0(\mathcal{I}_E(t))$.

THEOREM 1.2. *Fix $q \in \mathbb{P}^r$ and take a finite set $S \subset \mathbb{P}^n$ such that $\nu_d(S)$ irredundantly spans q. Assume $|S| < \binom{n+\lfloor d/2\rfloor}{n}$ and that S gives $|S|$ gives independent conditions to $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2\rfloor}$. Take any $A \subset \mathbb{P}^n$ such that $|A| = |S|$ and A induces an additive decomposition of f. Then $H^0(\mathcal{I}_A(\lfloor d/2\rfloor)) = H^0(\mathcal{I}_S(\lfloor d/2\rfloor))$.*

Theorem 1.2 does not assure that S is the only set evincing the rank of q, i.e. the uniqueness of the summands in an additive decomposition of f with $R(f)$ terms, but it shows where the other sets A giving potential additive decomposition with $R(f)$ summands may be located: they are contained in the base locus of $|\mathcal{I}_S(\lfloor d/2\rfloor)|$. The results in [3] (in particular [3, Theorem 6.2 and 6.3, Proposition 6.4]) for $d = 4$ show that non-uniqueness occurs if and only if the base locus of $|\mathcal{I}_S(2)|$ allows the existence of A.

In the last section we take $d = 4$. E. Angelini, L. Chiantini and N. Vannieuwenhoven consider the case $d = 4$ and $|S| = 2n + 1$ with an additional geometric property (linear general position or LGP for short; section 3 for its definition). For $d = 4$ and $|S| = 2n + 1$ they classified the set S in LGP for which identifiability holds (see Theorem 3.1 for a summary of [3, Theorems 6.2 and 6.3]). In section 3 using Theorem 1.2 we classify another family of sets S with $|S| = 2n + 1$ and for which identifiability holds (Theorem 3.2).

Remark 1.3: The results used to prove Theorem 1.1 (and summarized in Lemma 2.3 and Remark 2.4) work verbatim for a zero-dimensional scheme $A \subset \mathbb{P}^n$. The key is that in Lemma 2.3 and Remark 2.4 or in [6, Lemma 5.1] (or equivalently [9, Lemmas 2.4 and 2.5]) we may allow that one of the two schemes is not reduced. Under the assumption of part (a) of Theorem 1.1 the cactus rank of q (see [10, 12, 26] for its definition and its uses) is $|S|$. Under the assumptions of part (b) of Theorem 1.1 S is the only zero-dimensional subscheme of $\mathbb{P}^n$ evincing the cactus rank of q. However for our proofs it is important that S (i.e. the scheme to be tested) is a finite set, not a zero-dimensional scheme. Now assume that W is a zero-dimensional scheme and take $q \in \langle \nu_d(W)\rangle$ such that $q \notin \langle \nu_d(W')\rangle$ for any $W' \subsetneq W$. Assume that W is not reduced, that $\deg(W) \le \binom{n+\lfloor d/2-1\rfloor}{n}$ and that W gives $\deg(W)$ independent conditions to $\mathbb{C}[z_0,\dots,z_n]_{\lfloor d/2-1\rfloor}$. Quoting either [6, Lemma 5.1] or [9, Lemmas 2.4 and 2.5] we get that q has rank $> \deg(W)$.

Remark 1.4: The interested reader may check that the proof works with no modification if instead of $\mathbb{C}$ we take any algebraically closed field containing $\mathbb{Q}$. Since it uses only linear systems, it works over any field $K \supseteq \mathbb{Q}$ if as an additive decomposition of $f \in K[z_0,\dots,z_n]_d$ we take an expression (2) with $c_i \in K$ and $\ell_i \in K[z_0,\dots,z_n]_1$. Thus for the real field $\mathbb{R}$ when d is odd we may take the usual definition (1) of additive decomposition, while if d is even we allow $c_i \in \{-1, 1\}$. Theorem 1.1 applied to $\mathbb{C}$ says that $|S|$ is the complex rank of q, too, and in set-up of part (b) uniqueness holds even if we allow complex decompositions.

Remark 1.5: In the proofs of our results we use nothing about the form f or the point $q = [f] \in \mathbb{P}^r$. All our assumptions are on the set S and they apply to all $q \in \langle \nu_d(S)\rangle$ irredundantly spanned by $\nu_d(S)$. In all our results the set $\nu_d(S)$ is linearly independent (i.e. its elements are linearly independent) and hence the set of all $q \in \mathbb{P}^r$ irredundantly spanned by $\nu_d(S)$ is the complement in the $(|S| - 1)$-dimensional linear space $\langle \nu_d(S)\rangle$ of $|S|$ codimension 1 linear subspaces. To test that $\nu_d(S)$ irredundantly spans q it is sufficient to check the rank of a matrix with $|S|$ rows and $\binom{n+d}{n}$ columns. To the best of our knowledge this check (or a very similar one) must be done for all criteria of effectivity for forms ([3]).

2. The proofs of Theorems 1.1 and 1.2

Fix $q \in \mathbb{P}^r = \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d$. The *rank* $r_X(q)$ of q is the minimal cardinality of a finite set $S \subset \mathbb{P}^n$ such that $q \in \langle \nu_d(S) \rangle$. By the definition of Veronese embedding we have $r_X(q) = R(f)$ for any $f \in \mathbb{C}[z_0, \dots, z_n]_d$ such that $[f] = q$. Let $\mathcal{S}(X, q)$ denote the set of all $S \subset \mathbb{P}^n$ such that $q \in \langle \nu_d(S) \rangle$ and $|S| = r_X(q)$. We say that q is *identifiable with respect to* X or that q is X*-identifiable* if $|\mathcal{S}(X, q)| = 1$. By the construction of the order d Veronese embedding of X, $|\mathcal{S}(X, q)| = 1$ if and only if any form $f \in \mathbb{C}[z_0, \dots, z_n]_d$ with $[f] = q$ is identifiable. Recall that a finite subset $E \subset \nu_d(\mathbb{P}^n)$ irredundantly spans q if $q \in \langle E \rangle$ and $q \notin \langle E' \rangle$ for any $E' \subsetneq E$. Note that if E irredundantly spans a point of $\mathbb{P}^r$, then it is linearly independent, i.e. $\dim \langle E \rangle = |E| - 1$. If $E = \nu_d(A)$ for some $A \subset \mathbb{P}^n$, E is linearly independent if and only if A induces $|A|$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_d$. For each $S \in \mathcal{S}(X, q)$ the set $\nu_d(S)$ irredundantly spans q. For any $Z \subset \mathbb{P}^n$ and any $t \in \mathbb{Z}$ set $h^0(\mathcal{I}_Z(t)) := \dim H^0(\mathcal{I}_Z(t))$.

Remark 2.1: Fix an integer $t \geq 0$ and a finite subset A of $\mathbb{P}^n$. We write $h^1(\mathcal{I}_A(t))$ for the difference between $|A|$ and the number of independent conditions that A imposes to the $\binom{n+t}{n}$-dimensional vector space $\mathbb{C}[z_0, \dots, z_n]_t$. For any multiindex $\alpha = (a_0, \dots, a_n) \in \mathbb{N}^{n+1}$ set $z^\alpha := z_0^{a_0} \cdots z_n^{a_n}$ and $\|\alpha\| = a_0 + \cdots + a_n$. The integer $\|\alpha\|$ is the degree of the monomial z^α. The vector space $\mathbb{C}[z_0, \dots, z_n]_t$ of all degree t homogeneous polynomials in $z_0, \dots, z_n$ has the monomials z^α with $\|\alpha\| = t$ as a basis. We explain why to compute the non-negative integer $h^1(\mathcal{I}_A(t))$ we only need to compute the rank of the matrix with $|A|$ rows and $\binom{n+t}{n}$ columns. Since $h^1(\mathcal{I}_A(t)) = |A| - \binom{n+t}{n} + h^0(\mathcal{I}_A(t))$, it is sufficient to compute the integer $h^0(\mathcal{I}_A(t))$. Set $a := |A|$ and $b := \binom{n+t}{n}$. We order the points $p_1, \dots, p_a$ of A and the monomials z^α with $\|\alpha\| = t$. We call $w_1, \dots, w_b$ these monomials with the chosen ordering. The integer $a - h^1(\mathcal{I}_A(t))$ is the rank of the $a \times b$ matrix $M = (a_{ij})$ with as entry a_{ij} the value of w_j at p_i.

Remark 2.2: Fix $q \in \mathbb{P}^r = \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d \setminus \nu_d(\mathbb{P}^n)$ and take $A \subset \mathbb{P}^n$ such that $\nu_d(A)$ irredundanly spans q. The condition " $q \notin \nu_d(\mathbb{P}^n)$ " is equivalent to "$r_X(q) > 1$ ". Since $\nu_d(A)$ spans irredundantly at least one point of $\mathbb{P}^r$, it is linearly independent, i.e. $h^1(\mathbb{P}^r, \mathcal{I}_{\nu_d(A)}(1)) = 0$. Since $q \in \langle \nu_d(A) \rangle$ and $q \notin \nu_d(A)$, we have $h^1(\mathbb{P}^r, \mathcal{I}_{\nu_d(A) \cup \{q\}}(1)) > 0$. Since $h^1(\mathbb{P}^r, \mathcal{I}_{\nu_d(A)}(1)) = 0$ and $|\nu_d(A) \cup \{q\}| = |\nu_d(A)| + 1$, we have $h^1(\mathbb{P}^r, \mathcal{I}_{\nu_d(A) \cup \{q\}}(1)) = 1$.

Fix $f \in \mathbb{C}[z_0, \dots, z_n]_d \setminus \{0\}$ and let $q = [f] \in \mathbb{P}^r = \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d$, $r = \binom{n+d}{n} - 1$, be the point associated to f. Take $S \subset \mathbb{P}^n$ such that $\nu_d(S)$ irredundantly spans q. Fix any $A \subset \mathbb{P}^n$ evincing the rank of f. We have $|A| \leq |S|$. Set $Z := A \cup B$. Z is a finite subset of $\mathbb{P}^n$ and $|Z| \leq |A| + |S|$. To prove part (a) of Theorem 1.1 we need to prove that $|A| = |S|$. To prove part

(b) we need to prove that $A = S$. In the proof of part (a) we have $A \neq S$, because $|A| < |S|$. To prove part (b) of the theorem it is sufficient to get a contradiction from the assumption $A \neq S$.

We recall (with the same proof) [5, Lemma 1].

LEMMA 2.3. *Fix $q \in \mathbb{P}^r = \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d$ and assume the existence of $A, B \subset \mathbb{P}^n$ such that $\nu_d(A)$ and $\nu_d(B)$ irredundantly span q and $A \neq B$.*

Then $A \cup B$ does not impose $|A \cup B|$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_d$, i.e. $h^1(\mathcal{I}_{A\cup B}(d)) \neq 0$.

Proof. For all linear subspaces $U, W \subseteq \mathbb{P}^r$ the Grassmann's formula says that

$$\dim(U \cap W) + \dim(U + W) = \dim U + \dim W$$

with the convention $\dim \emptyset = -1$. Since $\nu_d(A)$ (resp. $\nu_d(B)$) irredundantly spans q, we have $\dim\langle\nu_d(A)\rangle = |A| - 1$ (resp. $\dim\langle\nu_d(B)\rangle = |B| - 1$). Since $A \neq B$, we have $A \cap B \subsetneq A$ and $A \cap B \subsetneq B$. Since $q \in \langle\nu_d(A)\rangle \cap \langle\nu_d(B)\rangle$ and $q \notin \langle\nu_d(A \cap B)\rangle$, we have $\langle\nu_d(A) \cap \langle\nu_d(B)\rangle \supsetneq \langle\nu_d(A \cap B)\rangle$. Since $\nu_d(A)$ and $\nu_d(B)$ are linearly independent and $\langle\nu_d(A)\rangle \cap \langle\nu_d(B)\rangle \supsetneq \langle\nu_d(A \cap B)\rangle$, the Grassmann's formula gives that $\nu_d(A \cup B)$ is not linearly independent, i.e. $A \cup B$ does not impose $|A \cup B|$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_d$. □

REMARK 2.4: We explain the particular case of [6, Lemma 5.1] or [9, Lemmas 2.4 and 2.5] we need. Fix $q \in \mathbb{P}^r = \mathbb{P}\mathbb{C}[z_0, \dots, z_n]_d$ and take finite sets $A, B \subset \mathbb{P}^n$ such that $\nu_d(A)$ and $\nu_d(B)$ irredundantly span q. In particular both A and B are linearly independent. Set $Z := A \cup B$. We fix $G \in \mathbb{C}[z_0, \dots, z_n]_t$, $1 \le t \le d$. We assume that $Z \setminus Z \cap G$ gives $|Z \setminus Z \cap G|$ independent conditions to $\mathbb{C}[z_0, \dots, z_n]_{d-t}$, i.e. we assume $h^1(\mathbb{P}^n, \mathcal{I}_{Z\setminus Z\cap G}(d - t)) = 0$. By either [6, Lemma 5.1] or [9, Lemmas 2.4 and 2.5] we have $A \setminus A \cap G = B \setminus B \cap G$. In particular if $A \subset G$, then $B \subset G$.

Proof of part (a) of Theorem 1.1: Recall that we have $\dim \mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2\rfloor} = \binom{n+\lfloor d/2\rfloor}{n}$. Since $|A| < |S| \le \binom{n+\lfloor d/2\rfloor}{n}$, there is $g \in \mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2\rfloor}$ such that $g(p) = 0$ for all $p \in A$. Let $G \subset \mathbb{P}^n$ be the degree $\lfloor d/2\rfloor$ hypersurface $\{g = 0\}$ of $\mathbb{P}^n$. Since $A \subset G$, we have $Z \setminus Z \cap G = S \setminus S \cap G$. Thus $Z \setminus Z \cap G$ gives independent conditions to forms of degree $\lfloor d/2\rfloor$. Thus it gives independent conditions to forms of degree $\lceil d/2\rceil = d - \lfloor d/2\rfloor$. Since $A \subset G$, Remark 2.4 gives $S \subset G$. Since this is true for all $g \in \mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2\rfloor}$ such that $g(p) = 0$ for all $p \in A$, we get that if $g_{|A} = 0$ and g has degree $\lfloor d/2\rfloor$, then $g_{|S} = 0$. Since S gives $|S|$ independent linear conditions to $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2\rfloor}$, A gives at least $|S|$ linear independent conditions to $\mathbb{C}[z_0, \dots, z_n]_{\lfloor d/2\rfloor}$, contradicting the inequality $|A| < |S|$. □

Proof of Theorem 1.2. To prove Theorem 1.2 we may assume $A \neq S$. Since $|A| = |S| < \binom{n+\lfloor d/2\rfloor}{n}$, there is $g \in \mathbb{C}[z_0,\dots,z_n]_{\lfloor d/2\rfloor}$ such that $g_{|A} \equiv 0$. The proof of part (a) of Theorem 1.1 gives $g_{|S} \equiv 0$. Thus $H^0(\mathcal{I}_A(\lfloor d/2\rfloor)) \subseteq H^0(\mathcal{I}_S(\lfloor d/2\rfloor))$. Since $H^0(\mathcal{I}_S(\lfloor d/2\rfloor))$ has codimension $|A|$ in $\mathbb{C}[z_0,\dots,z_n]_{\lfloor d/2\rfloor}$, we get $H^0(\mathcal{I}_A(\lfloor d/2\rfloor)) = H^0(\mathcal{I}_S(\lfloor d/2\rfloor))$. □

Proof of part (b) of Theorem 1.1: We have $H^0(\mathcal{I}_A(\lfloor d/2\rfloor)) = H^0(\mathcal{I}_S(\lfloor d/2\rfloor))$ by Theorem 1.2. To get $A = S$ it is sufficient to prove that for each $p \in \mathbb{P}^n \setminus A$ there is $g \in H^0(\mathcal{I}_S(\lfloor d/2\rfloor))$ such that $g(p) \neq 0$. Thus it is sufficient to prove that the sheaf $\mathcal{I}_S(\lfloor d/2\rfloor)$ is generated by its global sections. The assumption that S gives $|S|$ independent conditions to $\mathbb{C}[z_0,\dots,z_n]_{\lfloor d/2\rfloor-1}$ is translated in cohomological terms as $h^1(\mathbb{P}^n, \mathcal{I}_S(\lfloor d/2\rfloor - 1)) = 0$. The sheaf $\mathcal{I}_S(\lfloor d/2\rfloor)$ is generated by its global sections (and in particular for each $p \in \mathbb{P}^n \setminus S$ there is $f \in H^0(\mathcal{I}_S(\lfloor d/2\rfloor))$ such that $f(p) \neq 0$) by the Castelnuovo-Mumford's lemma ([20, Corollary 4.18], [24, Theorem 1.8.3]). □

3. The case $d = 4$

Set $X := \nu_d(\mathbb{P}^n) \subset \mathbb{P}^r$.

A finite set $S \subset \mathbb{P}^n$ is said to be in *linearly general position* (or in LGP, for short) if $\dim\langle A\rangle = \min\{n, |A|-1\}$ for each $A \subseteq S$. If $|S| \geq n+1$ the set S is in LGP if and only if each $A \subseteq S$ with $|A| = n+1$ spans $\mathbb{P}^n$.

In this section we take $d = 4$ and hence $r = \binom{n+4}{n} - 1$.

We recall a summary of [3, Theorems 6.2 and 6.3].

Theorem 3.1. *([3, Theorems 6.2 and 6.3]). Fix a finite set $S \subset \mathbb{P}^n$ in LGP such that $|S| = 2n+1$ and take $q \in \mathbb{P}^r$, $r = \binom{n+4}{n} - 1$, such that $\nu_4(S)$ irredundantly spans q.*

1. *q has rank $2n+1$.*
2. *Assume the existence of $B \subset \mathbb{P}^n$ such that $|B| = 2n+1$ and $B \neq S$. Then $B \cup S$ is contained in a rational normal curve of $\mathbb{P}^n$.*

We prove the following result.

Theorem 3.2. *Fix a finite set $S \subset \mathbb{P}^n$ such that $|S| = 2n+1$ and take $q \in \mathbb{P}^r$, $r = \binom{n+4}{n} - 1$, such that $\nu_4(S)$ irredundantly spans q. Assume that S is not in LGP, but there is $S' \subset S$ such that $|S'| = 2n$ and S' is in LGP. The point q has rank $2n+1$. Let e be the dimension of a minimal subspace $N \subset \mathbb{P}^n$ such that $|N \cap S| \geq e+2$. The point $q \in \mathbb{P}^r$ is identifiable if and only if $e \geq 2$. If $e = 1$, then* $\dim \mathcal{S}(X, q) = 1$.

To prove Theorem 3.2 we need some elementary observations.

Remark 3.3: Take $A \subset \mathbb{P}^m$, $m \ge 1$, such that $|A| = m+2$ and A is in LGP. It is classically known that any two such sets are projectively equivalent; we provide a linear algebra proof of this fact. We order the points $p_0, \dots, p_{m+1}$ of $|A|$. Since $p_0, \dots, p_m$ are $m+1$ linearly independent points, up to a change of homogeneous coordinates we may assume that $p_0, \dots, p_m$ are the $m+1$ coordinate points $(1 : 0 : \cdots : 0), \dots, (0 : \cdots : 0 : 1)$. Write $p_{m+1} = (w_0 : \cdots : w_m)$ for some $w_i \in \mathbb{C}$. The assumption that A is in LGP is equivalent to $w_i \ne 0$ for all i. We make the invertible projective transformation $z_i \mapsto {w_i}^{-1} z_i$, which leave fixed each p_i, $0 \le i \le m$, and maps p_{m+1} to the point $(1 : 1 : \cdots : 1)$. Thus each $B \subset \mathbb{P}^m$ in LGP such that $|B| = m+2$ is projectively equivalent to the set consisting of the coordinate points, plus the point $(a_0 : \cdots : a_m)$ with $a_i = 1$ for all i. In particular A is projectively equivalent to a general subset of $\mathbb{P}^m$ with cardinality $m+2$. Thus $h^1(\mathcal{I}_A(2)) = 0$.

Claim 1: The set A is the set-theoretic base locus of $|\mathcal{I}_A(2)|$ if and only if $m \ge 2$.

Proof of Claim 1: First assume $m = 1$. In this case we have $\mathcal{O}_{\mathbb{P}^1}(2)(-A) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and hence $h^0(\mathcal{O}_{\mathbb{P}^1}(2)(-A)) = 0$. Now assume $m = 2$. In this case Claim 1 is equivalent to say that 4 points of a plane, no 3 of them collinear, are the complete intersection of 2 conics; not only this is easy, but (since we proved that we may assume that A is general in $\mathbb{P}^2$), it is true because 2 general plane conics intersects transversally. Now assume $m > 2$. Fix $o \in \mathbb{P}^m \setminus A$. Let $A' \subset A$ be a subset of A such that $o \in \langle A' \rangle$ and with $|A'|$ minimal (it exists, because A spans $\mathbb{P}^m$). Since $o \notin A$, we have $m+1 \ge |A'| > 1$. Take $a \in A'$. Since A is in LGP, there is a hyperplane $H \subset \mathbb{P}^m$ such that $|A \cap H| = m$ (and hence H is spanned by $A \cap H$), $A' \setminus \{a\} \subset H$ and $a \notin H$. Set $\{a, b\} := A \setminus A \cap H$. Since $A' \cap H = A' \setminus \{a\}$ and the set A' is linearly independent, we have $H \cap \langle A' \rangle = \langle A' \setminus \{a\} \rangle$. Thus $o \notin H$. Assume for the moment $o \notin \langle \{a, b\} \rangle$. Thus $o \notin M$ for a general hyperplane $M \supseteq \langle \{a, b\} \rangle$. The hyperquadric $H \cup M$ contains A, but $o \notin H \cap M$. Hence $o \notin \mathcal{B}$. Now assume $o \in \langle \{a, b\} \rangle$. Since $|A'|$ is minimal and $|A'| > 1$, we have $|A'| = 2$. Write $A' = \{a, c\}$. Since $o \in \langle \{a, c\} \rangle \cap \langle \{a, b\} \rangle$ and $o \ne a$, the 3 points a, b, c are collinear, a contradiction.

Remark 3.4: Take $A \subset \mathbb{P}^m$, $m \ge 1$, such that $|A| = m+1$ and A spans $\mathbb{P}^m$. Up to a projective transformation we may assume that A is the union of the coordinates points of $\mathbb{P}^n$. As in Remark 3.3 by induction on m we see that $h^1(\mathcal{I}_A(2)) = 0$ and that A is the base locus of the linear system $|\mathcal{I}_A(2)|$.

Remark 3.5: Take $A \in \mathcal{S}(X, q)$ and any $A' \subsetneq A$, $A' \ne \emptyset$. Set $A'' := A \setminus A'$. In particular $|A| \ge 2$ and hence $q \notin X$. Since A evinces the X-rank of q, it is linearly independent and $h^1(\mathbb{P}^r, \mathcal{I}_{A \cup \{q\}}(1)) = 1$ (Remark 2.2). Since $A'' \subsetneq A$, we have $q \notin \langle A'' \rangle$. Thus $\langle A' \rangle \cap \langle A'' \cup \{q\} \rangle$ is a single point, q', and q' is the only element of $\langle A' \rangle$ such that $q \in \langle \{q'\} \cup A'' \rangle$. In the same way we see the existence of a single point $q'' \in \langle A'' \rangle$ such that $q \in \langle A' \cup \{q''\} \rangle$. We have $q \in \langle \{q', q''\} \rangle$. Since $A \in \mathcal{S}(X, q)$, we have $A' \in \mathcal{S}(X, q')$ and $A'' \in \mathcal{S}(X, q'')$. If we only assume that A irredundantly spans q the same proof gives the existence and uniqueness of q' and q'' such that A' irredundantly spans q' and A'' irredundantly spans q''.

LEMMA 3.6. *Let $H \subset \mathbb{P}^m$, $m \geq 2$, be a hyperplane. Take a finite set $S \subset \mathbb{P}^m$ such that $|S \setminus S \cap H| = 1$. Take homogeneous coordinates $z_0, \dots, z_m$ of $\mathbb{P}^m$ such that $H = \{z_m = 0\}$.*

(i) If $S \cap H$ imposes independent conditions to $\mathbb{C}[z_0, \dots, z_{m-1}]_2$, then S imposes independent conditions to $\mathbb{C}[z_0, \dots, z_m]_2$.

(ii) If $S \cap H$ is the base locus of $|\mathcal{I}_{S\cap H}(2)|$, then S is the base locus of $|\mathcal{I}_S(2)|$.

Proof. Set $\{p\} := S \setminus S \cap H$ and call $\mathcal{B}$ the base locus of $|\mathcal{I}_S(2)|$. We have the residual exact sequence of H:

$$0 \to \mathcal{I}_p(1) \to \mathcal{I}_S(2) \to \mathcal{I}_{S\cap H,H}(2) \to 0 \tag{3}$$

Since $\{p\}$ imposes independent conditions to $\mathbb{C}[z_0, \dots, z_m]_1$, we get part (i) and that the restriction map $\rho : H^0(\mathcal{I}_S(2)) \to H^0(H, \mathcal{I}_{S\cap H,H}(2))$ is surjective. Assume that $S \cap H$ is the base locus of $|\mathcal{I}_{S\cap H}(2)|$. Since ρ is surjective, we get $\mathcal{B} \cap H = S \cap H$. Fix $o \in \mathbb{P}^n \setminus H$ such that $o \neq p$. Take a hyperplane $M \subset \mathbb{P}^m$ such that $p \in M$ and $o \notin M$. The reducible quadric $H \cup M$ shows that $o \notin \mathcal{B}$. □

Proof of Theorem 3.2: Let $H \subset \mathbb{P}^n$ be a hyperplane containing N and spanned by points of S'. Since S' is in LGP and $|S| = |S'| + 1$, we have $|S \cap H| = n+1$, $|S' \cap H| = n$, $S \setminus S \cap H = S' \setminus S' \cap H$, and $|S' \setminus S' \cap H| = n$. Since S' is in LGP, $S' \setminus S' \cap H$ spans a hyperplane, M, and $S' \cap H \cap M = \emptyset$. Set $A := S' \cap H$ and $B := S' \cap M$. Note that $S \subset H \cup M$, $n \leq |M \cap S| \leq n+1$ and $|S \cap M| = n+1$ if and only if $S \setminus S' \subset H \cap M$, i.e. if and only if $N \subseteq H \cap M$. Set $\mathcal{B} := \{p \in \mathbb{P}^n \mid h^0(\mathcal{I}_{S\cup\{p\}}(2)) = h^0(\mathcal{I}_S(2))\}$. Since $S \subset H \cup M$ and $H \cup M$ is a quadric hypersurface, we have $S \subseteq \mathcal{B} \subseteq H \cup M$. Consider the residual exact sequences of H and M:

$$0 \to \mathcal{I}_{S\setminus S\cap H}(1) \to \mathcal{I}_S(2) \to \mathcal{I}_{S\cap H,H}(2) \to 0 \tag{4}$$

$$0 \to \mathcal{I}_{S\setminus S\cap M}(1) \to \mathcal{I}_S(2) \to \mathcal{I}_{S\cap M,M}(2) \to 0 \tag{5}$$

Note that $\mathcal{B}$ contains the base locus $\mathcal{B}_1$ of $\mathcal{I}_{S\cap H,H}(2)$ and the base locus $\mathcal{B}_2$ of $\mathcal{I}_{S\cap M,M}(2)$.

By Remark 3.4 we have $h^1(H, \mathcal{I}_{S\cap H,H}(2)) = h^1(M, \mathcal{I}_{S\cap M}(2)) = 0$. By the long cohomology exact sequence of (4) we get $h^1(\mathcal{I}_S(2)) = 0$. Theorem 1.1 gives that q has rank $2n+1$. By the long cohomology exact sequences of (4) and (5) the restriction maps $\rho : H^0(\mathcal{I}_S(2)) \to H^0(H, \mathcal{I}_{S\cap H,H}(2))$ and $\rho' : H^0(\mathcal{I}_S(2)) \to H^0(M, \mathcal{I}_{S\cap M,M}(2))$ are surjective. Thus $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Since $S \cap M$ is linearly independent, we have $\mathcal{B}_2 = S \cap M$. Take $F \in \mathcal{S}(X, q)$ such that $F \neq S$ (if any). In the case $e \geq 2$ we need to find a contradiction. In the case $e = 1$ we need a description of all F's sufficiently explicit to prove that $\dim \mathcal{S}(X, q) = 1$. More precisely, in the case $e = 1$ we will prove the existence of a subset $F_2 \subset F$ such that all $E \in \mathcal{S}(X, q)$ are of the form $F_2 \cup E_1$ with E_1 depending on E, F_2 the same for all $E \in \mathcal{S}(X, q)$ (and in particular $F_2 \subset S$) and E_1 coming from a bivariate form q' associated to q. Since q has rank $\leq 2n+1$, we have $|F| \leq 2n+1 < \binom{n+2}{2}$. Thus there is $G \in |\mathcal{I}_F(2)|$. Set $Z := S \cup F$. Fix any $G \in |\mathcal{I}_F(2)|$. Since $Z \setminus Z \cap G \subseteq S$ and $h^1(\mathcal{I}_S(2)) = 0$, we have $h^1(\mathcal{I}_{Z \setminus Z \cap G}(2)) = 0$. Since $F \subset G$, Lemma 2.3 and Remark 2.4 give $S \subset G$. Since this is true for all $G \in |\mathcal{I}_F(2)|$, we get $|\mathcal{I}_S(2)| \supseteq |\mathcal{I}_F(2)|$. Since $|S| \geq |F|$ and $h^1(\mathcal{I}_S(2)) = 0$, we get again $|F| = |S|$ and also that $|\mathcal{I}_S(2)| = |\mathcal{I}_F(2)|$. Since F is contained in the base locus of $|\mathcal{I}_F(2)|$, we get $F \subseteq \mathcal{B}$.

(a1) Assume $e \geq 2$. By Remark 3.3 $S \cap N$ is the base locus of $\mathcal{I}_{S \cap N}(2)$. Applying (if $e < n-1$) $n-1-e$ times Lemma 3.6 we get $\mathcal{B}_1 = S \cap H$. Thus $\mathcal{B} = S$. Hence $F \subseteq S$, a contradiction.

(a2) Assume $e = 1$. In this case $\mathcal{B}$ contains the line N. The proof of Lemma 3.6 gives $\mathcal{B}_1 = N \cup (S \cap H)$. By Theorem 1.2 we have $F \subset N \cap (S \setminus S \cap N)$. Thus $F = A_1 \cup A_2$ with $A_1 \subset N$, $A_2 \subseteq S \setminus S \cap N$ and $A_1 \cap A_2 = \emptyset$. We apply Remark 3.5 with $A = F$ and $A' = N \cap S$ and get $q' \in \langle \nu_d(S \cap N) \rangle$ and $q'' \in \langle \nu_4(S \setminus S \cap N) \rangle$ such that $q \in \langle \{q', q''\} \rangle$. Since $|S \cap N| = 3$, Sylvester's theorem q' has rank 3 with respect to degree 4 rational normal curve $\nu_4(N)$. We get $|F \cap N| \leq 3$. Since $|F| = |S|$, we get that each element of $\mathcal{S}(X, q)$ is the union of $S \setminus S \cap N$ and an element of $\mathcal{S}(X, q')$. By Sylvester's theorem ([21, §1.5]) we have $\dim \mathcal{S}(\nu_4(N), q') = 1$. A word about this case. We worked taking $F \neq S$. In principle if $\mathcal{S}(X, q)$ is a singleton we got a contradiction, not the proof that $\dim \mathcal{S}(X, q) = 1$. However, we may add $S \setminus S \cap N$ to any $E_1 \in \mathcal{S}(X, q')$ to get an element of $\mathcal{S}(X, q)$, and so it is never a singleton. $\square$

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