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The semilinear wave equation with non-monotone nonlinearity: a review

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Dedicated to Professor Jean Mawhin on his 75th birthday

Abstract. *We review recent results on the existence of weak 2π-periodic solutions in time and space for a class of semilinear wave equations with non-monotone nonlinearity. Similar results exist for Dirichlet-periodic boundary conditions but, for the sake of clarity, we exclude them in this presentation.*

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1. Introduction

In the study of the range of semilinear operators $L+N$, finding weak solutions to the wave equation

$$\Box u + g(u) := u_{tt} - u_{xx} + g(u) = f(x,t) \tag{1}$$

subject to double-periodic conditions

$$u(x,t) = u(x,t+2\pi) = u(x+2\pi,t) \quad \text{for all } x,t \in \mathbb{R}, \tag{2}$$

provides a rich source of open questions. Up to minor modifications, the results here reviewed extend to (1) subject to the Dirichlet-periodic condition

$$u(0,t) = u(\pi,t) = 0, \quad u(x,t) = u(x,t+2\pi) \quad \text{for all } x \in (0,\pi), t \in \mathbb{R}. \tag{3}$$

Professor Jean Mawhin is a pioneer in this field. His work points out the role of the interaction of the numerical range of N (i. e., the range of g') with the spectrum of L (i.e., the spectrum of $-\Box$, subject to either (2) or (3)), in the solvability of these problems. Such spectra are given by $\sigma(-\Box) = \{j^2 - k^2; j,k = 0,1,\ldots\}$ for condition (2) and by $\sigma_d(-\Box) = \{j^2 - k^2; j = 0,1,\ldots; k = 1,2,\ldots\}$ for condition (3).

For example, from [22] it follows that if g is monotone and

$$\left[\liminf_{|u|\to\infty}\frac{g(u)}{u}\ ,\ \limsup_{|u|\to\infty}\frac{g(u)}{u}\right]\cap\sigma(-\Box)=\emptyset, \tag{4}$$

then (1)–(2), as well as (1)–(3), has a solution. The same result may be obtained from related developments in [4, 7, 8, 25, 27, 29, 30]. Arguing as in Theorem 3 of [14] one sees that (4) may be extended to

$$\liminf_{|u|\to\infty}\frac{g(u)}{u}\in(\lambda_k,\lambda_{k+1}),\quad\text{and}\quad\limsup_{|u|\to\infty}\frac{g(u)}{u}<\nu\left(\liminf_{|u|\to\infty}\frac{g(u)}{u}\right), \tag{5}$$

where $\nu(a)>a$, $a\notin\sigma(-\Box)$ is the smallest value for which $\Box u+au_+-\nu(a)u_-=0$ subject to (2) has a weak solution. That is $(a,\nu(a))$ belongs to the Fucik spectrum of $\Box$ subject to (2).

Similar results occur in systems and wave equations in several space variables, see [2, 4, 5, 24, 32, 33]. All these works assume the range of g' not to include eigenvalues of infinite multiplicity in its interior. Note that only 0 is an eigenvalue of infinite multiplicity both for (1)-(2) and (1)-(3).

When the periodicity condition (2) is replaced by

$$u(x,t)=u(x,t+2\pi)=u(x+L,t)\quad\text{for all } x,t\in\mathbb{R}, \tag{6}$$

and L is not a rational multiple of π the spectrum $\sigma(\Box)$ may have multiple eigenvalues of infinite multiplicity and may not be a discrete. Here again professor Mawhin is a pioneer in the field with his work in [23, 21]. For additional analysis of this case the reader is referred to [28]. Little is known on the solvability of (1)-(6) when L is not a rational multiple of π. In [9] existence results for cases where $\sigma(\Box)$ is discrete and all the eigenvalues have finite multiplicity are found including cases where the range of g' may include multiple eigenvalues of infinite multiplicty.

If in (1) we replace $\Box$ by an elliptic operator, N need not be monotone as compactness arguments based on the absence of eigenvalues of infinite multiplicity suffice.

From now on we let $\Omega:=(0,2\pi)\times(0,2\pi)$ and

$$\begin{aligned}\alpha_{k,j}(x,t)&=\sin(kx)\cos(jt),\quad \beta_{k,j}(x,t)=\sin(kx)\sin(jt),\\ \gamma_{k,j}(x,t)&=\cos(kx)\cos(jt),\quad\text{and}\quad \delta_{k,j}(x,t)=\cos(kx)\sin(jt).\end{aligned} \tag{7}$$

Let K be the closed subspace of $L^2(\Omega)$ spanned by

$$\{\alpha_{k,k},\ \beta_{k,k},\gamma_{k,k},\delta_{k,k}\ ;\ k=0,1,2,\ldots\}.$$

That is, K is the null space of the wave operator $\Box$ subject to (2). If $v\in K$ then there are unique 2π-periodic null-average functions v_1 and v_2 and a unique number $\bar v$ such that $v(x,t)=\bar v+v_1(t+x)+v_2(t-x)$.

We let H denote the *Sobolev* space of functions u such that u as well as its first order partial derivatives belong to $L^2(\Omega)$. The norm in $L^2(\Omega)$ is denoted by $\|\ \|$ and the norm in H by $\|\ \|_1$. We let $Y = K^\perp \cap H$. We say that $u = y + v \in Y \oplus K$ is a weak solution of (1)-(2) if

$$\int_\Omega \{(y_t\hat{y}_t - y_x\hat{y}_x) - (g(u) - f)(\hat{y} + \hat{v})\}\, dxdt = 0, \tag{8}$$

for all $\hat{y} + \hat{v} \in Y \oplus K$.

2. Existence of forced vibrations

In [20, 35] it was established that $\Box + N$ subject to (2) has dense range in $L^2(\Omega)$ when $g'(u) \cap \sigma(-\Box) = \emptyset$ for u large. That is, for all f in a dense subset of $L^2(\Omega)$, the equation (1)-(2) has a weak solution. Note that here it is not assumed g to be monotone. More precisely, if there are constants $\alpha, \beta, c \in \mathbb{R}$, $\alpha \le \beta$, such that $\sigma(\Box) \cap [\alpha, \beta] = \emptyset$, that $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, and

$$-c + \frac{\alpha}{2}s^2 \le \int_0^s g(t)\, dt \le c + \frac{\beta}{2}s^2 \quad \text{for all } s \in \mathbb{R}, \tag{9}$$

then (1)-(2) has a solution for each f in a dense set of $L^2(\Omega)$. However, to date, it is not known if such a range (the set of all such f's) is all of $L^2(\Omega)$.

The arguments in [20, 35] do not provide a characterization of the f's for which (1)-(2) has a solution. Nevertheless, in [12, 15, 16], sufficient conditions for f to be in the range of $u \mapsto \Box(u) + g(u)$ are provided when

$$g(s) = \lambda s + h(s), \quad \text{with} \quad -\lambda \notin \sigma(\Box) \ \text{ and } \lim_{|u|\to+\infty} h'(u) = 0. \tag{10}$$

It is readily verified that functions satisfying (10) satisfy (9).

In order to find sufficient conditions on f for (1)-(2), or (1)-(3), to have a solution the concept of functions *flat on characteristics* was introduced in [16].

Definition 2.1. *We say that ϕ is not flat on characteristics if given $\epsilon > 0$ there exists $\delta > 0$ such that $m(\{x \in [0, \pi]; |\phi(x, r \pm x) - \rho| < \delta\}) < \epsilon$ for all $r, \rho \in \mathbb{R}$, where m stands for the one dimensional Lebesgue measure.*

In [12, Theorem 5.1] the following was proven.

Theorem 2.2. *Let $-\lambda \notin \sigma(\Box)$ and $f(x, t) = cq(x, t) \in L^p(\Omega)$, $p \ge 2$ and ϕ the solution to $\Box(\phi) + \lambda\phi = q(x, t)$, $\phi(x, t) = \phi(x + 2\pi, t) = \phi(x, t + 2\pi)$, $x, t \in \mathbb{R}$. If ϕ is not flat on characteristics then there exist c_0 such that for $|c| \ge c_0$ the equation (1)-(2) has a weak solution $u \in L^p(\Omega)$.*

Earlier versions of Theorem 2.2 are found in [15, 16] where the existence of bounded solutions is considered. The proofs in [12, 15, 16] are based on first establishing the existence of approximate solutions and then establishing the convergence of such approximations using the compactness of $(\Box + \lambda I)^{-1}$ on the range of $\Box$ and convergence in K in L^p using that the projection on K of such approximations are large due to the size of the parameter c.

3. Non-existence of continuous solutions

If $|g'|$ is bounded away from 0 (hence g is strictly monotone) and f is smooth, in [7, 29] it is shown smoothness of f implies smoothness of solutions to (1)-(2). As credited by P. Rabinowitz in [29], the ideas for showing such regularity go back to L. Nirenberg.

On the other hand, for non-monotone nonlinearities one cannot expect regularity of the solutions as shown by the following theorem and lemma.

Theorem 3.1. *Assume that $h(s) = g(s) - \lambda s$ is a differentiable function with support in $[0, D]$ for some $D > 0$, that $\lambda > 0$, that $-\lambda \notin \sigma(\Box)$ and that $h'(D/2) < -\lambda D/2$. Then there is $c_0 > 0$ such that if $|c| > c_0$ the problem (1)-(2) has no continuous solution for $f(x,t) = c\sin(x+t)$.*

For the proof of Theorem 3.1 the reader is referred to [10, Theorem 2.1].

In contrast with Theorem 3.1 we have the following existence result.

Lemma 3.2. *Let*

$$g(t) = \begin{cases} \tau_1 t + h(t) & \text{if } t \le 0 \\ \tau_2 t + h(t) & \text{if } t > 0, \end{cases} \tag{11}$$

with $\tau_1, \tau_2 > 0$, and h continuous such that

$$\lim_{|s|\to\infty} \frac{h(s)}{s} = 0. \tag{12}$$

If $f(x,t) = p(x+t)$ or $f(x,t) = p(x-t)$, with $p : \mathbb{R} \to \mathbb{R}$, $p \in L^2[0, 2\pi]$, and $p(\xi + 2\pi) = p(\xi)$ for all $\xi \in \mathbb{R}$, then the equation (1)–(2) has a solution.

Note that the above lemma allows for resonance $(-\tau_1, -\tau_2 \in \sigma(\Box))$ and jumping nonlinearities $(\tau_1 \neq \tau_2)$. Its proof goes as follows. One lets

$$\Gamma = \{\gamma : \mathbb{R} \to \mathbb{R}; \gamma \text{ is increasing, continuous and } \gamma(t) \le g(t) \text{ for all } t \in \mathbb{R}\}.$$

and

$$g_1(t) := \sup_{\gamma \in \Gamma} \gamma(t). \tag{13}$$

The function g_1 is continuous, non-decreasing, for all $\alpha \in \mathbb{R}$ the set $g^{-1}(\alpha)$ is a closed interval, and if $g(a) < g(t)$ for all $t > a$ then $g(a) = g_1(a)$.

For each $\xi \in \mathbb{R}$ there exists $a_\xi, b_\xi \in \mathbb{R}$ such that $g_1^{-1}(\{\xi)\}) = [a_\xi, b_\xi]$. Given $f(x,t) = p(x+t)$, we define $v(s) := b_{p(s)}$. Due to $\tau_1 > 0$, $\tau_2 > 0$, and (12), $v \in L^2(0, 2\pi)$. Also $g(v(\xi)) = p(\xi)$. Thus $u(x,t) = v(x+t) \in K$ and is a weak solution to (1)-(2). These solutions may have jump discontinuities along characteristic lines where g_1^{-1} is not single valued. Furthermore, such solutions need not be unique. For example, if $p(s) = \xi$ is constant in a segment $[c,d]$, and $a_\xi < b_\xi$ then defining, for any $y \in (c,d)$, $v_y(\zeta) = a_{p(\zeta)}$ for $\zeta \in [c,y)$, $v_y(\zeta) = b_{p(\zeta)}$ for $\zeta \in (y,d]$, and $u_y(x,t) = v_y(x+t)$ we have a continuum of solutions to (1)-(2).

4. Bifurcation

Finally we consider, subject to the periodicity condition (2), the one parameter equation

$$u_{tt} - u_{xx} + g(x,t,u,\lambda) = 0, \ \ x,t,u,\lambda \in R. \tag{14}$$

with $g(x,t,u) = g(x+2\pi,t) = g(x,t+2\pi)$. If $g(x,t,u,\lambda) = \lambda G(x,t,u)$, $G(x,t,u) = 0$, and $G_u(x,t,0) = 1$ one sees that $(0,\lambda_k)$ is a point of bifurcation for every $\lambda_k \in \sigma(-\Box)$. More precisely, there is a connected set of nonzero solutions to (14)-(2) containing $(0,\lambda_k)$ in its closure. This fact is proven imitating the arguments for the case in which $\Box$ is replaced by a second elliptic operator when $\lambda_k \neq 0$, and a more detailed analysis for $\lambda_k = 0$ as shown in [31].

Bifurcation from infinity. Recently, bifurcation from infinity was considered in [13] resulting in the following theorem.

THEOREM 4.1. *Let* $-\lambda_0 \in \sigma(\Box)$, $h : \mathbb{R} \to \mathbb{R}$ *a bounded continuous function. Suppose there exists* $M > 0$, $\gamma > 1$, *and* $A > 0$ *such that*

$$|h'(s)| \leq |s|^{-\gamma} \text{ for all } |s| \geq M, \text{ and } \lim_{s \to \pm\infty} h(s) = \pm A. \tag{15}$$

If $g(s) = \lambda s + h(s)$, *then there is* ϵ_0 *such that if* $0 < \lambda_0 - \lambda < \epsilon_0$ *the problem* (1)-(2) *has a nontrivial weak solution* $u_\lambda = v_\lambda + y_\lambda \in (K \oplus Y) \cap L^\infty(\Omega)$. *Furthermore, if* $\lambda \to \lambda_0$, *then* $\|v_\lambda\| + \|y_\lambda\|_1 \to \infty$.

For $\lambda_0 \neq 0$ the proof of Theorem 4.1 relies on the properties of sets of the form $\{(x,t); |p(x,t)| < \epsilon\}$, for p a trigonometric polynomial of a given degree, using the Nazarov-Turan lemma, see [19]. The case $\lambda_0 = 0$, relies on the fact that constant functions belongs to the kernel K. This case does not extend to the boundary condition (3) due to the absence of constant functions in the kernel of $\Box$ subject to this boundary condition.

Imperfect bifurcation. In [11], see also [6], the equation (14) for

$$g(x,t,u,\lambda) = \lambda(u+\lambda H)^{2k} + \lambda R(t,x,u+\lambda H) \tag{16}$$

subject to (2) and assuming that

$$\lim_{v\to 0} \frac{R_v(t,x,v)}{v^{2k-1}} = 0, \quad \text{and} \quad k \quad \text{a positive integer} \tag{17}$$

is considered. Sufficient conditions on $H \neq 0$ are provided for the existence of solutions that accumulate at $(0,0)$. Since $H \neq 0$ this is known as imperfect bifurcation.

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