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Remarks on nonautonomous bifurcation theory

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Dedicated to 75th birthday of Professor Jean Mawhin

Abstract. *We study some elementary bifurcation patterns when the bifurcation parameter is subjected to fast oscillations.*

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Foreword Professor Russell Johnson suddenly passed away on the 22 of July 2017, just after this manuscript had been accepted in its final form. Matteo Franca had the luck of being one of his PhD students and to profit of his vast knowledge and thoughtful advices since then, and wishes to take the chance to commemorate the excellent mathematician and magister to which he is indebted in several different ways.

1. Introduction

Problems of fast oscillations are of intrinsic mathematical interest and are of great importance in various areas of applied mathematics. It is well-known that, when the parameters of a mechanical system are subject to rapid oscillations, then the stability characteristics of that system may change in a substantial way. The interesting phenomenon of stabilization of a planar pendulum via vertical oscillations is a case in point [4], but it is far from being the only example. Others include the elimination of a Van der Pol oscillation, and the large-scale alteration of a stability diagram in a catalytic reactor [2, 3]. As a somewhat different example, we mention the quadruple ion trap (Paul trap), which is used as a component of a mass spectrometer.

In many problems of fast oscillations, one has a single parameter to deal with, namely the frequency of the oscillation. But in other problems there are one or more additional parameters, and these other parameters may determine a bifurcation pattern. In this paper we pose the question of how the presence of rapidly oscillating parameter disturbance can alter a bifurcation pattern. We will give partial answers to this question in the context of the simplest bifurcation patterns, namely the saddle-node, transcritical, pitchfork, and Andronov-Hopf scenarios. We will emphasize the case in which the fast oscillations are almost periodic in the sense of Bohr, or more generally of "strictly ergodic" type. Of course this is considerably more general than considering periodic fast oscillations. Thus for example we can deal with a situation in which a parameter is perturbed by two or more periodic terms whose periods are not commensurate, rather than by a single periodic term.

We are motivated in particular by the discussion in the paper [3], where the authors provide insights and methods which are valuable for the study of problems involving fast oscillations and stability. Incidentally, the authors of that paper seem rather dismissive of the scenario in which a parameter is slowly varied, and of that in which a parameter undergoes a stochastic disturbance; see also [2].

The starting point of this discussion is the problem

$$x' = x^2 - \varepsilon \qquad (x \in \mathbb{R}\,,\ \varepsilon \in \mathbb{R}). \tag{1}$$

Clearly equation (1) admits equilibria in $x = \pm\sqrt{\varepsilon}$ when $\varepsilon > 0$, where $x = -\sqrt{\varepsilon}$ is asymptotically stable and $x = +\sqrt{\varepsilon}$ is unstable. These merge as ε decreases through zero, and for $\varepsilon < 0$ all the solutions of (1) are unbounded in finite time.

Let us now modify (1) by adding in a rapidly oscillating term, to obtain

$$x' = x^2 - \varepsilon + \frac{\alpha}{\mu} f\left(\frac{t}{\mu}\right) \qquad ' = \frac{d}{dt}. \tag{2}$$

where α is a positive number, μ is a small positive parameter, and f is an almost periodic function with zero mean value. Note that the oscillations are not only rapid, but also indefinitely large as $\mu \to 0^+$.

To study the effect of the oscillatory term on the bifurcation pattern, we set $\tau = \frac{t}{\mu}$, so that (2) takes the form

$$\frac{dx}{d\tau} = \mu(x^2 - \varepsilon) + \alpha f(\tau)\,.$$

Following [2], we consider

$$\frac{dx}{d\tau} = \alpha f(\tau)\,. \tag{3}$$

Let us assume that $f(\tau)$ admits an almost periodic primitive $F(\tau)$; this is actually a highly nontrivial hypothesis, unless $f(\tau)$ is assumed to be periodic. There is no loss of generality in assuming that F has mean value zero. Then the general solution of (3) is of course

$$x = c + \alpha F(\tau) = h(\tau, c) \tag{4}$$

where c is an arbitrary constant. Still following [2], we make the change of variables

$$x = h(\tau, y) = y + \alpha F(\tau)\,,$$

which transforms (2) into

$$\frac{dy}{d\tau} = \mu\left\{[y + \alpha F(\tau)]^2 - \varepsilon\right\}\,. \tag{5}$$

At this point we apply the method of infinite-interval averaging see Fink [10] or Hale [15]. We will discuss the method in Section 2; here we merely state the result.

There is a neighborhood W of $y = 0$, which can be fixed independently of small $\mu > 0$, and an invertible change of variables in $w \in W$

$$y = w + \mu G(\tau, w)$$

such that, in the new variable w, (5) goes into

$$\frac{dw}{d\tau} = \mu\left\{w - \varepsilon + \alpha^2\overline{F^2}\right\} + o(\mu)\,. \tag{6}$$

Here the overbar indicates the mean value of the given function F. We note explicitly that $\overline{F^2}$ is the mean value of F^2; further the function G can be written explicitly:

$$G(\tau, w) = \mathrm{e}^{-\mu\tau}\int_{-\infty}^{\tau}\mathrm{e}^{\mu s}\big[2\alpha F(s)w + \alpha^2F^2(s)\big]\,ds$$

One can show that, if $\varepsilon > 0$ is fixed, $\varepsilon > \alpha^2\overline{F^2}$, and μ is chosen small enough (in dependence of ε), then (6) admits two almost periodic solutions, one of which is asymptotically stable and the other of which is unstable. If $\varepsilon < \alpha^2\overline{F^2}$, then for small μ, all solutions of (6) will leave some fixed neighborhood W of $w = 0$ in finite time.

At this point one must note that there has been a macroscopic change in the bifurcation pattern in (6) as compared to (2), namely the bifurcation point $\varepsilon = 0$ in (2) is transferred to $\varepsilon = \alpha^2\overline{F^2}$ in (6). Moreover, the very term "bifurcation pattern" is in the first moment not very well-defined, because the effect of the $o(\mu)$ terms on solutions of (6) may be quite pronounced if μ is not "small enough" relative to $\varepsilon - \alpha^2\overline{F^2}$, which of course takes its most interesting values near zero.

One may not find the "$\frac{\alpha}{\mu}$"-factor in front of f in (2) to be natural. Let us omit it and carry out the above calculations with f in place of $\frac{\alpha}{\mu}f$. We obtain the analogue of (6):

$$\frac{dw}{d\tau} = \mu\left\{w - \varepsilon + \mu^2\overline{F^2}\right\} + o(\mu^2)\,. \tag{7}$$

The bifurcation value of the averaged equation is $\varepsilon = \mu^2\overline{F^2}$, which tends to zero as $\mu \to 0^+$. However there is still an interesting issue as far as the μ-dependent terms are concerned, which we formulate as follows: set $\mu = \mu(\varepsilon)$ and ask "what happens" to the saddle-node pattern. We will take up this question in Section 3: rather remarkably it admits a reasonably clean-cut answer.

The reader may object at this point that one should be able to understand (2) in detail by applying repeated averaging, to obtain after $r+1$ steps

$$\frac{dx}{d\tau} = \mu\left\{y^2 - \varepsilon + \mu^2\overline{F^2} + \ldots + \mu^r f_r(y)\right\} + o(\mu^{r+1})\,.$$

To this it can be replied that, in the first place, repeated averaging is not always possible in the almost periodic, non-periodic world, in fact in many circumstances it is the "exceptional case". Second, if one envisions substituting a generic function such as $\mu = \varepsilon^{1/s}$ for μ with $s > r$, then one will still have to deal systematically with the $o(\mu^{1+r})$-term. So if μ tends to zero sufficiently slowly we will have to consider the "o"-term on the saddle-node bifurcation pattern.

This paper is structured as follows. In Section 2 we present a slightly generalized version of the Fink-Hale infinite-interval averaging theory. We make use of the Bebutov hull construction and other ideas of nonautonomous dynamics, which, in our opinion, clarify certain aspects of this method. Then, in Section 3, we discuss the classical bifurcation patterns when the relevant parameter is subjected to fast zero-mean oscillations. We will rediscuss the saddle node pattern, together with the transcritical, pitchfork, and Andronov-Hopf scenarios. Aside from presenting information on the bifurcation problems when fast oscillations are present, we want to illustrate what is now a rather extensive tool kit and body of results concerning nonautonomous differential systems. We make use of the basic Bebutov construction, some facts involving ergodic measures, exponential dichotomies, etc. We will also refer to previous papers concerning the nonautonomous bifurcation theory, as seems appropriate ([20, 25, 30]).

2. Preliminaries

In this section, we first present some facts from the field of topological dynamics (see [7, 35]) which will be useful in our discussion of infinite-interval averaging. Then we will describe our version of that averaging procedure.

Let P be a topological space. A real *flow* on P is determined by a family $\{\phi_t \mid t \in \mathbb{R}\}$ of homeomorphisms of P with the following properties:

- $\phi_0(p) = p$ for all $p \in P$;
- $\phi_t \circ \phi_s = \phi_{t+s}$ for all $t, s \in \mathbb{R}$;

- $\phi : P \times \mathbb{R} \to P$: $(t,p) \to \phi_t(p)$ is continuous.

Suppose now that P is a compact metric space, and let $\{\phi_t\}$ be a flow on P. A regular Borel probability measure ξ on P is said to be ϕ_t-invariant if $\xi(\phi_t(B)) = \xi(B)$ for each Borel set $B \subset P$ and for each $t \in \mathbb{R}$. An invariant measure is said to be $\{\phi_t\}$-*ergodic* if, in addition, the following indecomposibility condition holds: if $B \subset P$ is a Borel set, and if $\xi(B \triangle \phi_t(B)) = 0$ for each $t \in \mathbb{R}$, then either $\xi(B) = 0$ or $\xi(B) = 1$. Here $\triangle$ is the usual symmetric difference of sets: $A \triangle B = (A \backslash B) \cup (B \backslash A)$.

A famous theorem of Krylov and Bogoliubov ([23, 28]) states that, if P is a compact metric space, and $\{\phi_t\}$ is a flow on P, then there exists at least one $\{\phi_t\}$-ergodic measure ξ on P. If an ergodic measure ξ on P is the *only* $\{\phi_t\}$-ergodic measure on P the flow $(P, \{\phi_t\})$ is said to be *uniquely ergodic*. One can then apply a basic theorem of Birkhoff to the triple $(P, \{\phi_t\}, \xi)$, together with a refinement of that theorem. We state these results together.

THEOREM 2.1. *Let P be a compact metric space, let $\{\phi_t\}$ be a flow on P, and let ξ be a $\{\phi_t\}$-ergodic measure on P. If $h \in L^1(P,\xi)$, then*

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t h(\phi_s(p)) ds = \int_P h d\xi \tag{8}$$

for ξ-a.a. $p \in P$. If $h : P \to P$ is a continuous function and ξ is uniquely ergodic, then (8) *holds for all $p \in P$, and the limit is uniform in $p \in P$. That is, given $\varepsilon > 0$ there exists $T > 0$ such that, if $|t| \geq T$, then*

$$\left| \frac{1}{t} \int_0^t h(\phi_s(p)) ds - \int_P h d\xi \right| \leq \varepsilon \qquad (p \in P).$$

The first part of Theorem 2.1 can be stated and proved in the more general context of measurable flows; see e.g. [11]. The second part of the theorem is specific to a continuous flow $\{\phi_t\}$ defined on a compact space P [14].

Let P be a nonempty compact metric space. A flow $(P, \{\phi_t\})$ on P is said to be *minimal* if, for each $p \in P$, the *orbit* $\{\phi_t(p) \mid t \in \mathbb{R}\}$ is dense in P. A flow $(P, \{\phi_t\})$ is said to be *strictly ergodic* if it is minimal and admits a unique ergodic measure ξ.

Again let P be a nonempty compact metric space. A flow $(P, \{\phi_t\})$ on P is said to be *Bohr almost periodic*, or simply almost periodic, if there is a metric d on P, which is compatible with the topology on P, such that

$$d(\phi_t(p_1), \phi_t(p_2)) = d(p_1, p_2)$$

for all $p_1, p_2 \in P$ and for all $t \in \mathbb{R}$. If $(P, \{\phi_t\})$ is almost periodic, then for each $p \in P$ the *orbit closure* $\mathrm{cls}\{\phi_t(p) \mid t \in \mathbb{R}\} \subset P$ is strictly ergodic (in particular it is minimal), and in fact P is the union of its minimal sets. If $(P, \{\phi_t\})$ is an almost periodic minimal set, then one can give P the structure of a compact Abelian topological group with multiplication $*$ and dense subgroup $\mathbb{R}$ in such a way that $\phi_t(P) = p * t$ ($p \in P$, $t \in \mathbb{R}$). Let us note finally that, although a minimal almost periodic flow is strictly ergodic, the converse is not true, a fact which is illustrated by the Furstenberg flows [14].

We discuss a class of concrete minimal, almost periodic flows, namely the Kronecker flows. Let $d \geq 2$ and let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-torus. Let $\alpha_1, \ldots, \alpha_d$ be $\mathbb{Q}$-independent real numbers, let $p_1, \ldots, p_d$ be 1-periodic angular coordinates on $\mathbb{T}^d$, and set

$$\phi_t(p_1, \ldots, p_d) = (p_1 + \alpha_1 t, \ldots, p_d + \alpha_d t) \qquad \mathrm{mod}\ \mathbb{Z}^d.$$

Then the flow $(\mathbb{T}^d, \{\phi_t\})$ is minimal and almost periodic. The unique $\{\phi_t\}$-invariant measure is the normalized Haar measure on $\mathbb{T}^d$. More generally, if $\alpha_1, \ldots, \alpha_d$ satisfy exactly $k \in \{0, 1, 2, \ldots, d-1\}$ independent homogeneous $\mathbb{Q}$-linear relations, then $(\mathbb{T}^d, \{\phi_t\})$ laminates into a disjoint union of almost periodic minimal flows, each of which is flow-isomorphic to a $(d-k)$-dimensional Kronecker flow.

Next we give a brief discussion of the Bebutov construction, which actually consists of a family of mutually similar constructions. Consider a time-dependent differential system

$$\frac{dx}{dt} = f(t, x) \qquad t \in \mathbb{R}\,,\ x \in \mathbb{R}^d\,. \tag{9}$$

The general goal is to apply the methods of topological dynamics to study the solutions of (9). This can be done if f satisfies certain conditions, as we now indicate. We do not give proofs of the various assertions we make below: these are readily available in the literature (e.g., [34]) and in any case are usually quite easy to check directly.

First suppose that, for each compact subset $K \subset \mathbb{R}^d$, the restriction of f to $\mathbb{R} \times K$ is *uniformly* continuous. Then there exist:

(i) a compact metrizable space P which carries a flow $\{\phi_t\}$;

(ii) a continuous function $f_* : P \times \mathbb{R}^d \to \mathbb{R}^d$;

(iii) a point $p_* \in P$ such that $f(t, x) = f_*(\phi_t(p_*), x)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^d$.

The flow $\{\phi_t\}$ is induced by the translation in t, and the points of P are actually functions $p(t, x) = \lim_{n\to\infty} f(t + t_n, x)$ for appropriate sequences $\{t_n\} \subset \mathbb{R}$. Here the limit is taken in the compact-open topology on $\mathbb{R} \times \mathbb{R}^d$. One usually abuses notation at this point and writes f instead of f_* (but not p for p_*). The upshot is that equation (9) has been embedded into the family of differential equations

$$\frac{dx}{dt} = f(\phi_t(p), x) \qquad p \in P,\ x \in \mathbb{R}^n \tag{9p}$$

where (9) coincides with ($9p_*$).

Next suppose that each equation (9p) admits a unique global solution $x(t; x_0, p)$ for each initial value $x_0 \in \mathbb{R}^n$. Then the family of homeomorphisms

$$\psi_t : P \times \mathbb{R}^d \to P \times \mathbb{R}^d : (p, x_0) \to (\phi_t(p), x(t; x_0, p))$$

defines a flow on $P \times \mathbb{R}^d$. One speaks of a skew-product flow because the first factor does not depend on x_0. One can now use various techniques of topological dynamics to study the solutions of the various equation (9p), and in particular the solutions of equation (9), alias ($9p_*$).

One can take account of eventual smoothness properties of f in $x \in \mathbb{R}^d$, say of order $r \geq 1$, as follows. Let $l = (l_1, \ldots, l_d)$ be a multiindex of integers such that $0 \leq l_1, \ldots, l_d \leq l_1 + \ldots + l_d = |l| \leq r$. One requires that f together with all its partial derivatives $D^l_x f = D^{l_1}_{x_1} \ldots D^{l_d}_{x_d} f$ of order $|l| \leq r$ be uniformly continuous on sets of the form $\mathbb{R} \times K$ where $K \subset \mathbb{R}^d$ is compact. If this condition holds, then there exist:

(i) a compact metric space P with a flow $\{\phi_t\}$;

(ii) a continuous function $f_* : P \times \mathbb{R}^d \to \mathbb{R}^d$ such that $D^l_x f_* : P \times \mathbb{R}^d \to \mathbb{R}^d$ exists and is continuous for each multiindex $l = (l_1, \ldots, l_d)$ with $|l| \leq r$;

(iii) a point $p_* \in P$ such that $f(t, x) = f_*(\phi_t(p_*), x)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^d$.

One describes point (ii) by saying that f_* is of class C^r in x, uniformly in $p \in P$. In most of what we do below, we will simply assume that each function f_* which is encountered is C^∞ in x uniformly in $p \in P$.

Let us now turn to the theory of infinite-interval averaging, which we formulate in a context in which a Bebutov flow is present. The ideas discussed here are drawn from the presentation in Fink [10] and Hale [15]. First we recall a basic lemma [22], which extends somewhat Lemma 14.1 of Fink's book.

PROPOSITION 2.2. *Let P and X be compact metric spaces, and let $(P, \{\phi_t\})$ be a uniquely ergodic flow with unique invariant measure ξ. Let $f : P \times X \to \mathbb{R}^d$ be a continuous function, and let*

$$\overline{f}(x) = \int_P f(p, x) d\xi(p)$$

be the ξ-mean value of f. If μ is a positive number, define

$$F(p, x, \mu) = \int_{-\infty}^0 e^{\mu s} \left\{ f(\phi_s(p), x) - \overline{f}(x) \right\} ds \,.$$

Then there is a continuous positive function $\zeta : (0, \infty) \to (0, \infty)$ such that $\zeta(\mu) \to 0$ as $\mu \to 0^+$, and

$$|\mu F(p, x, \mu)| \leq \zeta(\mu) \qquad (\mu > 0,\ p \in P,\ x \in X) \,. \tag{10}$$

Let us remark that the function ζ need not tend to zero at a prearranged rate - to be explicit, one cannot write, say, $\zeta(\mu) = C\mu^s$ where C is a constant and $s > 0$. We give an illustrative example in Appendix B. Let us remark that, if $X \subset \mathbb{R}^m$ is the closure of a bounded open set, and if f is C^r on X uniformly in $p \in P$, then one can determine a continuous positive function $\zeta = \zeta(\mu)$ such that $\zeta(\mu) \to 0$ as $\mu \to 0^+$, and such that

$$|\mu D_x^l F(p, x, \mu)| \leq \zeta(\mu)$$

for all $p \in P$, $x \in X$, and for each multiindex $l = (l_1, \ldots, l_d)$ with $|l| \leq r$.

One can generalize these statements slightly by allowing f to depend continuously and smoothly on μ, for μ in some open interval containing $\mu = 0$. This can be seen by substituting $x \in \mathbb{R}^d$ by $(x, \eta) \in \mathbb{R}^{d+1}$, applying Proposition 2.2, then letting $\eta = \mu$.

Continuing the discussion, let $p \in P$, $x \in X$, and set

$$F_\mu(t, x) = F(\phi_t(p), x, \mu) = \mathrm{e}^{-\mu t} \int_{-\infty}^{t} \mathrm{e}^{\mu s} \left\{ f(\phi_s(p), x) - \overline{f}(x) \right\} ds. \tag{11}$$

Of course F_μ depends on p as well. Clearly

$$\frac{dF_\mu}{dt} = -\mu F_\mu + f(\phi_t(p), x) - \overline{f}(x)\,.$$

We can apply this observation to ODES with rapidly varying time dependence. Since smoothness issues are of no particular relevance at present, let us assume that $f : P \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d : (p, x, \mu) \to f(p, x, \mu)$ is a function which is C^∞ in (x, μ), uniformly in $p \in P$. Consider the family of differential equations

$$\frac{dx}{dt} = \mu f(\phi_t(p), x, \mu) \qquad x \in \mathbb{R}^d,\ \mu > 0 \tag{12p}$$

or equivalently

$$\frac{dx}{ds} = f(\phi_{\frac{s}{\mu}}(p), x, \mu) \qquad \frac{s}{\mu} = t \tag{13p}$$

where the rapid oscillations are displayed explicitly. For each fixed $p \in P$, make the following change of variables

$$x = y + \mu F_\mu(t, y) \tag{14}$$

where F_μ is defined in (11).

We state the averaging theorem of Fink and Hale in the context of the family (12p).

PROPOSITION 2.3. *If $\Delta > 0$, let B_Δ be the closed ball of radius Δ in $\mathbb{R}^d$ centered at the origin. Let P, $\{\phi_t\}$, f, F, F_μ be as above. There exist positive numbers Δ, Δ_0 and μ_0 with the following properties. First, for each $p \in P$ and $\mu \in (0, \mu_0)$, the transformation* (14) *is of class C^∞ on B_{Δ_0} and has a C^∞ inverse on B_Δ. Second, both the transformation* (14) *and its inverse are C^∞ on $B_{\Delta_0} \times (0, \mu_0)$ respectively $B_\Delta \times (0, \mu_0)$, uniformly in $p \in P$.*

We now make the change of variables (14) in equation (12p), for each $p \in P$, and obtain

$$\left(I + \mu \frac{\partial F_\mu}{\partial y}\right) \frac{dy}{dt} = \mu \left\{ \overline{f_\mu}(y) + \mu F_\mu(t, y) + f_\mu(\phi_t(p), x) - f_\mu(\phi_t(p), y) \right\} \quad \text{(15p)}$$

for all $0 < \mu \leq \mu_0$, $y \in B_\Delta$. Here I is the identity matrix, and we have written $f_\mu(\cdot) = f(\cdot, \mu)$. Now

$$f_\mu(p, x) - f_\mu(p, y) = \frac{\partial f_\mu}{\partial y}(p, y) \mu F_\mu + R$$

where the remainder R is of order $O(|\mu F_\mu|^2)$. Since μF_μ is of order $o(1)$ as $\mu \to 0^+$, it is natural to compare solutions of (12p) with those of the averaged equation

$$\frac{dy}{dt} = \mu \overline{f_\mu}(y) \,. \quad (16)$$

Generally speaking one does this on a case-by-case basis when dealing with problems on an infinite time interval.

It should be remarked that, in the case when f has only finitely many x-derivatives, there is in general a loss of smoothness of one degree in passing from (12p) to (15p).

It may happen that the function f in (12p) admits a primitive in the sense that there exists a continuous function $\tilde{F} : P \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ such that

$$\frac{d}{dt} \tilde{F}(\phi_t(p), x, \mu) = f(\phi_t(p), x, \mu) - \overline{f}(x, \mu)$$

identically in t, p, x, μ. In this case one replaces (14) by

$$x = y + \mu \tilde{F}_\mu(\phi_t(p), y)$$

where $\tilde{F}_\mu(\cdot) = \tilde{F}(\cdot, \mu)$. Then (15p) takes the form

$$\left(I + \mu \frac{\partial \tilde{F}_\mu}{\partial y}\right) \frac{dy}{dt} = \mu \left\{ \overline{f_\mu}(y) - \mu f_1(\phi_t(p), y, \mu) \right\}$$

where the non autonomous term $\mu f_1(p, y, \mu)$ is of order $O(\mu)$ as $\mu \to 0^+$ (and not just of order $o(1)$). It must be emphasized, however, that a primitive need not exist if one has a non-periodic time dependence, i.e., if $(P, \{\phi_t\})$ is not a periodic flow.

We consider one last issue, which regards the continuous/smooth convergence of the solutions of the rapidly oscillating equations (13p) as $\mu \to 0^+$. This issue has been treated in [9], and the results discussed there can be viewed as an amplification of classical theorems which are proved in the averaging theory on a finite interval [31].

We formulate a result which will be useful later. Consider the equations

$$\begin{aligned} \frac{dx}{ds} &= f(\phi_{\frac{s}{\mu}}(p), x, \mu) \qquad \mu > 0 \\ \frac{dx}{ds} &= \overline{f_0}(x) \qquad\qquad\quad \mu = 0 \end{aligned} \tag{17}$$

where $\overline{f_0}(x) = \int_P f(p, x, 0) d\xi(p)$. The first equation in (17) is equation (13p), and the second is (as we will see) the appropriate limiting equation. Assume that, for all $p \in P$, the solution $x(s; x_*, p, \mu)$ of equation (13p) with initial condition $x_* \in \mathbb{R}^2$ is defined for all $s \in (-\infty, \infty)$, and that the same condition holds for each solution $x_0(s, x_*)$ of the equation $\frac{dx}{ds} = \overline{f_0}(x)$. Define $\Psi : P \times \mathbb{R}^2 \times [0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ as follows:

$$\Psi(p, x_*, \mu, s) = \begin{cases} x(s; x_*, p, \mu) & \mu > 0 \\ x_0(s, x_*) & \mu = 0 \end{cases}$$

Proposition 2.4. *The map Ψ is continuous. Furthermore for each multiindex $k = (k_1, k_2)$ with integer components $k_1 \geq 0$, $k_2 \geq 0$, the derivative $D_x^k \Psi$ exists for all $p \in P$, $x_* \in \mathbb{R}^2$, $\mu \in [0, \infty)$ and $s \in \mathbb{R}$, and is continuous. That is*

$$\begin{aligned} \lim_{\mu \to 0^+} x(s; x_*, p, \mu) &= x_0(s; x_*) \\ \lim_{\mu \to 0^+} D_x^k x(s; x_*, p, \mu) &= D_x^k x_0(s; x_*) \end{aligned}$$

where the convergence is uniform in $(p, x_*, s) \in K \subset P \times \mathbb{R}^2 \times \mathbb{R}$ whenever K is compact.

A proof of this Proposition can be modelled on those of Propositions 2.5 and 2.6 in [9].

3. Analysis

In this section we analyze the elementary bifurcation patterns when the parameter is subjected to rapid oscillations. We will make use of the averaging method discussed in Section 2, of the methods used in [2, 3] for studying fast oscillation problems, and of various techniques from the field of Nonautonomous Dynamics.

3.1. The saddle-node pattern

The starting point is the differential equation

$$\frac{dx}{dt} = x^2 - \varepsilon + f\left(\frac{t}{\mu}\right) \qquad \mu > 0 \tag{18}$$

where μ is a small positive parameter. It is convenient to carry out a Bebutov construction on $f = f(\tau)$ where $\tau = \frac{t}{\mu}$. We assume that there exist a strictly ergodic flow $(P, \{\phi_\tau\})$, a continuous function $f_* : P \to \mathbb{R}$, and a point $p_* \in P$ such that $f(\tau) = f_*(\phi_\tau(p_*))$. In this way, equation (18) can be embedded in the family of equations

$$\frac{dx}{dt} = x^2 - \varepsilon + f_*(\phi_{\frac{t}{\mu}}(p)) \qquad \mu > 0,\, p \in P \tag{18p}$$

If for example $f(\tau)$ is Bohr almost periodic, then $(P, \{\phi_\tau\})$ is an almost periodic minimal flow. Let ξ be the normalized Haar measure on P; then ξ is the unique $\{\phi_\tau\}$-invariant measure on P. Assume that $\int_P f_* d\xi = 0$.

Let us carry out an a priori analysis of (18), based on the discussion in [19, pp. 170-172]. For this, fix $\mu > 0$ and make the change of variables

$$x = -\frac{\psi'}{\psi}\,, \qquad {}' = \frac{d}{dt}$$

which takes (18) to the form of a Schrödinger equation

$$-\frac{d^2\psi}{dt^2} = \left[f\left(\frac{t}{\mu}\right) - \varepsilon\right]\psi\,. \tag{19}$$

Clearly ε plays the role of an eigenvalue parameter in equation (19). We will want to consider in addition the family

$$-\frac{d^2\psi}{dt^2} = \left[f_*\left(\phi_{\frac{t}{\mu}}(p)\right) - \varepsilon\right]\psi\,. \tag{19p}$$

By hypothesis, the flow $(P, \{\phi_t\})$ is strictly ergodic, hence so is the flow of $(P, \{\phi_{\frac{t}{\mu}}\})$ for each $\mu > 0$. In this case, the following information is available; see, e.g., [18]. First, there is a critical value $\varepsilon = \varepsilon_c(\mu)$ such that, if $\varepsilon > \varepsilon_c(\mu)$, then the family of two-dimensional linear differential systems

$$\frac{d}{dt}\begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varepsilon - f_*\left(\phi_{\frac{t}{\mu}}(p)\right) & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi' \end{pmatrix} \tag{20p}$$

admits an exponential dichotomy over P. Second, if $\varepsilon < \varepsilon_c(\mu)$, then all nonzero solutions of (20p) rotate infinitely often around the origin in the $\begin{pmatrix} \psi \\ \psi' \end{pmatrix}$-plane, both as $t \to +\infty$ and as $t \to -\infty$.

It can be further be shown that, if $\varepsilon > \varepsilon_c(\mu)$, and if $Q_p : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the dichotomy projection at $p \in P$, then $\mathrm{Ker}Q_p \subset \mathbb{R}^2$ and $\mathrm{Im}Q_p \subset \mathbb{R}^2$ (viewed as lines passing through the origin in $\mathbb{R}^2$) make angles $\theta^{\pm}(p)$ with the vertical axis $\psi = 0$ which are bounded away from zero mod π. Let us note parenthetically that the statements of the previous two paragraphs do not require strict ergodicity of $(P, \{\phi_\tau\})$ but only minimality.

Returning to the x-variable via the transformation $x = -\frac{\psi'}{\psi}$, we see that, if $\varepsilon > \varepsilon_c(\mu)$, then there are two compact subsets $M_\pm \subset P \times \mathbb{R}$ which are invariant with respect to the local flow on $P \times \mathbb{R}^2$ induced by equations (18p). These sets determine an attractor-repeller pair, as follows from the fact that equations (20p) admit an exponential dichotomy for $\varepsilon > \varepsilon_c(\mu)$. In particular, if $\varepsilon > \varepsilon_c(\mu)$ and $p \in P$, then equation (18p) admits two globally defined bounded solutions, one of which attracts and the other of which repels nearby solutions.

On the other hand, if $\varepsilon < \varepsilon_c(\mu)$, then all solutions of (18p) are unbounded in finite time ($p \in P$). So one has an analogue of the saddle-node bifurcation pattern as ε increases through $\varepsilon_c(\mu)$, for each $\mu > 0$. However the analogy is not complete for the following reason. At $\varepsilon = \varepsilon_c(\mu)$ the flow on $P \times \mathbb{R}$ induced by equations (18p) does admit a unique minimal subset M_c, which would seem to correspond to the zero solution of (1) when $\varepsilon = 0$. However M_c need not be homeomorphic to P. In fact if $(P, \{\phi_t\})$ is a minimal almost periodic flow which is not periodic, it is possible to determine f_* in such a way that M_c with its flow is an almost automorphic but not almost periodic extension of $(P, \{\phi_{\frac{t}{\mu}}\})$, [35]. Examples of this phenomenon may be constructed, beginning with equation (20p), by using a method of Millionščikov ([27]; also Vinograd [36]). Nowadays one can also find such examples using other techniques; see e.g. [5].

Let us remark at this point that a bifurcation analysis similar to that we have just given can be carried out for a more general equation of the form

$$\frac{dx}{dt} = g(t, x, \varepsilon)$$

where ε is an appropriate bifurcation parameter, and g is almost periodic in t and concave as a function of x. See Núñez and Obaya [29].

Until now we have not taken account of the rapid oscillations present in equation (18). Their presence allows us to make several observations. The first one is

Proposition 3.1. *As $\mu \to 0^+$, the critical value $\varepsilon_c(\mu)$ tends to zero.*

Proof. The easiest way to prove this statement seems to be the following. Return to equations (20p). Since $\int_P f_* d\xi = 0$, the averaged form of this equation is the constant system

$$\frac{d}{dt}\begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi' \end{pmatrix}, \tag{21}$$

which has an exponential dichotomy whenever $\varepsilon > 0$. Fix a positive number ε. There exists a positive number $\mu_0 = \mu_0(\varepsilon)$ such that if $0 < \mu < \mu_0$, then the family (20p) admits an exponential dichotomy over P. This follows from Proposition 2.4 and the well known Sacker-Sell perturbation theorem [32].

In view of the previous analysis, this implies that $\limsup_{\mu\to 0^+} \varepsilon_c(\mu) \leq 0$. On the other hand, if $\varepsilon < 0$, we can analyze the ξ-rotation number $\alpha = \alpha(\mu)$ of the family (19p). See Appendix A for a discussion of the ξ-rotation number. By general results concerning the continuity of the rotation number, one has that $\lim_{\mu\to 0^+} \alpha(\mu)$ equals the rotation number of the constant system (21). This latter rotation number is strictly positive. So for small positive values of μ, the rotation number of $\alpha(\mu)$ is strictly positive. This means that $\varepsilon_c(\mu) \geq 0$ for small positive values of μ, hence $\liminf_{\mu\to 0^+} \varepsilon_c(\mu) \geq 0$. We conclude that indeed $\lim_{\mu\to 0^+} \varepsilon_c(\mu) = 0$. □

Let us now suppose that μ is a function of ε: $\mu = \mu(\varepsilon)$, which is continuous, positive when $\varepsilon \neq 0$, and such that $\lim_{\varepsilon\to 0} \mu(\varepsilon) = 0$. This is a particularly interesting situation because the oscillations "become fast" near $\varepsilon = 0$ in an ε-dependent way. We make a few remarks when μ depends on ε in this way and $\varepsilon \to 0$. We adopt the point of view that (18) with $\mu = \mu(\varepsilon)$ is a bifurcation problem with parameter ε.

We assume that the function $f(\tau)$ admits a bounded primitive $F(\tau)$: $F'(\tau) = f(\tau)$, which can be chosen to have mean value zero. It is well-known that, when these conditions hold, there is a continuous function $F_* : P \to \mathbb{R}$ such that $\int_P F_* d\xi = 0$ and

$$F_*(\phi_t(p)) - F_*(p) = \int_0^t f_*(\phi_\sigma(p))d\sigma \qquad (t \in \mathbb{R},\ p \in P)$$

Following [2] we consider the equation

$$\frac{dx}{d\tau} = \mu f(\tau) \qquad \tau = t/\mu$$

which has the general solution

$$x = \mu F(\tau) + c = h(\tau, c)$$

where c is an arbitrary constant. Setting $x = h(\tau, y)$ in (18) leads to

$$\frac{dy}{d\tau} = \mu \left\{ (y + \mu F)^2 - \varepsilon \right\} ,$$

and an application of the averaging procedure discussed in Section 2 leads to

$$\frac{dy}{dt} = \mu \left\{ y^2 - \varepsilon + \mu^2 \overline{F^2} + o(\mu)y + o(\mu^2) \right\} \tag{22}$$

where $\overline{F^2} = \int_P F_*^2 d\xi$. The averaged system is

$$\frac{dy}{dt} = \mu \left\{ y^2 - \varepsilon + \mu^2 \overline{F^2} \right\} . \tag{23}$$

The following observations are in order. First, if $\mu(\varepsilon) = |\varepsilon|^s$ for $s > 1/2$, then (23) admits a saddle-node bifurcation with critical value $\varepsilon = 0$. On the other hand, if $\mu(\varepsilon) = |\varepsilon|^{1/2}$ and if $\overline{F^2} > 1$, then no bifurcation occurs: (23) takes the form:

$$\frac{dy}{dt} = \mu \left\{ y^2 - \varepsilon + |\varepsilon| \overline{F^2} \right\}$$

and $-\varepsilon + |\varepsilon|\overline{F^2} > 0$ when $\varepsilon \neq 0$. Of course the same conclusion holds when $0 < s < 1/2$. To take account of the terms $o(\mu)y + o(\mu^2)$, note that if $|y| \leq |\varepsilon|^s$, these are dominated by $|\varepsilon|\overline{F^2} - \varepsilon$, so the full equation (22) also admits no bifurcation in $\varepsilon = 0$.

At this point one can envisage a curve $\mu = \mu(\varepsilon)$ whose graph intersects the critical curve in a transversal way in infinitely many points $(\mu_n, \varepsilon_n) \to (0, 0)$, so that there will be infinitely many switches in the direction of the bifurcation. One may ask about the dynamics of equation (18p) at such a point (μ_n, ε_n). The answer follows from the earlier considerations: there is a unique minimal subset $M_n \subset P \times \mathbb{R}$, which is an almost automorphic extension of P.

Remark 3.1. One can analyze the following somewhat more general bifurcation problem in a similar way:

$$\frac{dx}{dt} = ax^2 + 2bx + f - \varepsilon \tag{24}$$

where a, b, and f are functions of t/μ, and $\mu > 0$ is small. If $x = \frac{v}{u}$ and

$$J \frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \left[\begin{pmatrix} f & b \\ b & a \end{pmatrix} - \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} u \\ v \end{pmatrix} \tag{25}$$

then x satisfies (24). Here J is the antisymmetric matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now (25) can be viewed as a spectral problem of Atkinson type [1, 8] with spectral parameter $-\varepsilon$. Such a problem has a theory which is, in general terms, analogous to that of the Schrödinger equation (19). In a bit more detail, assume that there exist a strictly ergodic flow $(P, \{\phi_t\})$ with ergodic measure ξ, a point $p_* \in P$, and continuous functions a_*, b_*, $f_* : P \to \mathbb{R}$ such that $a(\tau) = a_*(\phi_\tau(p_*))$, $b(\tau) = b_*(\phi_\tau(p_*))$, $f(\tau) = f_*(\phi_\tau(p_*))$. There is a family of equations parametrized by $p \in P$, which corresponds to (25). Now assume that a_* is strictly positive on P. Then one can prove the existence of a critical curve $\varepsilon = \varepsilon_c(\mu)$, defined for $\mu > 0$, such that if $\varepsilon = \varepsilon_c(\mu)$ then equations (25) admit an exponential dichotomy over P, and if $\varepsilon < \varepsilon_c(\mu)$ then the ξ-rotation number of the family (25) is positive. So if $\mu > 0$, then a saddle-node type bifurcation occurs as ε decreases through $\varepsilon_c(\mu)$.

3.2. The transcritical pattern

The starting point is the equation

$$\frac{dx}{dt} = x\left(\varepsilon + f\left(\frac{t}{\mu}\right) - x\right) . \tag{26}$$

As before we write $\tau = \frac{t}{\mu}$, and assume that $f = f(\tau)$ has associated to it a Bebutov flow $(P, \{\phi_t\})$ which is strictly ergodic, with unique ergodic measure ξ. For example, $f(\tau)$ might be a Bohr-almost periodic function. There exists a point $p_* \in P$ and a continuous function $f_* : P \to \mathbb{R}$ such that $f(\tau) = f_*(\phi_t(p_*))$ for all $\tau \in \mathbb{R}$. We assume that $\int_P f_* d\xi = 0$. Introduce the family of equations

$$\frac{dx}{dt} = x\left(\varepsilon + f_*\left(\phi_{t/\mu}(p)\right) - x\right) . \tag{26p}$$

We can carry out a priori analysis of the family (26p), in the following way. Let

$$w = \frac{1}{x} ,$$

so that (26p) takes the form

$$\frac{dw}{dt} + \left[\varepsilon + f_*\left(\phi_{t/\mu}(p)\right)\right] w = 1 .$$

The substitution $w = \cot(\theta) = \frac{u}{v}$ leads to the family

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -a & 1 \\ 0 & a \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{27p}$$

where $a = \frac{1}{2}[\varepsilon + f_*]$ and one views θ as the angular coordinate in the (u, v)-plane. We will restrict attention to the sector $0 \leq \theta \leq \pi$, that is the closed upper half (u, v)-plane.

For each fixed $\mu > 0$ we can analyze the family (27p) along standard lines. Since $\int_P f_* d\xi = 0$, one has that the dynamical spectrum of (27p) reduces to $\{-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\}$ for all $\varepsilon \in \mathbb{R}$; this uses the unique ergodicity of $(P, \{\phi_t\})$. This implies that the family (27p) admits an exponential dichotomy over P whenever $\varepsilon \neq 0$ [21].

We can describe the dichotomy bundles as follows. One bundle B_0 is described by the relation $v = 0$ for all $\varepsilon \neq 0$. That is B_0 coincides with the product space $P \times \ell$, where $\ell \subset \mathbb{R}^2$ is the horizontal line $\left\{\begin{pmatrix} u \\ 0 \end{pmatrix} \mid u \in \mathbb{R}\right\}$. This bundle is unstable for $\varepsilon < 0$ and stable for $\varepsilon > 0$. The other bundle B_ε can be parametrized by a continuous map $\Theta : P \to (0, \pi)$ in the sense that

$$B_\varepsilon = \left\{\left(p, \begin{pmatrix} u \\ v \end{pmatrix}\right) \in P \times \mathbb{R}^2 \middle| \begin{pmatrix} u \\ v \end{pmatrix} = \left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\| \begin{pmatrix} \cos(\Theta(p)) \\ \sin(\Theta(p)) \end{pmatrix}\right\} .$$

Let I respectively II denote the first respectively the second quadrant in the (u, v) space. One can check that that, if $\epsilon < 0$, then B_ϵ lies in $P \times II$, while if $\epsilon > 0$ then B_ϵ lies in $P \times I$. One can further check that B_ϵ is stable for $\varepsilon < 0$ and unstable for $\varepsilon > 0$.

All this means that, for each $p \in P$, the x-equation (26p) admits the solution $x(t) \equiv 0$ together with a solution $x(t) = X(\phi_{t/\mu}(p))$, where $X : P \to \mathbb{R}$, $X(p) = \tan(\Theta(p))$ is continuous. It is clear that a clean analogue of the classical transcritical bifurcation pattern takes place at $\varepsilon = 0$, for each $\mu > 0$. In fact the solution $x \equiv 0$ is asymptotically stable for $\varepsilon < 0$ and unstable for $\varepsilon > 0$, while the solution $x(t) = X(\phi_{t/\mu}(p))$ is unstable for $\varepsilon < 0$ and asymptotically stable for $\varepsilon > 0$ ($p \in P$). Thus $\varepsilon = 0$ is a critical value of the parameter ε for each $\mu > 0$; we write $0 = \varepsilon = \varepsilon_c(\mu)$ to reflect this fact.

We have encountered what appears to be a typical difference between the bifurcation behavior of systems with "additive noise" (namely $\varepsilon + f$ in the saddle-node pattern), and "multiplicative noise" (namely $(\varepsilon + f)x$ in the transcritical pattern). That is, the critical curve $\varepsilon_c(\mu)$ needs not be constant in the first case, but is constant in the second case.

It is of interest to study the solutions of (26) and of the family (26p) at the critical value $\varepsilon(\mu) = 0$, for positive values of μ. The simplest case is that in which $f(\tau)$ admits a bounded primitive: thus $\sup_{\tau\in\mathbb{R}} |F(\tau)| < \infty$ and $F'(\tau) = f(\tau)$. Consider the problem

$$\frac{dx}{d\tau} = \mu f(\tau)x \qquad\qquad \tau = \frac{t}{\mu}$$

which has the general solution $x = h(\tau, c) = c\mathrm{e}^{\mu F(\tau)}$ with arbitrary constant c. Following [3], set

$$x = h(\tau, y) = y\mathrm{e}^{\mu F(\tau)}\,,$$

which transforms (26) into

$$\frac{dy}{d\tau} = \mu\mathrm{e}^{-\mu F(\tau)}\mathrm{e}^{2\mu F(\tau)}(-y^2) = -\mu\mathrm{e}^{\mu F(\tau)}y^2\,. \tag{28}$$

It is clear that all positive solutions of (28) tend to zero as $\tau \to \infty$, and so by boundedness of F, all positive solutions of (26) tend to zero as $\tau \to \infty$.

If f does not admit a bounded primitive, then the discussion of the positive solutions of (26) resp. (26p) requires a bit more effort. First of all, we claim that if $p \in P$, $x_0 > 0$, and $x(t)$ is the solution of (26p) such that $x(0) = x_0$, then $\liminf_{t\to\infty} x(t) = 0$. We sketch a proof. Suppose for contradiction that there exist $p_1 \in P$, $x^1 > 0$, and $\delta > 0$ such that, if $x^1(t)$ is the solution of ($26\mathrm{p}_1$) with $x^1(0) = x^1$, then $x^1(t) \geq \delta$ for all $t > 0$. It is clear that $x(t)$ is defined for all $t > 0$ and is uniformly bounded. Consider the local flow $\{\psi_t\}$ on $P \times \{0 \leq x < \infty\}$ defined by $\psi_t(p, x_0) = (\phi_t(p), x(t))$. Then the ω-limit set K

of (p_1, x^1) is a nonempty compact subset of $P \times \{0 < x < \infty\}$, and the local flow $\{\psi_t\}$ extends to a global flow on K. Since $(P, \{\phi_t\})$ is a minimal flow, we can conclude that, for each $p \in P$, there exists a solution $x_p(t)$ of (26p) with $x_p(0) > 0$ such that $x_p(t)$ is uniformly bounded above and $x_p(t) \geq \delta$ for all $t \in \mathbb{R}$.

Next let $\tau = t/\mu$, and let $f_* : P \to \mathbb{R}$ be the "extension of f to P" introduced earlier. There exists $p_* \in P$ such that $F(\tau) = \int_0^\tau f_*(\phi_\sigma(p_*))d\sigma$ is bounded above but unbounded below [33]. Write $x_*(\tau) = x_{p_*}(\mu\tau)$, so that

$$x_*(\tau) = \mathrm{e}^{\mu F(\tau)} \left[x_*(0) - \int_0^\tau \mathrm{e}^{-\mu F(\sigma)} x_*^2(\sigma) d\sigma \right].$$

Since $x_*(\tau) > 0$ for all $\tau > 0$, it is clear that $\liminf_{\tau\to\infty} x_*(\tau) = 0$, but this contradicts the hypothesis. The proof is complete.

It is not clear if "lim inf" can be replaced by "lim" in the above result. We conjecture that it cannot, though we do not have as yet a suitable example. On the other hand, if $\mu > 0$ is small, we can obtain more information by carrying out a Fink-Hale type averaging procedure, beginning with equation (26). Set

$$\begin{aligned} &\tau = t/\mu \\ &F_\mu(\tau) = \mathrm{e}^{-\mu\tau} \textstyle\int_{-\infty}^\tau \mathrm{e}^{\mu\sigma} f(\sigma) d\sigma \\ &x = y + \mu F_\mu(\tau) y \end{aligned}$$

so that (26) takes the form

$$(1 + \mu F_\mu) \frac{dy}{d\tau} = \mu \left\{ (1+f)\mu F_\mu y - (1 + \mu F_y)^2 y^2 \right\}. \tag{29}$$

From (29) we can draw the following conclusion. If $\mu > 0$ and $x_0 > 0$, let $x_\mu(t, x_0)$ be the solution of (26) such that $x_\mu(0, x_0) = x_0$. Let $x_* > 0$ be a positive number. Then there exists $\mu_* > 0$ such that, if $0 < \mu \leq \mu_*$ and if $x_0 \geq x_*$, then there is a corresponding number $\delta_* = \delta_*(x_*, \mu_*)$ with the property that $\frac{d}{dt} x_\mu(t, x_0) \leq -\delta_*$ for all t for which $x_\mu(t, x_0) \geq x_*$. See [20] for another approach to the nonautonomous transcritical bifurcation problems when fast oscillations are present.

3.3. The pitchfork bifurcation pattern

This bifurcation scenario can be studied essentially from the same point of view as the transcritical pattern. The starting point is the equation

$$\frac{dx}{dt} = x \left[\varepsilon + f\left(\frac{t}{\mu}\right) - x^2 \right] \tag{30}$$

together with the corresponding family

$$\frac{dx}{dt} = x \left[\varepsilon + f_*(\phi_{t/\mu}(p)) - x^2 \right]. \tag{30p}$$

Make the substitution $w = \frac{1}{2x^2}$ to obtain

$$\frac{dw}{dt} + 2(\varepsilon + f)w = 1\,,$$

then set $w = \cot(\theta) = \frac{u}{v}$ where

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\varepsilon - f & 1 \\ 0 & \varepsilon + f \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{31}$$

It is clear that we can study the rapidly oscillating bifurcation pattern using the arguments applied above. We summarize the conclusions that can be drawn. Set $\varepsilon = \varepsilon_c(\mu) = 0$ for each $\mu > 0$; this defines the critical curve. If $\varepsilon < \varepsilon_c(\mu)$, then all solutions of (30) tend to zero exponentially fast as $t \to \infty$. On the other hand, if $\varepsilon > \varepsilon_c(\mu)$, then the family

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -(\varepsilon + f_*\left(\phi_{t/\mu}(p)\right)) & 1 \\ 0 & \varepsilon + f_*\left(\phi_{t/\mu}(p)\right) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{31p}$$

admits an exponential dichotomy over P. The unstable bundle B_ε can be parametrized by a continuous map $\Theta : P \to \left(0, \frac{\pi}{2}\right)$ in the sense that

$$B_\varepsilon = \left\{\left(P, \begin{pmatrix} u \\ v \end{pmatrix}\right) \in P \times \mathbb{R}^2 \middle| \begin{pmatrix} u \\ v \end{pmatrix} = \left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\| \begin{pmatrix} \cos(\Theta(p)) \\ \sin(\Theta(p)) \end{pmatrix}\right\}.$$

Write $w(p) = \cot(\Theta(p))$, so that $w(p) > 0$ for all $p \in P$. Then the functions

$$X^{\pm}(p) = \frac{\pm 1}{\sqrt{2w(p)}}$$

give rise to solutions $x^{\pm}(t) = X^{\pm}\left(\phi_{t/\mu}(p)\right)$ of equation (30p) ($p \in P$). These solutions are exponentially asymptotically stable. We conclude that there is a clean analogue of the classical pitchfork bifurcation pattern as ε increases through zero. The behavior of solutions of the family (31p) when $\varepsilon = 0$ can be studied as was done in the transcritical case; we omit the details.

3.4. The Andronov-Hopf pattern

We first consider a Van der Pol oscillator which exhibits an AH (Andronov-Hopf) bifurcation when a parameter ε increases through zero. We will subject the parameter to rapid oscillations and analyze "what happens". Then we will make some remarks concerning equations with rapidly oscillating coefficients for which an AH-bifurcation takes place in the averaged equation. We will take account of the (few) general results known to us concerning the "nonautonomous Hopf bifurcation". For an introduction to the AH-bifurcation theory see [16].

Consider the equation

$$\frac{d^2x}{dt^2} - \left(\varepsilon + f\,(t/\mu)) - x^2\right)\frac{dx}{dt} + x = 0\,. \tag{32}$$

Compare this equation with (one version of) the Van der Pol oscillator:

$$\frac{d^2x}{dt^2} - \left(\varepsilon - x^2\right)\frac{dx}{dt} + x = 0\,. \tag{33}$$

We see that indeed the bifurcation parameter ε is subjected to fast oscillations. Let us note parenthetically that one sometimes refers to a different equation as that of Van der Pol, namely

$$\frac{d^2x}{dt^2} - \varepsilon(1-x^2)\frac{dx}{dt} + x = 0\,; \tag{34}$$

this equation does not admit an AH-bifurcation in $\varepsilon = 0$ because the origin $(x, x') = (0, 0)$ in the phase plane is a center. Incidentally in [3] the parameter ε in (34) is subjected to fast oscillations. We will only discuss the version (33) of the Van der Pol equation, or rather its perturbed form (32).

It is convenient to write equation (32) in phase coordinates $x_1 = x$, $x_2 = \frac{dx}{dt}$:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \left[\varepsilon + f\left(\frac{t}{\mu}\right)\right]x_2 - x_1^2x_2 - x_1 \end{pmatrix} \tag{35}$$

Let us write $\tau = t/\mu$ and apply the method of [3] to this equation. Consider the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu f(\tau)x_2 \end{pmatrix}. \tag{36}$$

Assume that $f(\tau)$ admits a bounded primitive $F(\tau)$ which has mean value zero: $\overline{F} = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau f(\sigma)d\sigma = 0$. The general solution of (36) is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \mathrm{e}^{\mu F(\tau)} \end{pmatrix} = h(\tau, c)$$

where $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. The substitution $x = h(\tau, y)$ takes (35) to the form

$$\frac{d}{d\tau}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mu\begin{pmatrix} y_2\mathrm{e}^{\mu F(\tau)} \\ (\varepsilon - y_1^2)y_2 - y_1\mathrm{e}^{-\mu F(\tau)} \end{pmatrix}. \tag{37}$$

Next write

$$\begin{aligned} a(\tau) &= \mathrm{e}^{\mu F(\tau)}\,, \quad \overline{a} = \lim_{\tau\to\infty}\tfrac{1}{\tau}\textstyle\int_0^\tau a(\sigma)d\sigma \\ b(\tau) &= \mathrm{e}^{-\mu F(\tau)}\,, \quad \overline{b} = \lim_{\tau\to\infty}\tfrac{1}{\tau}\textstyle\int_0^\tau b(\sigma)d\sigma \end{aligned}$$

Note that, if f is not identically zero, then $\overline{a} > 1$ and $\overline{b} > 1$ by Jensen's inequality. We make the further assumption that $a(\tau) - \overline{a}$ and $b(\tau) - \overline{b}$ admit bounded primitives $A(\tau)$ and $B(\tau)$. Write $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, then introduce the averaging transformation

$$y = z + \mu G(\tau, z) \qquad\qquad G(\tau, z) = \begin{pmatrix} A(\tau) z_2 \\ -B(\tau) z_1 \end{pmatrix};$$

equation (37) takes the form

$$\begin{aligned}\left[\mathbb{I} + \mu \begin{pmatrix} 0 & A(\tau) \\ -B(\tau) & 0 \end{pmatrix}\right] \frac{dz}{d\tau} = \mu \Bigg\{ & \begin{pmatrix} \overline{a} z_2 \\ -\overline{b} z_1 + (\varepsilon - z_1^2) z_2 \end{pmatrix} \\ & -\mu \begin{pmatrix} aBz_1 \\ bAz_2 + 2Az_1 z_2^2 + (\varepsilon - z_1^2) B z_1 \end{pmatrix} + o(\mu) \Bigg\}.\end{aligned} \tag{38}$$

A brief analysis of equation (38) leads to the following conclusions. First of all, the averaged system

$$\frac{dz}{d\tau} = \mu \begin{pmatrix} \overline{a} z_2 \\ (\varepsilon - z_1^2) z_2 - \overline{b} z_1 \end{pmatrix} \tag{39}$$

exhibits an AH-bifurcation in $\varepsilon = 0$. However, the rate of rotation along the AH-limit cycle is (to zeroeth order in ε) increased by a factor of $(\overline{a}\overline{b})^{1/2}$. This factor tends to 1 as $\mu \to 0^+$. That being said, it is worth considering what happens if the term $f\left(\frac{t}{\mu}\right)$ in (32) is replaced by $\frac{\alpha}{\mu} f\left(\frac{t}{\mu}\right)$ as in [2]. In that case the rate of rotation along the limit cycle in (39) becomes $(\overline{a}\overline{b})^{1/2}$, where now $\overline{a} = \overline{e^{\alpha F(\tau)}}$ and $\overline{b} = \overline{e^{-\alpha F(\tau)}}$. This quantity is larger than 1 (if $f \not\equiv 0$) and is μ-independent.

Second, the behavior of the solutions of the original equation (32) is of course not determined solely by those of equation (39), but is influenced also by the τ-dependent terms in (38). These terms are of order $O(\mu)$ as $\mu \to 0^+$. So one cannot expect to construct an analogue of the AH-theory for fixed $\mu > 0$, with bifurcation parameter ε: if ε is near zero, the $O(\mu)$-terms might wash away the structure needed to obtain any sort of "nonautonomous version of the limit cycle". On the other hand, if $\mu = O(\varepsilon^s)$ where $s > 1$, then one suspects that some analogue of the AH-pattern will be present for ε near zero. This is indeed the case, as we now explain in a more general context.

Consider a differential system

$$\frac{dx}{dt} = f\Big(\frac{t}{\mu}, x, \varepsilon\Big) \qquad x \in \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \tag{40}$$

where $\varepsilon \in \mathbb{R}$ and $\mu > 0$ are independent parameters, and $f(\cdot, 0, \cdot) = 0$. We assume that a Bebutov construction can be carried out on f, with the following results.

There exist a strictly ergodic flow $(P, \{\phi_\tau\})$ with unique ergodic measure ξ, a continuous function $f_* : P \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, and a point $p_* \in P$ such that $f_*(\phi_\tau(p_*), x, \varepsilon) = f(\tau, x, \varepsilon)$ for all $\tau \in \mathbb{R}$, $x \in \mathbb{R}^2$, $\varepsilon \in \mathbb{R}$. We further assume that f_* is a C^∞ function of (x, ε), uniformly in $p \in P$; that is, for each triple $(\alpha_1, \alpha_2, \alpha_3)$ of nonnegative integers the derivative

$$\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \frac{\partial^{\alpha_3}}{\partial x_3^{\alpha_3}} f_*$$

exists and is continuous on $P \times \mathbb{R}^2 \times \mathbb{R}$.

Set $\tau = t/\mu$, then introduce the family of differential systems

$$\frac{dx}{d\tau} = \mu f_*(\phi_\tau(p), x, \varepsilon)\,. \qquad (40p)$$

Write $\overline{f}(x, \varepsilon) = \int_P f_*(p, x, \varepsilon) d\xi(p)$. We assume that there exists a continuous function $F_* : P \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ such that for each $\tau \in \mathbb{R}$, $p \in P$, $x \in \mathbb{R}^2$ one has

$$F_*(\phi_\tau(p), x, \varepsilon) - F_*(p, x, \varepsilon) = \int_0^\tau \left[f_*(\phi_\sigma(p), x, \varepsilon) - \overline{f}(x, \varepsilon) \right] d\sigma\,.$$

Thus F_* is a "bounded continuous primitive of f_*". Finally we assume that F_* is C^∞ in (x, ε), uniformly in $p \in P$. When this conditions are fulfilled, the averaging transformation

$$x = y + \mu F_*(\phi_\tau(p), y, \varepsilon)$$

takes (40p) to the form

$$\frac{dy}{d\tau} = \mu \left\{ \overline{f}(y, \varepsilon) + \mu g_*(\phi_\tau(p), y, \varepsilon, \mu) \right\} \qquad (41p)$$

for a function g_* which is C^∞ in (x, ε, μ), uniformly in $p \in P$. Here in the first moment μ must be restricted to a neighborhood of zero, but we assume that g_* has been extended to all of $P \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$, so as to be C^∞ in (y, ε, μ) uniformly in $p \in P$.

Next we observe that, except for the small coefficient μ in front of the parenthesis $\{\cdot\}$ in (41p), equations (41p) have the form of the family of equations studied in [12]. We digress to recall some facts stated there. Consider the equations

$$\frac{dx}{dt} = f(x, \varepsilon) + \mu g(\phi_t(p), x, \varepsilon, \mu) \qquad (42p)$$

where f and g are C^∞ functions of all arguments, uniformly in $p \in P$. Suppose that $f(0,\varepsilon) = g(p,0,\varepsilon,\mu) = 0$ identically, and that

$$\frac{dx}{dt} = f(x,\varepsilon) \tag{43}$$

exhibits a supercritical AH bifurcation in $\varepsilon = 0$. More precisely it is required that $D_x f(0,\varepsilon)$ admits complex conjugate eigenvalues $\alpha(\varepsilon) \pm i\beta(\varepsilon)$ for ε near zero, where $\alpha(0) = 0$, $\alpha'(0) > 0$ and $\beta(0) > 0$. It is further required that the second Lyapunov coefficient is positive [24], or equivalently that the vague attractor condition be valid in $\varepsilon = 0$ [26].

One of the results of [12] can be stated as follows.

Theorem 3.2. *Suppose that $\mu = O(\varepsilon^s)$ for some $s > 1$. Then for sufficiently small ε, the family* (42p) *admits an integral manifold $M_\varepsilon \subset P \times \mathbb{R}^2$. If $\mu = 0$ this integral manifold reduces to $P \times C_\varepsilon$ where $C_\varepsilon \subset \mathbb{R}^2$ is the support of the AH-periodic orbit of system* (43)*. The flow on M_ε determined by the solutions of the family* (42p) *is isomorphic to a flow in a circle extension of P.*

For the precise meaning of the term "integral manifold" in this context see [12] and [37]. To say that M_ε is a circle extension of P amounts to saying that M_ε is homemorphic to $P \times \mathbb{S}^1$ where $\mathbb{S}^1$ is the circle. The flow on M_ε can be studied using results and ideas of [17], [6] and other papers. Actually the flow on M_ε turns to be a so-called suspension flow, which allows a substantial analysis to be made. In particular the classical Furstenberg flows [14] play an important role in this "nonautonomous Andronov-Hopf" bifurcation scenario. See [12] for details.

Let us now return to the family (41p), which we rewrite so as to emphasize the presence of the rapid oscillations:

$$\frac{dy}{d\tau} = \overline{f}(y,\varepsilon) + \mu g_*(\phi_{t/\mu}(p), y, \varepsilon, \mu)\,. \tag{43p}$$

It turns out that the integral manifold result valid for equations (42p) can be proved for the family (43p) as well; see the forthcoming paper [13] for a discussion of this point. We conclude that, at least if $\mu = O(\varepsilon^s)$ and $s > 1$, then a nonautonomous AH-bifurcation occurs in the family (41p) in the sense that the periodic orbit of $\frac{dy}{dy} = \overline{f}(y,\varepsilon)$ perturbs to a circle extension of P. It is not clear how to generalize the statement to the case $s = 1$.

A. Remarks on Atkinson problems

We state some basic facts concerning nonautonomous linear differential systems. Consider the equations

$$\frac{dx}{dt} = M(t)x \qquad\qquad x \in \mathbb{R}^d \tag{44}$$

where $M(\cdot)$ is a bounded uniformly continuous function defined on $\mathbb{R}$. We view $M(\cdot)$ as a point in the space $\mathcal{C} = C(\mathbb{R}, \mathcal{M}_n)$ of continuous maps from the reals into the set $\mathcal{M}_n$ of $n \times n$ real matrices. Give $\mathcal{C}$ the compact-open topology, and let $\phi_t(c) = c(t + \cdot)$ ($t \in \mathbb{R}$, $c \in \mathcal{C}$). Then $\{\phi_t \mid t \in \mathbb{R}\}$ is the translation flow on $\mathcal{C}$. Define $P = \mathrm{cls}\{\phi_t(M) \mid t \in \mathbb{R}\}$ and set $p_* = M \in P$, $M_*(p) = p(0)$. Then P is compact, $(P, \{\phi_t\})$ is a flow, and (44) is the equation corresponding to p_* of the family of linear differential systems

$$\frac{dx}{dt} = M_*(\phi_t(p))x \tag{44p}$$

Definition A.1. *We say that the family of equations* (44p) *admits an exponential dichotomy over P if there is a continuous projection-valued function $Q^2 = Q : P \to \mathcal{M}_n$ such that*

$$\begin{array}{rcll} |\Phi_p(t)Q(p)\Phi_p(s)^{-1}| & \leq & ke^{-\beta(t-s)} & t \geq s \\ |\Phi_p(t)(I - Q(p))\Phi_p(s)^{-1}| & \leq & ke^{\beta(t-s)} & t \leq s \end{array}$$

for positive constants k, β. Here $\Phi_p(t)$ is the $n \times n$ matrix solution of (44p) *such that $\phi_p(0)$ is the $n \times n$ identity matrix; $\Phi_p(t)$ is the fundamental matrix solution of equation* (44p).

If equations (44p) have an exponential dichotomy over P, then the stable and unstable bundles $\mathcal{B}_+$ and $\mathcal{B}_-$ are defined as follows:

$$\begin{array}{rcl} \mathcal{B}_+ & = & \{(p, x) \in P \times \mathbb{R}^d \mid x \in \mathrm{Im}\ Q(p)\} \\ \mathcal{B}_- & = & \{(p, x) \in P \times \mathbb{R}^d \mid x \in \mathrm{Ker}\ Q(p)\}\,. \end{array}$$

These vector bundles over the base space P are obviously invariant with respect to the linear skew-product flow $\{\psi_t \mid t \in \mathbb{R}\}$ on $P \times \mathbb{R}^d$ defined by $\psi_t(p, x) = (\phi_t(x), \Phi_p(t)x)$.

Next set $d = 2$. Let ξ be a $\{\phi_t\}$-ergodic measure on P. We define the *ξ-rotation number* of the family (44p). Introduce polar coordinates r, θ in the x-plane. For each $p \in P$, equation (44p) induces a differential equation for θ which does not depend on r:

$$\frac{d\theta}{dt} = g(\phi_t(p), \theta)\,. \tag{45p}$$

If $p \in P$ and $\theta \in \mathbb{R}$, then the solution $\theta(t)$ of (45p) is

$$\theta(t) = \theta_0 + \int_0^t g(\phi_s(p), \theta(s))ds\,.$$

Observe that $\frac{\theta(t)}{t} = \frac{1}{t}\int_0^t g(\phi_s(p), \theta(s))ds + o(1)$ as $|t| \to \infty$, so it is natural to compare the "average rotation" $\frac{\theta(t)}{t}$ with the time-averages of g, for various values of (p,θ_0). There is no a priori guarantee that these time-averages exist. However, using the Birkhoff ergodic theorem (Theorem 2.1), the following result can be proved.

PROPOSITION A.2. *There is a Borel set* $P_0 \subset P$ *of* ξ*-measure* 1*, such that if* $p_0 \in P$ *and* $\theta_0 \in \mathbb{R}$*, then* $\lim_{|t|\to\infty} \frac{\theta(t)}{t}$ *exists. The limit* $\alpha = \alpha_\xi$ *does not depend on the choice of* $p_0 \in P_0$ *and* $\theta_0 \in \mathbb{R}$*. If* ξ *is the only* $\{\phi_t\}$*-ergodic measure on* P*, then the limit exists for all* $(p_0,\theta_0) \in P \times \mathbb{R}$ *and does not depend on* (p_0,θ_0)*. In fact the limit is uniform in* $(p_0,\theta_0) \in P \times \mathbb{R}$*.*

For obvious reasons, the number α_ξ is called the ξ-rotation number of the family (44p).

One can apply the concepts of exponential dichotomy and rotation number to the study of Atkinson-type spectral problems. We briefly discuss this matter; the notation used below is suggested by the application of Remark 3.1. Again let P be a compact metric space with flow $\{\phi_t\}$. Let $a_*, b_*, f_* : P \to \mathbb{R}$ be continuous functions. Also let $\Gamma_* : P \to \mathcal{M}_2$ be a continuous function whose values are positive semi-definite matrices. Finally set $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Consider the family of differential systems:

$$J\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \left[\begin{pmatrix} f_* & b_* \\ b_* & a_* \end{pmatrix}(\phi_t(p)) - \varepsilon\Gamma_*(\phi_t(p))\right]\begin{pmatrix} u \\ v \end{pmatrix} \tag{46p}$$

where $-\varepsilon$ is to be viewed as a spectral parameter.

DEFINITION A.3. *Let* $\Phi_p(t)$ *be the fundamental matrix solution of* (46p) *when* $\varepsilon = 0$ *(*$p \in P$*). We say that the family* (46p) *satisfies the Atkinson condition if for each nonzero vector* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathbb{R}^2$ *and for each* $p \in P$ *the following condition holds:*

$$\int_{-\infty}^{\infty} \left\| \Gamma_*(\phi_t(p)) \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\|^2 dt > 0\,. \tag{45}$$

Here of course $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ *is the solution of* (46p) *with initial value* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$*.*

The Atkinson condition (45) means that the positive semidefinite matrix Γ_* "sees" each nonzero solution of (46p), for each $p \in P$. Note that, if $\Gamma_* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then equations 46p take the form (25) (with $\mu = 1$). It is easy to see that, if $\Gamma_* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and if $a_* > 0$ on P, then the Atkinson condition (45) holds for the family (46p).

As before, let ξ be a $\{\phi_t\}$-ergodic measure on P. The ξ-rotation number of the family (46p) is a function $\alpha = \alpha(\varepsilon)$ of ε. For a more general version of the following result (for linear systems of $2d$-dimensional Hamiltonian ODEs) see [8].

THEOREM A.4. *Suppose that the topological support of ξ equals P. Suppose that the family of differential systems* (46p) *satisfies the Atkinson condition* (45). *Then the function $\varepsilon \to \alpha(\varepsilon) : \mathbb{R} \to \mathbb{R}$ is continuous and monotone increasing. The family* (46p) *admits an exponential dichotomy over P at $\varepsilon = \varepsilon_0$ if and only if ε_0 is an element of an open interval $I \subset \mathbb{R}$ such that $\alpha(\varepsilon)$ is constant on I.*

We can apply Theorem A.4 to the situation discussed in Remark 3.1 by setting $\Gamma_* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and by assuming that $a_* > 0$ on P. It turns out that, in this case, $\alpha(\varepsilon) = 0$ for all sufficiently large ε. Since $\alpha(\varepsilon)$ is continuous and monotone nonincreasing, it is natural to define $\varepsilon_c = \min\{\varepsilon \in \mathbb{R} \mid \alpha(\varepsilon) = 0\}$. Then Theorem A.4 states that, if $\varepsilon > \varepsilon_c$, then the family (46p) admits an exponential dichotomy over P. If $\varepsilon > \varepsilon_c$ and $p \in P$, let Q_p be the dichotomy projection of (46p). It turns out that the strict positivity of a_* implies that neither the image nor the kernel of Q_p can contain a vertical vector $\begin{pmatrix} 0 \\ v \end{pmatrix} \in \mathbb{R}^2$. So, reasoning as in Section 3, we can conclude that, if $\varepsilon > \varepsilon_c$, then the equation (24) admits two bounded solutions, one of which is attracting and the other is repelling.

B. Fink averaging: an example

In Section 2 we stated that the function $\zeta(\mu)$ of equation (10) cannot, in general, be chosen to be of order $O(\mu^s)$ for any $s > 0$. We will give an example to illustrate this point. We will construct a quasi-periodic function $f(t)$ which has mean value zero, and which has the following properties:

- $\mu \int_{-\infty}^{0} \mathrm{e}^{\mu t} f(t) dt$ is not $O(\mu^s)$ as $\mu \to 0^+$ if $s > 0$;
- $f'(t)$ is a quasi-periodic function.

This means that, in the quasi-periodic averaging theory outlined in Section 2, the time-dependent quantity $\mu F_\mu(t, y)$ of (14) and (15p) cannot be made to be $O(\mu^s)$ for any $s > 0$. In this sense the $o(1)$-estimate on μF_μ cannot be improved.

To begin the construction, let $P = \mathbb{R}^2/\mathbb{Z}^2$ be the standard 2-torus with angular coordinates θ_1, θ_2 mod 1. Introduce the Kronecker flow

$$\phi_t(\theta_1, \theta_2) = (\theta_1 + t, \theta_2 + \omega t) \mod 1$$

where $\omega \in (0, 1)$ is an irrational number with the following property: there are sequences $\{m_k \mid k \geq 1\}$ and $\{n_k \mid k \geq 1\}$ of integers such that $n_k \to \infty$ and

$$|m_k + n_k \omega| \leq \mathrm{e}^{-n_k} \quad (k = 1, 2, \ldots) . \tag{46}$$

If $(m,n) \in \mathbb{Z}^2$ write

$$\varepsilon_{mn}(\theta_1,\theta_2) = \mathrm{e}^{2\pi i m\theta_1}\mathrm{e}^{2\pi i n\theta_2}\,,$$

and set

$$f_*(\theta_1,\theta_2) = \sum_{(m,n)\neq(0,0)} f_{m,n}\varepsilon_{mn}(\theta_1,\theta_2) \tag{47}$$

where the coefficients $f_{m,n}$ satisfy $f_{-m,-n} = \overline{f_{m,n}}$ for all integer values of m, n. Here and below the overbar indicates the complex conjugate. The coefficients $f_{m,n}$ will be chosen so that the right-hand side of (47) converges uniformly on P. Let ξ be the normalized Haar measure on P, and note that $\int_P f_* d\xi = 0$.

Next choose positive real numbers β_k such that $\sum_{k=1}^{\infty} \beta_k^{-1} < \infty$. Set

$$\mu_k = \beta_k^{-\beta_k} \quad (k = 1,2,\dots)\,,$$

then choose integers m_k, n_k such that

$$|m_k + n_k\omega| \leq \mu_k^{\beta_k} = \beta_k^{-\beta_k^2} \quad (k = 1,2,\dots)\,.$$

This can be done by using (46) to choose n_k such that $n_k > \beta_k^2 \ln(\beta_k)$. Set

$$\begin{aligned} f_{m_k,n_k} = \beta_k^{-1} = f_{-m_k,-n_k} & \qquad (k = 1,2,\dots) \\ f_{m,n} = 0 & \qquad \text{other values of } (m,n) \in \mathbb{Z}^2\,. \end{aligned}$$

Finally define $f(t) = f_*(t,\omega t)$ so that $f(t)$ is obtained by evaluating f_* along the orbit through $p_* = (0,0) \in P$.

Observe that, for each $\mu > 0$,

$$\int_{-\infty}^{0} \mathrm{e}^{\mu s} f(s)ds = \sum_{k=1}^{\infty} f_{m_k,n_k}\left[\frac{1}{\mu + 2\pi i(m_k + n_k\omega)} + \frac{1}{\mu - 2\pi i(m_k + n_k\omega)}\right]$$

since f_{m_k,n_k} is real. For each $\ell = 1,2,\dots$ choose $\mu = \mu_\ell$ and note that

$$\mu_\ell \int_{-\infty}^{0} \mathrm{e}^{\mu_\ell s} f(s)ds = \sum_{k=1}^{\infty} f_{m_k,n_k} \frac{2|\mu_\ell|^2}{|\mu_\ell|^2 + |2\pi(m_k + n_k\omega)|^2}\,.$$

If $k = \ell$, then the corresponding term of the series is real, positive, and greater than

$$f_{m_\ell,n_\ell} = \beta_\ell^{-1} = \mu_\ell^{s_\ell}$$

where $s_\ell = \frac{1}{\beta_\ell} \to 0$ as $\ell \to \infty$. We used the fact that $m_\ell + n_\ell\omega$ is small compared to μ_ℓ for $\ell = 1,2,\dots$. If $k \neq \ell$, then the corresponding term of the series is positive. So we can conclude that $\mu \int_{-\infty}^{0} \mathrm{e}^{\mu s} f(s)ds$ cannot be $O(\mu^s)$ as $\mu \to 0^+$, for any number $s > 0$.

Concerning the derivative of $f(t)$, we note that

$$f'(t) = -4\pi \sum_{k=1}^{\infty} (m_k + n_k\omega) f_{m_k,n_k} \sin[2\pi(m_k + n_k\omega)t]\,.$$

Since $|m_k + n_k\omega| < \mathrm{e}^{-(|m_k|+|n_k|)/2}$, we see that $f'(t)$ extends to an analytic function, call it $f'_*(\theta_1, \theta_2)$, on the torus P. So $f'(t)$ is certainly a quasi-periodic function.

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