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A note on finite group-actions on surfaces containing a hyperelliptic involution

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Abstract. *By topological methods using the language of orbifolds, we give a short and efficient classification of the finite diffeomorphism groups of closed orientable surfaces of genus $g \geq 2$ which contain a hyperelliptic involution; in particular, for each $g \geq 2$ we determine the maximal possible order of such a group.*

Keywords: hyperelliptic Riemann surface, hyperelliptic involution, finite diffeomorphism group.
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1. Introduction

Every finite group occurs as the isometry group of a closed hyperbolic 3-manifold [7]; on the other hand, the class of isometry groups ofhyperbolic, hyperelliptic 3-manifolds (i.e., hyperbolic 3-manifolds which are 2-fold branched coverings of S^3, branched along a knot or link) is quite restricted but a complete classification turns out to be difficult (see [9]). More generally one can ask: what are the finite groups which act on a closed 3-manifold and contain a hyperelliptic involution, i.e. an involution with quotient space S^3? Due to classical results for hyperelliptic Riemann surfaces, the situation is much better understood in dimension 2; motivated by the 3-dimensional case, in the present note we discuss the situation for surfaces. All surfaces in the present paper will be orientable, and all finite group-actions orientation-preserving.

Let F_g be a closed orientable surface of genus $g \geq 2$; we call a finite group G of diffeomorphisms of F_g *hyperelliptic* if G contains a hyperelliptic involution, i.e. an involution with quotient space S^2. The quotient F_g/G is a 2-orbifold (a closed surface with a finite number of branch points), and such a 2-orbifold can be given the structure of a hyperbolic 2-orbifold by uniformizing it by a Fuchsian group (see [12, Chapter 6]). Lifting the hyperbolic structure to F_g, this becomes a hyperbolic surface such that the group G acts by isometries. In particular, G acts as a group of conformal automorphisms of the underlying Riemann surface F_g; if G contains a hyperelliptic involution,

F_g is a hyperelliptic Riemann surface. A hyperelliptic Riemann surface has a unique hyperelliptic involution, with $2g+2$ fixed points, which is central in its automorphism group (see [4, Section III.7] for basic facts about hyperelliptic Riemann surfaces, and [10, Chapter 13] for the language of orbifolds). In particular, a hyperelliptic involution h in a finite group of diffeomorphisms G of F_g is unique and central, and the factor group $\bar{G} = G/\langle h \rangle$ acts on the quotient-orbifold $F_g/\langle h \rangle \cong \mathcal{S}^2(2^{2g+2})$, which denotes the 2-sphere with $2g+2$ hyperelliptic branch points of order 2, and G permutes the set $\mathcal{B}$ of the $2g+2$ hyperelliptic branch points. Note that any two hyperelliptic involutions of a surface F_g are conjugate by a diffeomorphism (since they have the same quotient $\mathcal{S}^2(2^{2g+2})$) and, if distinct, generate an infinite dihedral group of diffeomorphisms.

Conversely, if $\bar{G}$ is a finite group acting on the orbifold $\mathcal{S}^2(2^{2g+2})$ (in particular, mapping the set $\mathcal{B}$ of hyperelliptic branch points to itself), then the set of all lifts of elements of $\bar{G}$ to F_g defines a group G with $G/\langle h \rangle \cong \bar{G}$ and $F_g/G = \mathcal{S}^2(2^{2g+2})/\bar{G}$. The finite groups $\bar{G}$ which admit an orientation-preserving action on the 2-sphere S^2 are cyclic $\mathbb{Z}_n$ with quotient-orbifold $\mathcal{S}^2(n,n)$, dihedral $\mathbb{D}_{2n}$ of order $2n$ with quotient $\mathcal{S}^2(2,2,n)$, tetrahedral $\mathbb{A}_4$ of order 12 with quotient $\mathcal{S}^2(2,3,3)$, octahedral $\mathbb{S}_4$ of order 24 with quotient $\mathcal{S}^2(2,3,4)$, or dodecahedral A_5 of order 60 with quotient $\mathcal{S}^2(2,3,5)$.

In the following theorem we classify large hyperelliptic group-actions; however, the methods apply easily also to arbitrary actions, see Remark 2.3.

THEOREM 1.1. *Let G be a finite group of diffeomorphisms of a closed orientable surface F_g of genus $g \geq 2$ containing a hyperelliptic involution; suppose that $|G| \geq 4g$ and that G is maximal, i.e. not contained in a strictly larger finite group of diffeomorphisms of F_g. Then G belongs to one of the following cases (see 2.1 for the definitions of the groups $A_{8(g+1)}$ and B_{8g}):*

$$\begin{array}{lll}
G = A_{8(g+1)}, & \bar{G} \cong \mathbb{D}_{4(g+1)}, & F_g/G = \mathcal{S}^2(2,4,2g+2); \\
G = B_{8g}, & \bar{G} \cong \mathbb{D}_{4g}, & F_g/G = \mathcal{S}^2(2,4,4g); \\
G \cong \mathbb{Z}_{4g+2}, & \bar{G} \cong \mathbb{Z}_{2g+1}, & F_g/G = \mathcal{S}^2(2,2g+1,4g+2); \\
|G| = 120, & \bar{G} \cong \mathbb{A}_5, & g = 5,9,14,15,20,24,29,30; \\
|G| = 48, & \bar{G} \cong \mathcal{S}_4, & g = 2,3,5,6,8,9,11,12; \\
|G| = 24, & \bar{G} \cong \mathbb{A}_4, & g = 4.
\end{array}$$

In each of the cases, up to conjugation by diffeomorphisms of F_g there is a unique group G for each genus g (see Sections 2.3 and 2.4 for the quotient orbifolds in the last three cases).

COROLLARY 1.2. *Let $m_h(g)$ denote the maximal order of a hyperelliptic group of diffeomorphisms of a closed orientable surface of genus $g \geq 2$; then $m_h(g) = 8(g+1)$, with the exceptions $m_h(2) = m_h(3) = 48$ and $m_h(5) = m_h(9) = 120$.*

The maximal order $m(g)$ of a finite group of diffeomorphisms of closed surface of genus $g \geq 2$ is not known in general; there is the classical Hurwitz bound $m(g) \leq 84(g-1)$ [6] which is both optimal and non-optimal for infinitely many values of g. Considering hyperelliptic groups as in Theorem 1.1 one has $m(g) \geq 8(g+1)$, and Accola and Maclachlan proved that $m(g) = 8(g+1)$ for infinitely many values of g, see Remark 2.2 in Section 2.

The group $G \cong \mathbb{Z}_{4g+2}$ in Theorem 1.1 realizes the unique action of a cyclic group of maximal possible order $4g+2$ on a surface of genus $g \geq 2$, see Remark 2.1.

2. Proof of Theorem 1.1

2.1. Dihedral groups

Let $\bar{G} = \mathbb{D}_{2n}$ be a dihedral group of order $2n$ acting on the orbifold $\mathcal{S}^2(2^{2g+2})$. The action of $\mathbb{D}_{2n}$ on the 2-sphere has one orbit $\mathcal{O}_2$ consisting of the two fixed points of the cyclic subgroup $\mathbb{Z}_n$ of $\mathbb{D}_{2n}$, two orbits $\mathcal{O}_n$ and $\mathcal{O}'_n$ each of n elements whose union is the set of $2n$ fixed point of the n reflections in the dihedral group $\mathbb{D}_{2n}$, and regular orbits $\mathcal{O}_{2n}$ with $2n$ elements. We consider various choices for the set $\mathcal{B}$ of $2g+2$ hyperelliptic branch points in $\mathcal{S}^2(2^{2g+2})$.

i) $\mathcal{B} = \mathcal{O}_n, \quad n = 2g+2, \quad \mathcal{S}^2(2^{2g+2})/\bar{G} = \mathcal{S}^2(2, 4, 2g+2)$.

We define $A_{8(g+1)}$ as the group G of order $8(g+1)$ of all lifts of elements of $\bar{G}$ to the 2-fold branched covering F_g of $\mathcal{S}^2(2^{2g+2})$. It is easy to find a presentation of $A_{8(g+1)}$: considering the central extension $1 \to \mathbb{Z}_2 = \langle h \rangle \to A_{8(g+1)} \to \mathbb{D}_{4(g+1)} \to 1$ and the presentation $\mathbb{D}_{4(g+1)} = < \bar{x}, \bar{y} \mid \bar{x}^2 = \bar{y}^2 = (\bar{x}\bar{y})^{2(g+1)} = 1 >$, one obtains the presentation $A_{8(g+1)} = < x, y, h \mid h^2 = 1,\ [x,h] = [y,h] = 1,\ y^2 = h,\ x^2 = y^4 = (xy)^{2(g+1)} = 1 >$.

ii) $\mathcal{B} = \mathcal{O}_n \cup \mathcal{O}_2, \quad n = 2g$ even, $\quad \mathcal{S}^2(2^{2g+2})/\bar{G} = \mathcal{S}^2(2, 4, 4g)$.

The lift G of $\bar{G}$ to F_g defines a group B_{8g} of order $8g$, with presentation $B_{8g} = < x, y, h \mid h^2 = 1,\ [x,h] = [y,h] = 1,\ y^2 = (xy)^{2g} = h,\ x^2 = y^4 = (xy)^{4g} = 1 >$.

iii) $\mathcal{B} = \mathcal{O}_n \cup \mathcal{O}'_n, \quad n = g+1, \quad \mathcal{S}^2(2^{2g+2})/\bar{G} = \mathcal{S}^2(4, 4, g+1)$.

This orbifold has an involution with quotient $\mathcal{S}^2(2, 4, 2(g+1))$ which lifts to $\mathcal{S}^2(2^{2g+2})$, hence G is a subgroup of index 2 in $A_{8(g+1)}$.

iv) $\mathcal{B} = \mathcal{O}_{2n}, \quad n = g+1, \quad \mathcal{S}^2(2^{2g+2})/\bar{G} = \mathcal{S}^2(2, 2, 2, g+1)$.

This orbifold has an involution with quotient $\mathcal{S}^2(4,2,2(g+1))$ which lifts to $\mathcal{S}^2(2^{2g+2})$, and G is a subgroup of index 2 in $A_{8(g+1)}$.

v) $\mathcal{B} = \mathcal{O}_n \cup \mathcal{O}'_n \cup \mathcal{O}_2, \quad n = g, \quad \mathcal{S}^2(2^{2g+2})/\bar{G} = \mathcal{S}^2(4,4,2g)$.

Again there is an involution, with quotient $\mathcal{S}^2(2,4,4g)$, hence G is a subgroup of index 2 of B_{8g}.

vi) $\mathcal{B} = \mathcal{O}_{2n} \cup \mathcal{O}_2, \quad n = g, \quad \mathcal{S}^2(2^{2g+2})/\bar{G} = \mathcal{S}^2(2,2,2,2g)$.

Dividing out a further involution one obtains $\mathcal{S}^2(4,2,4g)$, and G is a subgroup of index 2 in B_{8g}.

Note that for any other choice of $\mathcal{B}$ one obtains groups G of order less than $4g$.

Remark 2.1: Incidentally, by results of Accola [1] and Maclachlan [8], for infinitely many values of g the groups $A_{8(g+1)}$ in i) realize the maximal possible order of a group acting on a surface of genus $g \geq 2$. Moreover, the group $A_{8(g+1)}$ has an abelian subgroup $\mathbb{Z}_{2g+2} \times \mathbb{Z}_2$ of index two which realizes the maximal possible order of an abelian group acting on a surface of genus $g \geq 2$ (see [12, 4.14.27]).

2.2. Cyclic groups

Next we consider the case of a cyclic group $\bar{G} = \mathbb{Z}_n$. There are two orbits with exactly one point, the fixed points of $\mathbb{Z}_n$, all other orbits have n points (regular orbits).

If $\mathcal{B}$ consists of a regular orbit and exactly one of the two fixed points of $\mathbb{Z}_n$, with $n+1 = 2g+2$, $n = 2g+1$ odd and $\mathcal{S}^2(2^{2g+2})/\bar{G} = \mathcal{S}^2(2,2g+1,2(2g+1))$, then the 2-fold branched covering of $\mathcal{S}^2(2^{2g+2})$ is a surface of genus g on which a cyclic group $G \cong \mathbb{Z}_{4g+2}$ acts.

If $\mathcal{B}$ consists of one regular orbit, then $n = 2g+2$, $\mathcal{S}^2(2^{2g+2})/\bar{G} = \mathcal{S}^2(2,2g+2,2g+2)$ which is a 2-fold branched covering of $\mathcal{S}^2(2,4,2g+2)$, hence $G \cong \mathbb{Z}_{2g+2} \times \mathbb{Z}_2$ is a subgroup of index 2 in $A_{8(g+1)}$.

If $\mathcal{B}$ consists of a regular orbit and the two fixed points of $\mathbb{Z}_n$, then $n+2 = 2g+2$, $n = 2g$, $\mathcal{S}^2(2^{2g+2})/\bar{G} = \mathcal{S}^2(2,4g,4g)$ which is a 2-fold cover of $\mathcal{S}^2(2,4,4g)$, and $G \cong \mathbb{Z}_{4g}$ is a subgroup of index 2 in B_{8g}.

Remark 2.2: By a result of Wiman [11], $4g+2$ is the maximal possible order of a cyclic group-action on a surface of genus $g \geq 2$, and the action of $G \cong \mathbb{Z}_{4g+2}$ above is the unique action of a cyclic group realizing this maximal order (see [5], [12, 4.14.27]). The group $G \cong \mathbb{Z}_{2g+2} \times \mathbb{Z}_2$ instead realizes the maximum order of an abelian group-action on a surface of genus $g \geq 2$, see Remark 2.1.

2.3. Dodecahedral groups

Now let $\bar{G} = \mathbb{A}_5$ be the dodecahedral group acting on S^2. The orbits of the action are $\mathcal{O}_{12}$ consisting of the 12 fixed points of the 6 subgroups $\mathbb{Z}_5$ of $\mathbb{A}_5$ (the centers of the 12 faces of a regular dodecahedron), $\mathcal{O}_{20}$ consisting of the twenty fixed points of the 10 subgroups $\mathbb{Z}_3$ (the 20 vertices of a regular dodecahedron), $\mathcal{O}_{30}$ consisting of the 30 fixed points of the 15 involutions (the centers of the 30 edges of a regular dodecahedron), and regular orbits $\mathcal{O}_{60}$. The list of the different choices of $\mathcal{B}$, the genera g and the corresponding quotient-orbifolds are as follows:

$$\begin{array}{lll}
\mathcal{B} = \mathcal{O}_{12}: & g = 5, & \mathcal{S}^2(2^{12})/\bar{G} \cong \mathcal{S}^2(2,3,10); \\
\mathcal{B} = \mathcal{O}_{20}: & g = 9, & \mathcal{S}^2(2^{20})/\bar{G} \cong \mathcal{S}^2(2,6,5); \\
\mathcal{B} = \mathcal{O}_{30}: & g = 14, & \mathcal{S}^2(2^{30})/\bar{G} \cong \mathcal{S}^2(4,3,5); \\
\mathcal{B} = \mathcal{O}_{60}: & g = 29, & \mathcal{S}^2(2^{60})/\bar{G} \cong \mathcal{S}^2(2,2,3,5); \\
\mathcal{B} = \mathcal{O}_{12} \cup \mathcal{O}_{20}: & g = 15, & \mathcal{S}^2(2^{32})/\bar{G} \cong \mathcal{S}^2(2,6,10); \\
\mathcal{B} = \mathcal{O}_{12} \cup \mathcal{O}_{30}: & g = 20, & \mathcal{S}^2(2^{42})/\bar{G} \cong \mathcal{S}^2(4,3,10); \\
\mathcal{B} = \mathcal{O}_{20} \cup \mathcal{O}_{30}: & g = 24, & \mathcal{S}^2(2^{50})/\bar{G} \cong \mathcal{S}^2(4,6,5); \\
\mathcal{B} = \mathcal{O}_{12} \cup \mathcal{O}_{20} \cup \mathcal{O}_{30}: & g = 30, & \mathcal{S}^2(2^{62})/\bar{G} \cong \mathcal{S}^2(4,6,10).
\end{array}$$

These are exactly the genera in the Theorem for the case $\bar{G} \cong A_5$. For $g = 5$, 9, 15 and 29, the group G is isomorphic to $\mathbb{A}_5 \times \mathbb{Z}_2$, for $g = 14$, 20, 24 and 30 to the binary dodecahedral group $\mathbb{A}_5^*$ (these are the two central extensions of $\mathbb{A}_5$).

REMARK 2.3: For each of the finite groups $\bar{G}$ acting on S^2 one can easily produce a complete list of the possible quotient orbifolds $F_g/G = \mathcal{S}^2(2^{2g+2})/\bar{G}$ (i.e., without the restriction $|G| \geq 4g$ in the Theorem). For the case of $\bar{G} = \mathbb{A}_5$, the possible quotient-orbifolds are as follows (where $m \geq 0$ denotes the number of regular orbits $\mathcal{O}_{60}$ in the set $\mathcal{B}$ of hyperelliptic branch points).

$$\begin{array}{llll}
\mathcal{S}^2(2^m,2,3,5), & \mathcal{S}^2(2^m,2,3,10), & \mathcal{S}^2(2^m,2,6,5), & \mathcal{S}^2(2^m,4,3,5), \\
\mathcal{S}^2(2^m,2,6,10), & \mathcal{S}^2(2^m,4,3,10), & \mathcal{S}^2(2^m,4,6,5), & \mathcal{S}^2(2^m,4,6,10).
\end{array}$$

Each of these orbifolds defines a unique action of $\bar{G} \cong \mathbb{A}_5$ on an orbifold $\mathcal{S}^2(2^{2g+2})$ and of $G \cong \mathbb{A}_5 \times \mathbb{Z}_2$ or $\mathbb{A}_5^*$ on a surface F_g, and this gives the complete classification of the actions of the groups G of type $\bar{G} \cong \mathbb{A}_5$, up to conjugation.

2.4. Octahedral and tetrahedral groups

Finally, playing the same game for the groups $\mathbb{S}_4$ and $\mathbb{A}_4$, one produces similar lists also for these cases. This gives the list of genera for the groups G of type $\mathbb{S}_4$ in the Theorem; the groups G of type $\mathbb{A}_4$ are all subgroups of index 2 in groups G of type $\mathbb{S}_4$ except for $g = 4$ (with $\mathcal{B} = \mathcal{O}_4 \cup \mathcal{O}_6$ and quotient-orbifold $\mathcal{S}^2(3, 4, 6)$). The groups G are isomorphic to one of the two central extensions $\mathbb{A}_4 \times \mathbb{Z}_2$ and $\mathbb{A}_4^*$ of $\mathbb{A}_4$, or to one of four central extensions of $\mathbb{S}_4$.

Note added for the revised version. The referee provided the two additional references [2] and [3] in which similar results are obtained, in an algebraic language and by different algebraic methods. The main point of the present note is a short, topological approach to the classification: After the preliminary fact from complex analysis that a hyperelliptic involution of a Riemann surface is unique and central in its automorphism group, the methods in the present note are purely topological, offering a short and efficient topological approach to the classification.

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