A note on finite group-actions on surfaces containing a hyperelliptic involution

BRUNO P. ZIMMERMANN

Abstract. By topological methods using the language of orbifolds, we give a short and efficient classification of the finite diffeomorphism groups of closed orientable surfaces of genus \( g \geq 2 \) which contain a hyperelliptic involution; in particular, for each \( g \geq 2 \) we determine the maximal possible order of such a group.

Keywords: hyperelliptic Riemann surface, hyperelliptic involution, finite diffeomorphism group.

MS Classification 2010: 57M60, 57S17, 30F10.

1. Introduction

Every finite group occurs as the isometry group of a closed hyperbolic 3-manifold [7]; on the other hand, the class of isometry groups of hyperbolic, hyperelliptic 3-manifolds (i.e., hyperbolic 3-manifolds which are 2-fold branched coverings of \( S^3 \), branched along a knot or link) is quite restricted but a complete classification turns out to be difficult (see [9]). More generally one can ask: what are the finite groups which act on a closed 3-manifold and contain a hyperelliptic involution, i.e. an involution with quotient space \( S^3 \)? Due to classical results for hyperelliptic Riemann surfaces, the situation is much better understood in dimension 2; motivated by the 3-dimensional case, in the present note we discuss the situation for surfaces. All surfaces in the present paper will be orientable, and all finite group-actions orientation-preserving.

Let \( F_g \) be a closed orientable surface of genus \( g \geq 2 \); we call a finite group \( G \) of diffeomorphisms of \( F_g \) hyperelliptic if \( G \) contains a hyperelliptic involution, i.e. an involution with quotient space \( S^2 \). The quotient \( F_g/G \) is a 2-orbifold (a closed surface with a finite number of branch points), and such a 2-orbifold can be given the structure of a hyperbolic 2-orbifold by uniformizing it by a Fuchsian group (see [12, Chapter 6]). Lifting the hyperbolic structure to \( F_g \), this becomes a hyperbolic surface such that the group \( G \) acts by isometries. In particular, \( G \) acts as a group of conformal automorphisms of the underlying Riemann surface \( F_g \); if \( G \) contains a hyperelliptic involution,
$F_g$ is a hyperelliptic Riemann surface. A hyperelliptic Riemann surface has a unique hyperelliptic involution, with $2g + 2$ fixed points, which is central in its automorphism group (see [4, Section III.7] for basic facts about hyperelliptic Riemann surfaces, and [10, Chapter 13] for the language of orbifolds). In particular, a hyperelliptic involution $h$ in a finite group of diffeomorphisms $G$ of $F_g$ is unique and central, and the factor group $\bar{G} = G/(h)$ acts on the quotient-orbifold $\bar{F}_g/(h) \cong S^2(2g+2)$, which denotes the 2-sphere with $2g + 2$ hyperelliptic branch points of order 2, and $G$ permutes the set $B$ of the $2g + 2$ hyperelliptic branch points. Note that any two hyperelliptic involutions of a surface $F_g$ are conjugate by a diffeomorphism (since they have the same quotient $S^2(2g+2)$) and, if distinct, generate an infinite dihedral group of diffeomorphisms.

Conversely, if $G$ is a finite group acting on the orbifold $S^2(2g+2)$ (in particular, mapping the set $B$ of hyperelliptic branch points to itself), then the set of all lifts of elements of $\bar{G}$ to $F_g$ defines a group $G$ with $G/(h) \cong \bar{G}$ and $F_g/G = S^2(2g+2)/\bar{G}$. The finite groups $\bar{G}$ which admit an orientation-preserving action on the 2-sphere $S^2$ are cyclic $\mathbb{Z}_n$ with quotient-orbifold $S^2(n, n)$, dihedral $D_{2n}$ of order $2n$ with quotient $S^2(2, 2, n)$, tetrahedral $A_4$ of order 12 with quotient $S^2(2, 3, 3)$, octahedral $S_4$ of order 24 with quotient $S^2(2, 3, 4)$, or dodecahedral $A_5$ of order 60 with quotient $S^2(2, 3, 5)$.

In the following theorem we classify large hyperelliptic group-actions; however, the methods apply easily also to arbitrary actions, see Remark 2.3.

**Theorem 1.1.** Let $G$ be a finite group of diffeomorphisms of a closed orientable surface $F_g$ of genus $g \geq 2$ containing a hyperelliptic involution; suppose that $|G| \geq 4g$ and that $G$ is maximal, i.e. not contained in a strictly larger finite group of diffeomorphisms of $F_g$. Then $G$ belongs to one of the following cases (see 2.1 for the definitions of the groups $A_{8(g+1)}$ and $B_{8g}$):

- $G = A_{8(g+1)}$, $\bar{G} \cong D_{4g+1}$; $F_g/G = S^2(2, 4, 2g + 2)$;
- $G = B_{8g}$, $\bar{G} \cong D_{4g}$; $F_g/G = S^2(2, 4, 4g)$;
- $G \cong \mathbb{Z}_{4g+2}$, $\bar{G} \cong \mathbb{Z}_{2g+1}$; $F_g/G = S^2(2, 2g + 1, 4g + 2)$;
- $|G| = 120$, $\bar{G} \cong A_5$, $g = 5, 9, 14, 15, 20, 24, 29, 30$;
- $|G| = 48$, $\bar{G} \cong S_4$, $g = 2, 3, 5, 6, 8, 9, 11, 12$;
- $|G| = 24$, $\bar{G} \cong A_4$, $g = 4$.

In each of the cases, up to conjugation by diffeomorphisms of $F_g$ there is a unique group $G$ for each genus $g$ (see Sections 2.3 and 2.4 for the quotient orbifolds in the last three cases).

**Corollary 1.2.** Let $m_h(g)$ denote the maximal order of a hyperelliptic group of diffeomorphisms of a closed orientable surface of genus $g \geq 2$; then $m_h(g) = 8(g + 1)$, with the exceptions $m_h(2) = m_h(3) = 48$ and $m_h(5) = m_h(9) = 120$. 

The maximal order \( m(g) \) of a finite group of diffeomorphisms of closed surface of genus \( g \geq 2 \) is not known in general; there is the classical Hurwitz bound \( m(g) \leq 84(g-1) \) [6] which is both optimal and non-optimal for infinitely many values of \( g \). Considering hyperelliptic groups as in Theorem 1.1 one has \( m(g) \geq 8(g+1) \), and Accola and Maclachlan proved that \( m(g) = 8(g+1) \) for infinitely many values of \( g \), see Remark 2.2 in Section 2.

The group \( G \cong \mathbb{Z}_{4g+2} \) in Theorem 1.1 realizes the unique action of a cyclic group of maximal possible order \( 4g+2 \) on a surface of genus \( g \geq 2 \), see Remark 2.1.

2. Proof of Theorem 1.1

2.1. Dihedral groups

Let \( \tilde{G} = \mathbb{D}_{2n} \) be a dihedral group of order \( 2n \) acting on the orbifold \( S^2(2^{2g+2}) \).

The action of \( \mathbb{D}_{2n} \) on the 2-sphere has one orbit \( O_2 \) consisting of the two fixed points of the cyclic subgroup \( \mathbb{Z}_n \) of \( \mathbb{D}_{2n} \), two orbits \( O_n \) and \( O'_{n} \), each of \( n \) elements whose union is the set of \( 2n \) fixed point of the \( n \) reflections in the dihedral group \( \mathbb{D}_{2n} \), and regular orbits \( O_{2n} \) with \( 2n \) elements. We consider various choices for the set \( B \) of \( 2g+2 \) hyperelliptic branch points in \( S^2(2^{2g+2}) \).

i) \( B = O_n, \quad n = 2g + 2, \quad S^2(2^{2g+2}) / \tilde{G} = S^2(2, 2, 4g + 2) \).

We define \( A_{8(g+1)} \) as the group \( G \) of order \( 8(g+1) \) of all lifts of elements of \( \tilde{G} \) to the 2-fold branched covering \( F_g \) of \( S^2(2^{2g+2}) \). It is easy to find a presentation of \( A_{8(g+1)} \): considering the central extension \( 1 \to \mathbb{Z}_2 = \langle h \rangle \to A_{8(g+1)} \to \mathbb{D}_{4(g+1)} \to 1 \) and the presentation \( \mathbb{D}_{4(g+1)} = \langle x, y | x^2 = y^2 = (xy)^{2g+1} = 1 \rangle \), one obtains the presentation \( A_{8(g+1)} = \langle x, y, h | h^2 = 1, [x, h] = [y, h] = 1, y^2 = h, x^2 = y^4 = (xy)^{2(g+1)} = 1 \rangle \).

ii) \( B = O_n \cup O_2, \quad n = 2g \) even, \( S^2(2^{2g+2}) / \tilde{G} = S^2(2, 4, 4g) \).

The lift \( G \) of \( \tilde{G} \) to \( F_g \) defines a group \( B_{8g} \) of order \( 8g \), with presentation \( B_{8g} = \langle x, y, h | h^2 = 1, [x, h] = [y, h] = 1, y^2 = (xy)^g = h, x^2 = y^4 = (xy)^g = 1 \rangle \).

iii) \( B = O_n \cup O'_{n}, \quad n = g + 1, \quad S^2(2^{2g+2}) / \tilde{G} = S^2(4, 4, g + 1) \).

This orbifold has an involution with quotient \( S^2(2, 4, 2(g+1)) \) which lifts to \( S^2(2^{2g+2}) \), hence \( G \) is a subgroup of index 2 in \( A_{8(g+1)} \).

iv) \( B = O_{2n}, \quad n = g + 1, \quad S^2(2^{2g+2}) / \tilde{G} = S^2(2, 2, 2, g + 1) \).
This orbifold has an involution with quotient $S^2(4, 2, 2(g + 1))$ which lifts to $S^2(2^{2g+2})$, and $G$ is a subgroup of index 2 in $A_{8(g+1)}$.

v) $B = \mathcal{O}_n \cup \mathcal{O}_n' \cup \mathcal{O}_2$, $n = g$, $S^2(2^{2g+2})/\bar{G} = S^2(4, 4, 2g)$.

Again there is an involution, with quotient $S^2(2, 4, 4g)$, hence $G$ is a subgroup of index 2 of $B_{8g}$.

vi) $B = \mathcal{O}_{2n} \cup \mathcal{O}_2$, $n = g$, $S^2(2^{2g+2})/\bar{G} = S^2(2, 2, 2, 2g)$.

Dividing out a further involution one obtains $S^2(4, 2, 4g)$, and $G$ is a subgroup of index 2 in $B_{8g}$.

Note that for any other choice of $B$ one obtains groups $G$ of order less than $4g$.

**Remark 2.1:** Incidentally, by results of Accola [1] and Maclachlan [8], for infinitely many values of $g$ the groups $A_{8(g+1)}$ in i) realize the maximal possible order of a group acting on a surface of genus $g \geq 2$. Moreover, the group $A_{8(g+1)}$ has an abelian subgroup $\mathbb{Z}_{2g+2} \times \mathbb{Z}_2$ of index two which realizes the maximal possible order of an abelian group acting on a surface of genus $g \geq 2$ (see [12, 4.14.27]).

### 2.2. Cyclic groups

Next we consider the case of a cyclic group $\bar{G} = \mathbb{Z}_n$. There are two orbits with exactly one point, the fixed points of $\mathbb{Z}_n$, all other orbits have $n$ points (regular orbits).

If $B$ consists of a regular orbit and exactly one of the two fixed points of $\mathbb{Z}_n$, with $n + 1 = 2g + 2$, $n = 2g + 1$ odd and $S^2(2^{2g+2})/\bar{G} = S^2(2, 2g + 1, 2(2g+1))$, then the 2-fold branched covering of $S^2(2^{2g+2})$ is a surface of genus $g$ on which a cyclic group $G \cong \mathbb{Z}_{4g+2}$ acts.

If $B$ consists of one regular orbit, then $n = 2g + 2$, $S^2(2^{2g+2})/\bar{G} = S^2(2, 2g + 2, 2g + 2)$ which is a 2-fold branched covering of $S^2(2, 2g + 2)$, hence $G \cong \mathbb{Z}_{2g+2} \times \mathbb{Z}_2$ is a subgroup of index 2 in $A_{8(g+1)}$.

If $B$ consists of a regular orbit and the two fixed points of $\mathbb{Z}_n$, then $n + 2 = 2g + 2$, $n = 2g$, $S^2(2^{2g+2})/\bar{G} = S^2(2, 4g, 4g)$ which is a 2-fold cover of $S^2(2, 4, 4g)$, and $G \cong \mathbb{Z}_{4g}$ is a subgroup of index 2 in $B_{8g}$.

**Remark 2.2:** By a result of Wiman [11], $4g + 2$ is the maximal possible order of a cyclic group-action on a surface of genus $g \geq 2$, and the action of $G \cong \mathbb{Z}_{4g+2}$ above is the unique action of a cyclic group realizing this maximal order (see [5].
The group $G \cong \mathbb{Z}_{2g+2} \times \mathbb{Z}_2$ instead realizes the maximum order of an abelian group-action on a surface of genus $g \geq 2$, see Remark 2.1.

### 2.3. Dodecahedral groups

Now let $\bar{G} \cong A_5$ be the dodecahedral group acting on $S^2$. The orbits of the action are $\mathcal{O}_{12}$ consisting of the 12 fixed points of the 6 subgroups $\mathbb{Z}_5$ of $A_5$ (the centers of the 12 faces of a regular dodecahedron), $\mathcal{O}_{20}$ consisting of the twenty fixed points of the 10 subgroups $\mathbb{Z}_3$ (the 20 vertices of a regular dodecahedron), $\mathcal{O}_{30}$ consisting of the 30 fixed points of the 15 involutions (the centers of the 30 edges of a regular dodecahedron), and regular orbits $\mathcal{O}_{60}$. The list of the different choices of $\mathcal{B}$, the genera $g$ and the corresponding quotient-orbifolds are as follows:

$$
\begin{align*}
B = \mathcal{O}_{12} : & \quad g = 5, \quad S^2(2^{12})/\bar{G} \cong S^2(2,3,10); \\
B = \mathcal{O}_{20} : & \quad g = 9, \quad S^2(2^{20})/\bar{G} \cong S^2(2,6,5); \\
B = \mathcal{O}_{30} : & \quad g = 14, \quad S^2(2^{30})/\bar{G} \cong S^2(4,3,5); \\
B = \mathcal{O}_{60} : & \quad g = 29, \quad S^2(2^{60})/\bar{G} \cong S^2(2,2,3,5); \\
B = \mathcal{O}_{12} \cup \mathcal{O}_{20} : & \quad g = 15, \quad S^2(2^{32})/\bar{G} \cong S^2(2,6,10); \\
B = \mathcal{O}_{12} \cup \mathcal{O}_{30} : & \quad g = 20, \quad S^2(2^{42})/\bar{G} \cong S^2(4,3,10); \\
B = \mathcal{O}_{20} \cup \mathcal{O}_{30} : & \quad g = 24, \quad S^2(2^{50})/\bar{G} \cong S^2(4,6,5); \\
B = \mathcal{O}_{12} \cup \mathcal{O}_{20} \cup \mathcal{O}_{30} : & \quad g = 30, \quad S^2(2^{62})/\bar{G} \cong S^2(4,6,10).
\end{align*}
$$

These are exactly the genera in the Theorem for the case $\bar{G} \cong A_5$. For $g = 5, 9, 15$ and 29, the group $G$ is isomorphic to $A_5 \times \mathbb{Z}_2$, for $g = 14, 20, 24$ and 30 to the binary dodecahedral group $A_5^* \cong \mathbb{Z}_{2g+2}$ (these are the two central extensions of $A_5$).

**Remark 2.3:** For each of the finite groups $\bar{G}$ acting on $S^2$ one can easily produce a complete list of the possible quotient orbifolds $F_g/\bar{G} = S^2(2^{2g+2})/\bar{G}$ (i.e., without the restriction $|G| \geq 4g$ in the Theorem). For the case of $\bar{G} = A_5$, the possible quotient-orbifolds are as follows (where $m \geq 0$ denotes the number of regular orbits $\mathcal{O}_{60}$ in the set $\mathcal{B}$ of hyperelliptic branch points).

$$
\begin{align*}
S^2(2^m,2,3,5), \quad S^2(2^m,2,3,10), \quad S^2(2^m,2,6,5), \quad S^2(2^m,4,3,5), \\
S^2(2^m,2,6,10), \quad S^2(2^m,4,3,10), \quad S^2(2^m,4,6,5), \quad S^2(2^m,4,6,10).
\end{align*}
$$

Each of these orbifolds defines a unique action of $\bar{G} \cong A_5$ on an orbifold $S^2(2^{2g+2})$ and of $G \cong A_5 \times \mathbb{Z}_2$ or $A_5^*$ on a surface $F_g$, and this gives the complete classification of the actions of the groups $G$ of type $\bar{G} \cong A_5$, up to conjugation.
2.4. Octahedral and tetrahedral groups

Finally, playing the same game for the groups $S_4$ and $A_4$, one produces similar lists also for these cases. This gives the list of genera for the groups $G$ of type $S_4$ in the Theorem; the groups $G$ of type $A_4$ are all subgroups of index 2 in groups $G$ of type $S_4$ except for $g = 4$ (with $B = O_4 \cup O_6$ and quotient-orbifold $S^2(3, 4, 6)$). The groups $G$ are isomorphic to one of the two central extensions $A_4 \times \mathbb{Z}_2$ and $A_4^*$ of $A_4$, or to one of four central extensions of $S_4$.

Note added for the revised version. The referee provided the two additional references [2] and [3] in which similar results are obtained, in an algebraic language and by different algebraic methods. The main point of the present note is a short, topological approach to the classification: After the preliminary fact from complex analysis that a hyperelliptic involution of a Riemann surface is unique and central in its automorphism group, the methods in the present note are purely topological, offering a short and efficient topological approach to the classification.

References


Author’s address:

Bruno P. Zimmermann  
Università degli Studi di Trieste  
Dipartimento di Matematica e Geoscienze  
Via Valerio 12/1  
34127 Trieste, Italy  
E-mail: zimmer@units.it

Received June 6, 2018  
Revised October 1, 2018  
Accepted October 3, 2018