

On Fröberg-Macaulay conjectures for algebras

MATS BOIJ AND ALDO CONCA

ABSTRACT. Macaulay's theorem and Fröberg's conjecture deal with the Hilbert function of homogeneous ideals in polynomial rings over a field K . In this short note we present some questions related to variants of Macaulay's theorem and Fröberg's conjecture for K -subalgebras of polynomial rings.

Keywords: Hilbert functions, Macaulay theorem.
MS Classification 2010: 13D40, 14M25.

1. Introduction

Macaulay's theorem and Fröberg's conjecture deal with the Hilbert function of homogeneous *ideals* in polynomial rings S over a field K . In this short note we present some questions related to variants of Macaulay's theorem and Fröberg's conjecture for K -subalgebras of polynomial rings. In details, given a subspace V of forms of degree d we consider the K -subalgebra $K[V]$ of S generated by V . What can be said about Hilbert function of $K[V]$? The analogy with the ideal case suggests several questions. To state them we start by recalling Macaulay's theorem, Fröberg's conjecture and Gotzmann's persistence theorem for ideals. Then we present the variants for K -subalgebras along with some partial results and examples.

2. Macaulay's theorem and Fröberg's conjecture for ideals

Let K be a field and $S = K[x_1, \dots, x_n]$ be the polynomial ring equipped with its standard grading, i.e., with $\deg x_i = 1$ for $i = 1, \dots, n$. Then $S = \bigoplus_{j=0}^{\infty} S_j$ where S_j denotes the K -vector space of homogeneous polynomials of degree j . Given positive integers d, u such that $u \leq \dim S_d$ let $G(u, S_d)$ be the Grassmannian of all u -dimensional K -subspaces of S_d . For a given subspace $V \in G(u, S_d)$, the homogeneous components of the ideal $I = (V)$ of S generated by V are the vector spaces $S_j V$, i.e., the vector spaces generated by the elements fg with $f \in S_j$ and $g \in V$.

QUESTION 2.1. *What can be said about the dimension of $S_j V$ in terms of $u = \dim V$?*

2.1. Lower bound

Macaulay's theorem on Hilbert functions, see [10], provides a lower bound for $\dim S_j V$ given $\dim V$. It asserts that there exists a subspace $L \in G(u, S_d)$ such that

$$\dim S_j L \leq \dim S_j V$$

for every j and every $V \in G(u, S_d)$. Furthermore $\dim S_j L$ can be expressed combinatorially in terms of d and u by means of the so-called *Macaulay expansion*, see [2, 13] for details. The vector space L can be taken to be generated by the largest u monomials of degree d with respect to the lexicographic order. Such an L is called the *lex-segment* (vector space) associated to the pair d and u and it is denoted by $\text{Lex}(u, S_d)$.

2.2. Persistence

A vector space $L \in G(u, S_d)$ is called *Gotzmann* if it satisfies

$$\dim S_1 L = \dim S_1 \text{Lex}(u, S_d),$$

i.e., if

$$\dim S_1 L \leq \dim S_1 V,$$

for all $V \in G(u, S_d)$. Gotzmann's persistence theorem [8] asserts that if $L \in G(u, S_d)$ is Gotzmann then $S_1 L$ is Gotzmann as well. In particular if L is Gotzmann one has

$$\dim S_j L \leq \dim S_j V,$$

for all j and all $V \in G(u, S_d)$.

2.3. Upper bound

Clearly, the upper bound for $\dim S_j V$ is given by the $\dim S_j W$ for a "general" W in $G(u, S_d)$. More precisely, there exists a non-empty Zariski open subset U of $G(u, S_d)$ such that for every $V \in G(u, S_d)$, for every $j \in \mathbb{N}$ and every $W \in U$ one has

$$\dim S_j V \leq \dim S_j W.$$

Fröberg's conjecture predicts the values of the upper bound $\dim S_j W$. For a formal power series $f(z) = \sum_{i=0}^{\infty} f_i z^i \in \mathbb{Z}[[z]]$ one denotes $[f(z)]_+$ the series

$\sum_{i=0}^{\infty} g_i z^i$, where $g_i = f_i$ if $f_j \geq 0$ for all $j \leq i$ and $g_i = 0$ otherwise. Given n, u and d one considers the formal power series:

$$\sum c_i z^i = \left[\frac{(1 - z^d)^u}{(1 - z)^n} \right]_+$$

and then Fröberg’s conjecture asserts that $\dim S_j W = \dim S_{j+d} - c_{d+j}$ for all j . It is known to be true in these cases:

- (1) $n \leq 3$ and any u, d, j , [1, 7],
- (2) $u \leq n + 1$ and any d, j , [12],
- (3) $j = 1$ and any n, u, d , [9]

and it remains open in general. See [11] for some recent contributions.

3. Macaulay’s theorem and Fröberg’s conjecture for subalgebras

For any subspace $V \in G(u, S_d)$ we can consider the K -subalgebra $K[V] \subseteq S$ generated by V . Indeed, $K[V]$ is the coordinate ring of the closure of the image of the rational map $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{u-1}$ associated to V .

The homogeneous component of degree j of $K[V]$ is the vector space V^j , i.e., the K -subspace of S_{jd} generated by the elements of the form $f_1 \cdots f_j$ with $f_1, \dots, f_j \in V$.

QUESTION 3.1. *What can be said about the dimension of V^j ? In other words, what can be said about the Hilbert function of the K -algebra $K[V]$?*

DEFINITION 3.2. *For positive integers, n, d, u and j , define*

$$L(n, d, u, j) = \min\{\dim V^j : V \in G(u, S_d)\}$$

and

$$M(n, d, u, j) = \max\{\dim V^j : V \in G(u, S_d)\}.$$

3.1. Lower bound

Recall that a monomial vector space W is said to be *strongly stable* if $m x_i / x_j \in W$ for every monomial $m \in W$ and $i < j$ such that $x_j | m$. Intersections, sums and products of strongly stable vector spaces are strongly stable. Given monomials $m_1, \dots, m_c \in S_d$ the smallest strongly stable vector space containing them is denoted by $\text{St}(m_1, \dots, m_c)$ and it is called the strongly stable vector space generated by m_1, \dots, m_c .

PROPOSITION 3.3. *Given n, d, u and j there exists a strongly stable vector space $W \in G(u, S_d)$ such that*

$$L(n, d, u, j) = \dim W^j$$

independently of the field K .

Proof. Given a term order $<$ on S for every $V \in G(u, S_d)$ one has $\text{in}(V)^j \subseteq \text{in}(V^j)$ for every j . Hence one has $\dim V_0^j \leq \dim V^j$ where $V_0 = \text{in}(V)$. Therefore the lower bound $L(n, d, u, j)$ is achieved by a monomial vector space. Comparing the vector space dimension of monomial algebras is a combinatorial problem and hence we may assume the base field has characteristic 0. Applying a general change of coordinates, we may put V in “generic coordinates” and hence $\text{in}(V)$ is the generic initial vector space of V with respect to some term order. Being such it is Borel fixed. Since the base field has characteristic 0, we have that $\text{in}(V)$ is strongly stable. Therefore the lower bound $L(n, d, u, j)$ is achieved by a strongly stable vector space. \square

EXAMPLE 3.4: For $n = 3, d = 4, u = 7$ there are 3 strongly stable vector spaces:

- 1) $\text{St}\{xy^3, x^2z^2\} = \langle xy^3, x^2z^2, x^2yz, x^2y^2, x^3z, x^3y, x^4 \rangle$
– the Lex Segment
- 2) $\text{St}\{xy^2z\} = \langle xy^2z, xy^3, x^2yz, x^2y^2, x^3z, x^3y, x^4 \rangle$
- 3) $\text{St}\{y^4, x^2yz\} = \langle y^4, xy^3, x^2yz, x^2y^2, x^3z, x^3y, x^4 \rangle$
– the RevLex Segment

In this case 2) and 3) turns out to give rational normal scrolls of type (3, 2) and (4, 1) respectively and they give the minimal possible Hilbert function in all values.

EXAMPLE 3.5: For $n = 3, d = 5$ and $u = 12$, there are five strongly stable subspaces of S_d :

$$\begin{aligned} W_1 = \text{St}\{x^2z^2, xy^3z\} &= \langle x^5, x^4y, x^4z, x^3y^2, x^3yz, x^3z^2, x^2y^3, x^2y^2z, \\ &\quad x^2yz^2, x^2z^3, xy^4, xy^3z \rangle, \\ W_2 = \text{St}\{xy^2z^2\} &= \langle x^5, x^4y, x^4z, x^3y^2, x^3yz, x^3z^2, x^2y^3, x^2y^2z, \\ &\quad x^2yz^2, xy^4, xy^3z, xy^2z^2 \rangle, \\ W_3 = \text{St}\{x^2z^3, y^5\} &= \langle x^5, x^4y, x^4z, x^3y^2, x^3yz, x^3z^2, x^2y^3, x^2y^2z, \\ &\quad x^2yz^2, x^2z^3, xy^4, y^5 \rangle, \end{aligned}$$

$$W_4 = \text{St}\{x^2z^2y, xy^3z, y^5\} = \langle x^5, x^4y, x^4z, x^3y^2, x^3yz, x^3z^2, x^2y^3, x^2y^2z, x^2yz^2, xy^4, xy^3z, y^5 \rangle,$$

$$W_5 = \text{St}\{x^3z^2, y^4z\} = \langle x^5, x^4y, x^4z, x^3y^2, x^3yz, x^3z^2, x^2y^3, x^2y^2z, xy^4, xy^3z, y^5, y^4z \rangle.$$

In this case, neither the Lex segment, W_1 , nor the RevLex segment, W_5 , achieve the minimum Hilbert function. The Hilbert series are given by

$$\begin{aligned} \text{HS}_{K[W_1]}(z) = \text{HS}_{K[W_5]}(z) &= \frac{1 + 9z + 3z^2}{(1 - z)^3}, \\ \text{HS}_{K[W_2]}(z) = \text{HS}_{K[W_4]}(z) &= \frac{1 + 9z + 2z^2}{(1 - z)^3}, \\ \text{HS}_{K[W_3]}(z) &= \frac{1 + 9z + 5z^2}{(1 - z)^3}. \end{aligned}$$

and the minimum turns out to be $L(3, 5, 12, j) = \dim W_2^j = \dim W_4^j = 6j^2 + 5j + 1$, for $j \geq 1$.

QUESTIONS 3.6. (1) Does there exist a (strongly stable) subspace $W \in G(u, S_d)$ such that $L(n, d, u, j) = \dim W^j$ for every j ?

(2) Given n, d, u, j can one characterize combinatorially the strongly stable subspace(s) W with the property $L(n, d, u, j) = \dim W^j$?

(3) Persistence: Assume $W \in G(u, S_d)$ satisfies $L(n, d, u, 2) = \dim W^2$. Does it satisfy also $L(n, d, u, j) = \dim W^j$ for all j ?

REMARK 3.7: For $n = 2$ there exists only one strongly stable vector space in $G(u, S_d)$, i.e. $\langle x^d, x^{d-1}y, \dots, x^{d-u+1}y^{u-1} \rangle$ (which is both the Lex and RevLex segment) and the questions in 3.6 have all straightforward answers.

REMARK 3.8: It is proved in [5] that Lex-segments, RevLex-segments and principal strongly stable vector spaces define normal and Koszul toric rings (in particular Cohen-Macaulay). Furthermore in [6] it is proved that a strongly stable vector spaces with two strongly stable generators define a Koszul toric ring. On the other hand, there are examples of strongly stable vector spaces with a non-Cohen-Macaulay and non-quadratic toric ring, see [3, Example 1.3].

3.2. Upper bound

As in the ideal case, the upper bound is achieved by a general subspace W , i.e., for W in a non-empty Zariski open subset of $G(u, S_d)$.

QUESTION 3.9. What can be said about the value $M(n, d, u, j)$?

Obviously,

$$M(n, d, u, j) \leq \min \left\{ \dim S_{jd}, \binom{u-1+j}{u-1} \right\} \quad (1)$$

and the naive expectation is that equality holds in (1), i.e., if f_1, \dots, f_u are general forms of degree d , then the monomials of degree j in the f_i 's are either linearly independent or they span S_{jd} . It turns out that in nature things are more complex than expected at first. First of all, if $u > n$ then equality in (1) would imply that for a generic W one would have $W^j = S_{jd}$ for large j . This fact can be stated in terms of projections of the d -th Veronese variety: the projection associated to W is an isomorphism. Recall that a generic linear projection of a smooth projective variety of dimension m from some projective space where it is embedded, into a projective space of dimension c is an isomorphism if $c \geq 2m + 1$. Hence we have that if $u \geq 2n$ then equality in (1) holds at least for large j . On the other hand, for $n + 1 \leq u < 2n$ equality in (1) should not be expected unless one knows that the corresponding projection of the Veronese variety behaves in an unexpected way.

Summing up, the most natural question turns out to be:

QUESTION 3.10. *Assume that $u \geq 2n$. Is it true that*

$$M(n, d, u, j) = \min \left\{ \dim S_{jd}, \binom{u-1+j}{u-1} \right\}$$

holds for all j ?

The answer turns out to be negative as the following example shows:

PROPOSITION 3.11. *For any space W generated by eight quadrics in four variables the dimension of W^2 is at most 34 independently of the base field K . That is:*

$$M(4, 2, 8, 2) \leq 34 < \min \left\{ \dim S_4, \binom{7+2}{7} \right\} = 35.$$

REMARK 3.12: This assertion was proven in [4, Theorem 2.4] using a computer algebra calculation. Here we present a more conceptual argument.

Proof. Firstly we may assume that K has characteristic 0 and is algebraically closed. Secondly we may assume that W is generic. The 8-dimension space of quadrics W is apolar to a 2-dimension space of quadrics, call it V . A pair of generic quadrics can be put simultaneously in diagonal form, i.e., that V is generated by $x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$. See for example [14]. Indeed, it is sufficient that V contains a quadric of rank 4 since that can be put into the form $x_1^2 + x_2^2 + x_3^2 + x_4^2$ and the other form can then be

diagonalized preserving the first. As a consequence, after a change of coordinates W contains $x_i x_j$ with $1 \leq i < j \leq 4$. Since $(x_1 x_4)(x_2 x_3) = (x_1 x_2)(x_3 x_4)$ and $(x_1 x_3)(x_2 x_4) = (x_1 x_2)(x_3 x_4)$ we have at least two independent relations among the 36 generators of W^2 . Therefore $\dim W^2 \leq 34$. \square

More precisely one has:

PROPOSITION 3.13. *One has $M(4, 2, 8, 2) = 34$ independently of the base field K .*

Proof. We have already argued that $M(4, 2, 8, 2) \leq 34$.

Therefore it is enough to describe an 8-dimension space of quadrics W in 4 variables such that $\dim W^2 = 34$. We set

$$W_0 = \langle x_i x_j : 1 \leq i < j \leq 4 \rangle$$

and

$$F = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 \text{ and } G = b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + b_4 x_4^2.$$

Then we set $W_1 = \langle F, G \rangle$ and then

$$W = W_0 + W_1.$$

We consider two conditions on the coefficients a_1, a_2, \dots, b_4 :

CONDITIONS 3.14. (1) *All the 2-minors of*

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

are non-zero.

(2) *The matrix*

$$\begin{pmatrix} a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 \\ a_1 b_1 & a_2 b_2 & a_3 b_3 & a_4 b_4 \end{pmatrix}$$

has rank 3.

We observe that W_0^2 is generated by the 19 monomials of degree 4 and largest exponent ≤ 2 . Then we note that if W_1 contains a quadric q supported on x_i^2, x_j^2 and x_h^2 with i, j, h distinct and $k \notin \{i, j, h\}$ then $x_k x_i q = x_k x_i^3 \pmod{W_0^2}$ and similarly for j and h . This implies that if Condition (1) holds then $W_0^2 + W_0 W_1$ is generated by the 31 monomials different from x_1^4, \dots, x_4^4 . Assuming that Condition (1) holds, we have that the matrix representing F^2, G^2, FG in $S_4/W_0^2 + W_0 W_1$ is exactly the one appearing in Condition (2). Then F^2, G^2, FG are linearly independent $\pmod{W_0^2 + W_0 W_1}$ if and only if Condition (2) holds. Summing up, if Conditions (1) and (2) hold then

$\dim W^2 = 34$. Finally we observe that for $F = x_1^2 + x_3^2 + x_4^2$ and $G = x_2^2 + \alpha x_3^2 + x_4^2$ the conditions (1) and (2) are satisfied provided $\alpha \neq 0$ and $\alpha \neq 1$. Hence this (conceptual) argument works for any field but $\mathbb{Z}/2\mathbb{Z}$. Over $\mathbb{Z}/2\mathbb{Z}$ one can consider the space W generated by $x_1^2, x_2^2, x_3^2, x_1x_3, x_2x_4, x_3x_4, x_2x_3 + x_1x_4, x_1x_2 + x_4^2$ and check with the help of a computer algebra system that $\dim W^2 = 34$. \square

As far as we know the case discussed in Proposition 3.11 is the only known case where $u \geq 2n$ and the actual value of $M(n, d, u, j)$ is smaller than

$$\min \left\{ \dim S_{jd}, \binom{u-1+j}{u-1} \right\}.$$

Acknowledgements

The first author was partially supported by VR2013-4545 and the second author was partially supported by INdAM-GNSAGA.

We thank Winfried Bruns and David Eisenbud for helpful discussions on the material of this paper.

REFERENCES

- [1] D.J. ANICK, *Thin algebras of embedding dimension three*, J. Algebra, **100** (1986), no. 1, 235–259,
- [2] W. BRUNS AND J. HERZOG, *Cohen-Macaulay rings*, Cambridge Stud. Adv. Math., vol. 39, Cambridge University Press, Cambridge, 1993.
- [3] W. BRUNS AND A. CONCA, *Linear resolutions of powers and products*, Singularities and computer algebra, Springer, Cham, 2017, 47–69.
- [4] L. CHIANTINI, F. ORECCHIA, AND I. RAMELLA *Maximal rank and minimal generation of some parametric varieties*, J. Pure Appl. Algebra **186** (2004), no. 1, 21–31.
- [5] E. DE NEGRI, *Toric rings generated by special stable sets of monomials*, Math. Nachr. **203** (1999), 31–45.
- [6] M. DIPASQUALE, C. FRANCISCO, J. MERMIN, J. SCHWEIG, AND G. SOSA, *The Rees algebra of a two-Borel ideal is Koszul*, Proc. Amer. Math. Soc., to appear.
- [7] R. FRÖBERG, *An inequality for Hilbert series of graded algebras*, Math. Scand. **56** (1985), no. 2, 117–144.
- [8] G. GOTZMANN, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Z. **158** (1978), no. 1, 61–70 (German).
- [9] M. HOCHSTER AND D. LAKSOV, *The linear syzygies of generic forms*, Comm. Algebra **15** (1987), no. 1-2, 227–239.
- [10] F. S. MACAULAY, *Some properties of enumeration in the theory of modular systems*, Proc. London Math. Soc. (2) **26** (1927), 531–555.
- [11] G. NENASHEV, *A note on Fröberg’s conjecture for forms of equal degrees*, C. R. Math. Acad. Sci. Paris **355** (2017), no. 3, 272–276.

- [12] R. P. STANLEY, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM J. Algebraic Discrete Methods **1** (1980), no.2, 168–184.
- [13] B. SURY, *Macaulay expansion*, Amer. Math. Monthly **121** (2014), no.4, 359–360.
- [14] M. J. WONENBURGER, *Simultaneous diagonalization of symmetric bilinear forms*, J. Math. Mech. **15** (1966), 617–622.

Authors' addresses:

Mats Boij
KTH - Royal Institute of Technology
SE-100 44 Stockholm, Sweden
E-mail: boij@kth.se

Aldo Conca
Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
I-16146 Genova, Italy
E-mail: conca@dima.unige.it

Received October 26, 2018
Accepted November 5, 2018