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On the singular 1-dimensional planar sheaves supported on quartics

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Abstract. *In the case of the fine Simpson moduli spaces of* 1-*dimensional planar sheaves supported on quartics, the subvariety of sheaves that are not locally free on their support is connected, singular, and has codimension* 2.

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1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic zero, let V be a vector space over $\Bbbk$ of dimension 3, and let $\mathbb{P}_2 = \mathbb{P}V$ be the corresponding projective plane. Fix a linear polynomial $P(m) = dm + c$ with integer coprime coefficients and consider the Simpson [13] moduli space $M := M_P(X)$ of stable sheaves on X with Hilbert polynomial P. As shown in [9], M is a fine moduli space, it is a smooth irreducible projective variety of dimension $d^2 + 1$. A generic sheaf in M is a line bundle on its Fitting support, which is a planar projective curve of degree d.

Singular sheaves

In general M contains a closed subvariety M' of sheaves that are not locally free on their support. Since M is irreducible, the complement M_B of M' is an open dense subset whose points are sheaves that are locally free on their support. So, one could consider M as a compactification of M_B. We call the sheaves from the boundary $M' = M \setminus M_B$ *singular*. As one can see on the following examples for $d \leqslant 3$, the boundary M' does not have the minimal codimension in general.

First examples

Notice that twisting with $\mathcal{O}_{\mathbb{P}_2}(1)$ gives the isomorphism of the moduli spaces $M_{dm+c}(\mathbb{P}_2) \cong M_{dm+c+d}(\mathbb{P}_2)$. Moreover, by the duality result from [12], there is the isomorphism $M_{dm+c}(\mathbb{P}_2) \cong M_{dm-c}(\mathbb{P}_2)$ given by $\mathcal{F} \mapsto \mathscr{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}_2})$. As shown in [14], two moduli spaces $M_{dm+c}(\mathbb{P}_2)$ and $M_{d'm+c'}(\mathbb{P}_2)$ are isomorphic if and only if f $d = d'$ and $c = \pm c' \mod d$. Therefore, for fixed d, it is enough to understand $d/2 + 1$ different moduli spaces.

For $d = 1$, M_{m+c} is a fine moduli space that consists of twisted structure sheaves $\mathcal{O}_L(c-1)$ of lines L in $\mathbb{P}_2$. Therefore, each M_{m+c} is just the dual projective plane $\mathbb{P}_2^* = \mathbb{P}V^*$. In this case there are no singular sheaves.

For $d = 2$ and $c = 2\beta + 1$, M_{2m+c} is a fine moduli space whose points are the isomorphism classes of twisted structure sheaves $\mathcal{O}_C(\beta)$ of planar conics $C \subseteq \mathbb{P}_2$. In this case M_{2m+c} is isomorphic to the space of conics $\mathbb{P}S^2V^*$. As in the previous case the subvariety M'_{2m+c} of singular sheaves is empty.

The situation changes for $d = 3$. For $c \in \mathbb{Z}$ with $\gcd(3, c) = 1$ all moduli spaces M_{3m+c} are isomorphic to the universal plane cubic curve and M'_{3m+c} is a smooth subvariety of codimension 2 isomorphic to the universal singular locus of a cubic curve. A construction that interprets in this case the blow-up of M along M' as a compactification of M_B by an irreducible divisor consisting of vector bundles of curves in certain reducible surfaces was given in [8]. Since it explicitly uses the properties of M', it seems important to understand the geometry of M' in order to perform a similar modification for other moduli spaces of planar 1-dimensional sheaves.

The main result of the paper

The cases with $d \leqslant 3$ were the only cases where the boundary M' has been completely understood. In this note we study the subvariety of singular sheaves in the case of $M = M_{4m+c}(\mathbb{P}_2)$, $\gcd(4, c) = 1$, i. e., for the fine Simpson moduli spaces, which consist entirely of stable points and parameterize the isomorphism classes of sheaves. As already mentioned above, it is enough to consider the case $c = -1$.

We describe all possible singular sheaves in M, the main result of the paper is summarized in the following:

PROPOSITION 1.1. *Let M be the Simpson moduli space of stable sheaves with Hilbert polynomial $P(m) = 4m + c$, $\gcd(4, c) = 1$. Let $M' \subseteq M$ be the subvariety of singular sheaves. Then M' is a singular (path-)connected subvariety of codimension 2.*

We use the merits of computer algebra computations: the most important computations in the paper are performed using SINGULAR [1]. At the same time we comment on the restrictedness of computer algebra methods due to the complexity of the involved algorithms.

Structure of the paper

In Section 2 we give a detailed description of the stratification from [4] of the moduli space M into an open stratum M_0 and its closed complement M_1. In Section 3 we describe the open stratum of M as an open subvariety of a projective bundle over the space of Kronecker modules $N = N(3; 2, 3)$. In Section 4 we give a characterization of singular sheaves in M_0 and study the fibres of M_0 over N, which allows us to demonstrate in Section 5 the assertions of Proposition 1.1. In Section 6 we study, for an isomorphism class $[\mathcal{E}]$ in M_0, how being singular is related with the singularities of the support of $\mathcal{E}$. The computations with Singular [1] used in the paper (the code and its output) are presented in Appendix A.

2. Description of $M_{4m-1}(\mathbb{P}_2)$

Let M be the Simpson moduli space of stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $4m - 1$. In [4] it has been shown that M can be decomposed into two strata M_1 and M_0 such that M_1 is a closed subvariety of M of codimension 2 and M_0 is its open complement.

2.1. Closed stratum.

The closed stratum M_1 is a closed subvariety of M of codimension 2 given by the condition $h^0(\mathcal{E}) \neq 0$ (more precisely $h^0(\mathcal{E}) = 1$). It consists of the isomorphism classes of sheaves with locally free resolutions

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{\left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right)} \mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{E} \to 0, \tag{1}$$

where z_1 and z_2 are linear independent linear forms on $\mathbb{P}_2$. M_1 is a geometric quotient of the variety of injective matrices $\left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right)$ as above by the non-reductive group

$$(\mathrm{Aut}(2\mathcal{O}_{\mathbb{P}_2}(-3)) \times \mathrm{Aut}(\mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2}))/\mathbb{k}^*$$

(cf. [5]). M_1 is isomorphic to the universal quartic plane curve

$$\{(p, C) \mid C \subseteq \mathbb{P}_2 \text{ is a quartic plane curve, } p \in C\}.$$

The latter can be explained as follows. The sheaves with resolution (1) are exactly the non-trivial extensions

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathbb{k}_p \to 0, \tag{2}$$

where $C = C_A = Z(\det A)$ is the quartic curve defined by the determinant of A and $p = p_A = Z(z_1, z_2)$ is the point on C defined by two linear independent linear forms z_1 and z_2.

2.2. Open stratum.

The open stratum M_0 is the complement of M_1 given by the condition $h^0(\mathcal{E}) = 0$, it consists of the cokernels $\mathcal{E}_A$ of the injective morphisms

$$\mathcal{O}_{\mathbb{P}_2}(-3) \oplus 2\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} 3\mathcal{O}_{\mathbb{P}_2}(-1) \tag{3}$$

with

$$A = \begin{pmatrix} q_0 & q_1 & q_2 \\ z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{pmatrix}$$

such that the (2×2)-minors of the linear part of A are linear independent. Equivalently, the Kronecker module

$$\alpha = \begin{pmatrix} z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{pmatrix} \tag{4}$$

is stable (cf. [6, Lemma 1], [2, Proposition 15]).

2.2.1. Twisted ideals of 3 non-collinear points of C

If the maximal minors of α are coprime, then $\mathcal{E}_A \cong \mathcal{I}_Z(1)$, where $\mathcal{I}_Z$ is the ideal sheaf of the zero dimensional subscheme $Z \subseteq C$ of length 3 defined by the maximal minors of α. In this case the isomorphism class of $\mathcal{E} = \mathcal{E}_A$ is a part of the exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_C(1) \to \mathcal{O}_Z \to 0 \tag{5}$$

and is uniquely defined by Z and C.

Let M_{00} denote the open subscheme of all such sheaves in M_0.

2.2.2. Extensions

If the maximal minors of α have a linear common factor, say l, then $f = \det(A) = l \cdot h$ and $\mathcal{E}_A$ is in this case a non-split extension

$$0 \to \mathcal{O}_L(-2) \to \mathcal{E}_A \to \mathcal{O}_{C'} \to 0, \tag{6}$$

where $L = Z(l)$, $C' = Z(h)$.

For fixed L and C' the subscheme of the isomorphism classes of non-trivial extensions (6) can be identified with $\mathbb{k}^2$.

Let M_{01} denote the closed subscheme of M_0 of all such sheaves. Notice that M_{01} is locally closed in M.

2.2.3. M_0 **as a geometric quotient.**

M_0 is the geometric quotient of the variety of injective matrices as in (3) by the group

$$G' = \mathrm{Aut}(\mathcal{O}_{\mathbb{P}_2}(-3) \oplus 2\mathcal{O}_{\mathbb{P}_2}(-2)) \times \mathrm{Aut}(3\mathcal{O}_{\mathbb{P}_2}(-1)).$$

As shown in [11] M_0 can be seen as an open subvariety in the projective quotient $\mathbb{B}$ of the variety of all semistable matrices (3) by the same group.

3. Description of M_0 as an open subvariety in $\mathbb{B}$

3.1. Kronecker modules

Let $\mathbb{V}$ be the affine variety of Kronecker modules

$$2\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\Phi} 3\mathcal{O}_{\mathbb{P}_2}. \tag{7}$$

There is a natural group action of $G = (\mathrm{GL}_2(\Bbbk) \times \mathrm{GL}_3(\Bbbk))/\Bbbk^*$ on $\mathbb{V}$. Since $\gcd(2,3) = 1$, all semistable points of this action are stable and G acts freely on the open subset $\mathbb{V}^s$ of stable points. A Kronecker module (7) is stable if its maximal minors are linear independent quadratic forms. There exists a geometric quotient $N = N(3;2,3) = \mathbb{V}^s/G$, which is a smooth projective variety of dimension 6. For more details consult [7, Section 6] and [2, Section III].

The cokernel of a stable Kronecker module $\Phi \in \mathbb{V}^s$ is an ideal of a zero-dimensional scheme Z of length 3 if the maximal minors of Φ are coprime. In this case there is a locally free resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{\Phi} 3\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix}} \mathcal{O}_{\mathbb{P}_2} \to \mathcal{O}_Z \to 0 \tag{8}$$

and, moreover, Z does not lie on a line. Let $\mathbb{V}_0$ denote the open subvariety of $\Phi \in \mathbb{V}^s$ of Kronecker modules with coprime maximal minors. Let $N_0 \subseteq N$ be the corresponding open subvariety in the quotient space.

This way one obtains a morphism from $N_0 \subseteq N$ to the Hilbert scheme H of zero-dimensional subschemes of $\mathbb{P}_2$ of length 3, which sends a class of $\Phi \in \mathbb{V}^s$ to the zero scheme of its maximal minors. Since, by Hilbert-Burch theorem, every zero dimensional scheme of length 3 that does not lie on a line has a minimal resolution of type (8), this gives an isomorphism between N_0 and the open subvariety $H_0 \subseteq H$ consisting of Z that do not lie on a line.

Let $N' = N \setminus N_0$, then N' is the quotient of the variety of Kronecker modules (7) whose maximal minors have a common linear factor.

Since every matrix representing a point in N' is equivalent to a matrix $\left(\begin{smallmatrix} z_0 & 0 & z_1 \\ 0 & z_0 & z_2 \end{smallmatrix}\right)$ with linear independent linear forms z_0, z_1, z_2, one can see that N' is isomorphic to $\mathbb{P}_2^* = \mathbb{P}V^*$, the space of lines in $\mathbb{P}_2$, such that a line corresponds to the common linear factor of the minors of the corresponding Kronecker module.

The complement H' of H_0 is an irreducible hypersurface (cf. [3, p. 46], [7]). The isomorphism $H_0 \to N_0$ can be extended to the morphism $H \xrightarrow{\pi} N$ that describes H as the blowing up of N along N'. The fibre over $L \in \mathbb{P}_2^*$ consists of those $Z \in H$ lying on L, i. e., the fibre over L is $L^{[3]} \cong \mathbb{P}_3$, the Hilbert scheme of 3 points on L.

3.2. $\mathbb{B}$ as a projective bundle over N

Let us provide here the argument from the proof of [11, Proposition 7.7].

Consider two vector spaces $\mathbb{U}_1 = 2\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ and $\mathbb{U}_2 = 3\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2))$. One identifies elements of $\mathbb{V} \times \mathbb{U}_2$ with morphisms (3) by

$$(\Phi, Q) \mapsto \begin{pmatrix} Q \\ \Phi \end{pmatrix}.$$

Both $\mathbb{V} \times \mathbb{U}_1$ and $\mathbb{V} \times \mathbb{U}_2$ are trivial vector bundles over $\mathbb{V}$. Consider the morphism

$$\mathbb{V} \times \mathbb{U}_1 \xrightarrow{F} \mathbb{V} \times \mathbb{U}_2, \quad (\Phi, L) \mapsto \begin{pmatrix} L \cdot \Phi \\ \Phi \end{pmatrix}.$$

Since the matrices from $\mathbb{V}^s$ have linear independent maximal minors, F is injective over $\mathbb{V}^s$. Therefore, $\mathbb{V}^s \times \mathbb{U}_1 \xrightarrow{F} \mathbb{V}^s \times \mathbb{U}_2$ is a vector subbundle and hence the cokernel E of F is a vector bundle of rank 12 on $\mathbb{V}^s$.

The group action of $\mathrm{GL}_2(\Bbbk) \times \mathrm{GL}_3(\Bbbk)$ on $\mathbb{V}^s \times \mathbb{U}_1$ and $\mathbb{V}^s \times \mathbb{U}_2$ induces a group action of $\mathrm{GL}_2(\Bbbk) \times \mathrm{GL}_3(\Bbbk)$ on E and hence an action of $G = (\mathrm{GL}_2(\Bbbk) \times \mathrm{GL}_3(\Bbbk))/\Bbbk^*$ on the projective bundle $\mathbb{P}E$. Finally, since the stabilizer of $\Phi \in \mathbb{V}^s$ under the action of G is trivial, G acts trivially on the fibres of $\mathbb{P}E$ and thus $\mathbb{P}E$ descends to a projective $\mathbb{P}_{11}$-bundle

$$\mathbb{B} \xrightarrow{\nu} N = N(3; n-1, n) = \mathbb{V}^s/G,$$

which is exactly the geometric quotient of $\mathbb{V}^s \times \mathbb{U}_2 \setminus \operatorname{Im} F$ with respect to G' mentioned above.

3.2.1. The fibres of $\mathbb{B} \xrightarrow{\nu} N$ over N_0

A fibre over a point from N_0 can be seen as the space of plane quartics through the corresponding subscheme of 3 non-collinear points. Indeed, consider a point from N_0 given by a Kronecker module $\left(\begin{smallmatrix} z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{smallmatrix}\right)$ with coprime minors

d_0, d_1, d_2. The fibre over such a point consists of the orbits of injective matrices

$$\begin{pmatrix} q_0 & q_1 & q_2 \\ z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{pmatrix}, \quad q_0, q_1, q_2 \in S^2V^*,$$

under the group action of G'. In particular such a fibre is contained in M_{00}. If two matrices

$$\begin{pmatrix} q_0 & q_1 & q_2 \\ z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{pmatrix}, \quad \begin{pmatrix} Q_0 & Q_1 & Q_2 \\ z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{pmatrix}$$

lie in the same orbit of the group action, then their determinants are equal up to a multiplication by a non-zero constant. Vice versa, if the determinants of two such matrices are equal, $q - Q = (q_0 - Q_0, q_1 - Q_1, q_2 - Q_2)$ lies in the syzygy module of $\begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix}$, which is generated by the rows of $\begin{pmatrix} z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{pmatrix}$ by Hilbert-Burch theorem. This implies that $q - Q$ is a combination of the rows and thus the matrices lie on the same orbit.

3.2.2. M_0 and flags of subschemes on $\mathbb{P}_2$.

Let $\mathbb{B}_0$ denote the restriction of $\mathbb{B}$ to N_0. Then $\mathbb{B}_0$ coincides with M_{00} as the fibres over N_0 are contained in M_0.

Let $\mathbb{P}S^4V^* = \mathbb{P}_{14}$ be the space of plane quartics. Let

$$M \xrightarrow{\mu} \mathbb{P}S^4V^* = \mathbb{P}_{14}, \quad [\mathcal{E}] \mapsto \mathrm{Supp}(\mathcal{E}),$$

be the morphism sending an isomorphism class of sheaf $\mathcal{E}$ to its support. Then its restriction to M_0 is induced by the equivariant morphism that sends a matrix (3) defining a point in M_0 to the quartic determined by its determinant.

$\mathbb{B}_0$ is isomorphic to the image of the injective morphism

$$\mathbb{B}_0 \xrightarrow{\mu \times \nu} \mathbb{P}(S^4V^*) \times N_0 \cong \mathbb{P}(S^4V^*) \times H_0, \tag{9}$$

which coincides with the subvariety of pairs (C, Z) with $Z \subseteq C$. It is isomorphic to the open subscheme $H_0(3,4) \subseteq H(3,4)$ of the Hilbert flag-scheme of flags $Z \subseteq C \subseteq \mathbb{P}_2$ (zero-dimensional subscheme Z of length 3 on a curve $C \subseteq \mathbb{P}_2$ of degree 4) such that Z does not lie on a line.

3.2.3. The fibres of $\mathbb{B} \xrightarrow{\nu} N$ over N'

A fibre over $L \in N'$ can be seen as the join $J(L^*, \mathbb{P}S^3V^*) \cong \mathbb{P}_{11}$ of $L^* \cong \mathbb{P}_1$ and the space of plane cubic curves $\mathbb{P}(S^3V^*) \cong \mathbb{P}_9$. To see this assume $L = Z(x_0)$, i. e., L is given by $\begin{pmatrix} x_0 & 0 & x_1 \\ 0 & x_0 & x_2 \end{pmatrix}$. Then the fibre over L is given by the orbits of matrices

$$\begin{pmatrix} q_0(x_1, x_2) & q_1(x_1, x_2) & q_2(x_0, x_1, x_2) \\ x_0 & 0 & x_1 \\ 0 & x_0 & x_2 \end{pmatrix} \tag{10}$$

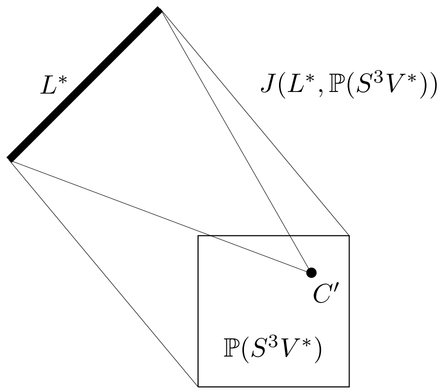
and can be identified with the projective space $\mathbb{P}(2H^0(L, \mathcal{O}_L(2)) \oplus S^2V^*)$. Rewrite the matrix (10) as

$$\begin{pmatrix} l\cdot x_2 - b(x_1,x_2) & -l\cdot x_1 - cx_2^2 & a(x_0,x_1,x_2) \\ x_0 & 0 & x_1 \\ 0 & x_0 & x_2 \end{pmatrix}, \quad l(x_1,x_2) = \xi_1 x_1 + \xi_2 x_2, \quad \xi_1, \xi_2 \in \Bbbk.$$

Its determinant equals $x_0(a(x_0,x_1,x_2)\cdot x_0 + b(x_0,x_1)\cdot x_1 + c\cdot x_2^3)$. This allows to reinterpret the fibre as the projective space

$$\mathbb{P}(H^0(L,\mathcal{O}_L(1)) \oplus S^3V^*) \cong J(L^*, \mathbb{P}S^3V^*).$$

The intersection of the fibre with M_0 is $J(L^*, \mathbb{P}(S^3V^*)) \setminus L^*$. It is a rank 2 vector bundle over $\mathbb{P}(S^3V^*)$ whose fibre over a cubic curve $C' \in \mathbb{P}S^3V^*$ is identified with the set of the isomorphism classes of sheaves from M_{01} defined by (6) with fixed L and C'. This fibre corresponds to the projective plane joining C' with L^* inside the join $J(L^*, \mathbb{P}(S^3V^*))$. In the notations of the example above ξ_1 and ξ_2 are the coordinates of this affine plane.



The points of $J(L^*, \mathbb{P}(S^3V^*)) \setminus L^*$ parameterize the extensions (6) from M_{01} with fixed L.

3.2.4. Description of the complement of M_0 in $\mathbb{B}$.

Let $\mathbb{B}' = \mathbb{B} \setminus M_0$. Then $\mathbb{B}'$ is a union of lines L^* from each fibre over N' (as explained above), it is isomorphic to the tautological $\mathbb{P}_1$-bundle over $N' = \mathbb{P}_2^*$

$$\{(L,x) \in \mathbb{P}_2^* \times \mathbb{P}_2 \mid L \in \mathbb{P}_2^*, x \in L\}. \tag{11}$$

The fibre $\mathbb{P}_1$ of $\mathbb{B}'$ over, say, line $L = Z(x_0) \subseteq \mathbb{P}_2$ can be identified with the space of classes of matrices (3) with zero determinant

$$\begin{pmatrix} \xi\cdot x_2 & -\xi\cdot x_1 & 0 \\ x_0 & 0 & x_1 \\ 0 & x_0 & x_2 \end{pmatrix}, \quad \xi = \alpha x_1 + \beta x_2, \quad \langle \alpha, \beta \rangle \in \mathbb{P}_1.$$

Let N_c be the open subset of N_0 that corresponds to 3 different (and hence non-collinear) points. Under the isomorphism $N_0 \cong H_0$ it corresponds to the open subvariety $H_c \subseteq H_0$ of the non-collinear configurations of 3 points on $\mathbb{P}_2$.

Let $M_c = \mathbb{B}_c$ be the restriction of $\mathbb{B}$ to N_c. Then $M_c \subseteq M_{00} \subseteq M_0 \subseteq M$ are inclusions of open subvarieties of M.

4. The subvariety of singular sheaves

Let M_1' and M_0' denote the intersections of the subvariety $M' = M'_{4m-1}$ of singular sheaves with M_1 and M_0 respectively.

4.1. Characterization of singular sheaves

4.1.1. Singular sheaves in M_1

As shown in [8], the subvariety M_1' coincides with the universal singular locus

$$\{(p, C) \mid C \subseteq \mathbb{P}_2 \text{ is a quartic plane curve, } p \in \operatorname{Sing}(C)\},$$

which is a smooth subvariety of M_1 of codimension 2.

4.1.2. Singular sheaves in M_0.

LEMMA 4.1. *The sheaf $\mathcal{E}_A$ from M_0 is singular if and only if the ideal $\mathcal{I}_{min} = \mathcal{I}_{min}(A)$ generated by all (2×2)-minors of A defines a non-empty scheme.*

Proof. If there are no zeros of $\mathcal{I}_{min}$, then at every point of $\mathbb{P}_2$ at least one of the (2×2)-minors is invertible, hence using invertible elementary transformations one can bring A to the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det A \end{pmatrix}$$

and therefore $\mathcal{E}$ is locally isomorphic to $\mathcal{O}_C$, $C = Z(\det A) = \operatorname{Supp} \mathcal{E}$.

If p is a zero point of $\mathcal{I}_{min}$, then the rank of A is at most 1 at p. Therefore, the dimension of $\mathcal{E}(p) = \mathcal{E}_p/\mathfrak{m}_p\mathcal{E}_p$ is at least 2. Since the rank of $\mathcal{E}$ (on support) is 1, we conclude that $\mathcal{E}$ is a singular sheaf. □

4.2. M_0' and computer algebra

Lemma 4.1 suggests the following approach to study M_0' using computer algebra.

Put $\mathbb{A} := \operatorname{Hom}(\mathcal{O}_{\mathbb{P}_2}(-3) \oplus 2\mathcal{O}_{\mathbb{P}_2}(-2), 3\mathcal{O}_{\mathbb{P}_2}(-1)) \cong \Bbbk^{36}$ and let $\mathbb{W}_0 \subseteq \mathbb{A}$ be the quasi-affine variety of injective matrices (3) such that $M_0 \cong \mathbb{W}_0/G'$

as mentioned in 2.2.3. Consider the ideal $I \subseteq \Bbbk(\mathbb{A})[x_0, x_1, x_3]$ of (2×2)-minors of the universal matrix on $\mathbb{A}$. Then eliminating the variables x_0, x_1, x_2 from the saturation ideal $I : (x_0, x_1, x_2)^\infty$, we will obtain the ideal $J = (I : (x_0, x_1, x_2)^\infty) \cap \Bbbk[\mathbb{A}]$ defining the subvariety in $\mathbb{A}$ of the matrices whose cokernels are singular sheaves. Having this, one computes the dimension of the zero scheme of J, its singularities, etc.

Though all actions with the ideals mentioned above are implemented in different systems of computer algebra, the complexity of the involved algorithms have not even made it possible for us to compute J. Therefore, we are going to study first the fibres of M_0' over N.

4.3. Fibres of M_0' over N

Let us consider the restriction of ν to M_0' and describe its fibres. There are the following possible cases:

1. fibres over $N_c \cong H_c$, i. e., over 3 different non-collinear points;
2. fibres over $Z \in N_0$ consisting of a simple point and a double point;
3. fibres over curvilinear triple points $Z \in N_0$;
4. fibres over non-curvilinear triple points $Z \in N_0$;
5. fibres over N'.

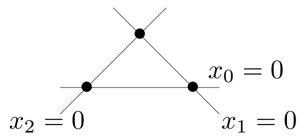
The corresponding fibres will be referred to as fibres of type (1), (2), (3), (4), and (5) respectively.

4.3.1. Fibres of type (1)

Let $Z \in H_c \cong N_c$ be a non-collinear configuration of 3 points in $\mathbb{P}_2$. Then, after applying an appropriate coordinate change, we can assume without loss of generality that Z is the union of three points $\mathrm{pt}_0 = \langle 1, 0, 0 \rangle$, $\mathrm{pt}_1 = \langle 0, 1, 0 \rangle$, $\mathrm{pt}_2 = \langle 0, 0, 1 \rangle$, the corresponding Kronecker module is

$$\Phi = \begin{pmatrix} x_0 & x_1 & 0 \\ x_0 & 0 & x_2 \end{pmatrix},$$

whose minors $d_0 = x_1 x_2$, $d_1 = -x_0 x_2$, $d_2 = -x_0 x_1$ generate the ideal I_Z of Z.



The fibre of ν over the class of Φ in N_0 consists of the orbits of the matrices

$$A = \begin{pmatrix} q_0(x_0, x_1, x_2) & q_1(x_0, x_2) & q_2(x_0, x_1) \\ x_0 & x_1 & 0 \\ x_0 & 0 & x_2 \end{pmatrix}.$$

The coefficients of

$$\begin{aligned} q_0 &= a_0x_0^2 + a_1x_0x_1 + a_2x_0x_2 + a_3x_1^2 + a_4x_1x_2 + a_5x_2^2, \\ q_1 &= b_0x_0^2 + b_2x_0x_2 + b_5x_2^2, \\ q_2 &= c_0x_0^2 + c_1x_0x_1 + c_3x_1^2 \end{aligned} \tag{12}$$

can be seen as the projective coordinates of the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{11}$.

The ideal that defines the subvariety corresponding to the singular sheaves is computed by eliminating the variables x_0, x_1, x_2 from the saturation of $\mathcal{I}_{min}$ with respect to the non-essential maximal ideal (x_0, x_1, x_2). We perform the computations using the computer algebra system Singular (cf. [1]).

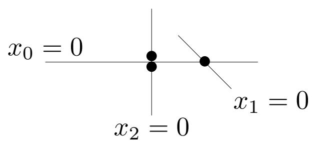
We get the ideal (see A.1 for computations)

$$(b_0, c_0) \cap (a_3, c_3) \cap (a_5, b_5),$$

i. e., the fibre of M_0' over $[\Phi]$ is a union of 3 components, each being a projective subspace in $\mathbb{P}_{11}$ of codimension 2. The components lie in a general position: each two components intersect along a projective subspace of codimension 4 and the intersection of all three of them is a projective subspace of codimension 6.

4.3.2. Fibres of type (2)

Let $Z \in H_0 \setminus H_c$ be a non-collinear configuration of a simple point pt_1 and a double non-collinear point at pt_2. The double point is defined by the underlying simple point pt_2 and a tangent vector at pt_2. Since Z does not lie on a line, the tangent vector should be normal to the line joining pt_1 and pt_2. Therefore, after applying an appropriate coordinate change, we can assume without loss of generality that $\mathrm{pt}_1 = \langle 0, 0, 1\rangle$, $\mathrm{pt}_2 = \langle 0, 1, 0\rangle$, and the tangent vector at pt_2 is parallel to the line given by x_2.



The ideal of Z equals $(x_0, x_1) \cap (x_0^2, x_2)$, the corresponding Kronecker module can be taken to be

$$\Phi = \begin{pmatrix} x_0 & x_1 & 0 \\ 0 & x_0 & x_2 \end{pmatrix}$$

The fibre of ν over the class of Φ in N_0 consists of the orbits of the matrices

$$A = \begin{pmatrix} q_0(x_0,x_1,x_2) & q_1(x_0,x_2) & q_2(x_0,x_1) \\ x_0 & x_1 & 0 \\ 0 & x_0 & x_2 \end{pmatrix}.$$

The coefficients of q_0, q_1, q_2 as in (12) can be seen as the projective coordinates of the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{11}$.

The fibre of M_0' over $[\Phi] \in N$ is given by the ideal

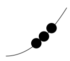
$$(a_3^2, c_3^2, a_3c_3, a_1c_3 - a_3c_1) \cap (a_5, b_5)$$

whose radical is $(a_3, c_3) \cap (a_5, b_5)$, which means that the fibre consists of two components each of which is a projective subspace of $\nu^{-1}([\Phi]) \cong \mathbb{P}_{11}$ of codimension 2. For computations see A.2.

4.3.3. Fibres of type (3)

Let Z be a triple curvilinear point. Without loss of generality, applying an appropriate coordinate change if necessary, we can assume that Z is supported at pt $= \langle 1,0,0\rangle$ and the ideal of Z in the affine coordinates $x = x_1/x_0$, $y = x_2/x_0$ is in this case

$$(y^3, x - sy - t^{-1}y^2), \quad s \in \Bbbk, \quad t \in \Bbbk^*.$$



The corresponding Kronecker module can be taken to be

$$\Phi = \begin{pmatrix} x_2 + 2stx_0 & x_1 - sx_2 & tx_0 \\ x_1 + sx_2 & 0 & x_2 \end{pmatrix}.$$

The fibre of ν over the class of Φ in N_0 consists of the orbits of the matrices

$$A = \begin{pmatrix} q_0(x_0,x_1,x_2) & q_1(x_0,x_2) & q_2(x_0,x_1) \\ x_2 + 2stx_0 & x_1 - sx_2 & tx_0 \\ x_1 + sx_2 & 0 & x_2 \end{pmatrix}. \tag{13}$$

The coefficients of q_0, q_1, q_2 as in (12) can be seen as the projective coordinates of the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{11}$.

The fibre of M_0' over $[\Phi] \in N$ is given by the ideal whose radical is

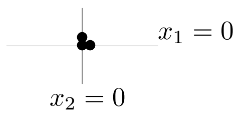
$$(b_0, a_0 - 2sc_0),$$

which means that the fibre consists of one component which is a projective subspace of codimension 2 in $\nu^{-1}([\Phi]) \cong \mathbb{P}_{11}$. For computations see A.3.

4.3.4. Fibres of type (4)

Let Z be a non-curvilinear triple point. After a change of coordinates we may assume that Z is supported at $\mathrm{pt} = \langle 1, 0, 0\rangle$. Since there is only one non-curvilinear triple point at a given point of a smooth surface, the ideal of Z equals (x_1^2, x_1x_2, x_2^2), the corresponding Kronecker module can be taken to be

$$\Phi = \begin{pmatrix} x_2 & x_1 & 0 \\ x_1 & 0 & x_2 \end{pmatrix}.$$



The fibre of ν over the class of Φ in N_0 consists of the orbits of the matrices

$$A = \begin{pmatrix} q_0(x_0, x_1, x_2) & q_1(x_0, x_2) & q_2(x_0, x_1) \\ x_2 & x_1 & 0 \\ x_1 & 0 & x_2 \end{pmatrix}.$$

By Lemma 4.1 all such matrices define singular sheaves since all (2×2)-minors vanish at pt. Therefore, M_0' is a $\mathbb{P}_{11}$-bundle over the locus of non-curvilinear triple points.

4.3.5. Fibres of type (5)

Let $[\Phi] \in N'$, then without loss of generality

$$\Phi = \begin{pmatrix} x_0 & 0 & x_1 \\ 0 & x_0 & x_2 \end{pmatrix}$$

and the fibre of ν over $[\Phi]$ consists of the orbits of the matrices (10). By Lemma 4.1 the sheaf defined by

$$\begin{pmatrix} q_0(x_1, x_2) & q_1(x_1, x_2) & q_2(x_0, x_1, x_2) \\ x_0 & 0 & x_1 \\ 0 & x_0 & x_2 \end{pmatrix}$$

is singular if and only if the quadratic forms

$$q_0(x_1, x_2) = a_3x_1^2 + a_4x_1x_2 + a_5x_2^2 \quad \text{and} \quad q_1(x_1, x_2) = b_3x_1^2 + b_4x_1x_2 + b_5x_2^2$$

have a common zero. The latter holds if and only if the resultant of q_0 and q_1

$$R = R(q_0, q_1)(a_3, a_4, a_5, b_3, b_4, b_5)$$

vanishes. Since R is an irreducible homogeneous polynomial of degree 4 in variables a_3, a_4, a_5, b_3, b_4, b_5, the fibres over N' are open subsets of irreducible hyper-surfaces of degree 4 in $\mathbb{P}_{11}$. These subsets are obtained by throwing away the points corresponding to matrices with zero determinant, i. e., the line L^* (cf. 3.2.3), which is contained in the hypersurface.

5. Main result

The information about the fibres of M'_0 over N obtained in the previous section allows to prove Proposition 1.1.

5.1. Dimension

We showed that the fibres of M'_0 over N are generically 9-dimensional, the fibres are more than 9-dimensional only over a subvariety of N of dimension 2. Therefore, the dimension of M'_0 (and thus of M') is 15, i. e., M' has codimension 2 in M.

5.2. Singularities

Notice that the computation from 4.3.1 works also locally over the base. Let us make this clear in the case of $\Bbbk = \mathbb{C}$, i. e., in the analytic category with analytic topology.

Let us vary the points

$$p_0 = \langle 1, p_1^{(0)}, p_2^{(0)} \rangle, \quad p_1 = \langle p_0^{(1)}, 1, p_2^{(1)} \rangle, \quad p_2 = \langle p_0^{(2)}, p_1^{(2)}, 1 \rangle$$

in disjoint neighborhoods in $\mathbb{P}_2$ of points $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$, $\langle 0,0,1\rangle$ respectively. Assume moreover that p_0, p_1, p_2 are always non-collinear. Then $p_1^{(0)}$, $p_2^{(0)}$, $p_0^{(1)}$, $p_2^{(1)}$, $p_0^{(2)}$, $p_1^{(2)}$ are local coordinates of N around the class of the Kronecker module

$$\Phi = \begin{pmatrix} x_0 & x_1 & 0 \\ x_0 & 0 & x_2 \end{pmatrix}.$$

Denote by U_{p_0,p_1,p_2} the corresponding neighborhood of $[\Phi]$.

Let $\bar{x}_i$, $i = 0,1,2$, be a linear form that defines the line not passing through p_i and passing through the other two points.

The fibre of ν over the class of

$$\bar{\Phi} = \begin{pmatrix} \bar{x}_0 & \bar{x}_1 & 0 \\ \bar{x}_0 & 0 & \bar{x}_2 \end{pmatrix}$$

consists of the orbits of the matrices

$$A = \begin{pmatrix} \bar{q}_0(\bar{x}_0,\bar{x}_1,\bar{x}_2) & \bar{q}_1(\bar{x}_0,\bar{x}_2) & \bar{q}_2(\bar{x}_0,\bar{x}_1) \\ \bar{x}_0 & \bar{x}_1 & 0 \\ \bar{x}_0 & 0 & \bar{x}_2 \end{pmatrix}.$$

The coefficients of

$$\begin{aligned}\bar{q}_0 &= a_0\bar{x}_0^2 + a_1\bar{x}_0\bar{x}_1 + a_2\bar{x}_0\bar{x}_2 + a_3\bar{x}_1^2 + a_4\bar{x}_1\bar{x}_2 + a_5\bar{x}_2^2,\\ \bar{q}_1 &= b_0\bar{x}_0^2 + b_2\bar{x}_0\bar{x}_2 + b_5\bar{x}_2^2,\\ \bar{q}_2 &= c_0\bar{x}_0^2 + c_1\bar{x}_0\bar{x}_1 + c_3\bar{x}_1^2\end{aligned}$$

can be seen as the projective coordinates of the fibre $\nu^{-1}([\bar{\Phi}]) \cong \mathbb{P}_{11}$, this gives a trivialization of $\mathbb{B}$ around $[\Phi]$. As in 4.3.1 we conclude that M' over U_{p_0,p_2,p_3} is a trivial bundle with the fibre computed in 4.3.1. Therefore, $M'_c = M' \cap M_c$ is a bundle over N_c with this singular fibre, which shows that M' is singular.

Remark 5.1. *Our argument shows that the singularities of M'_0 over N_c lie in codimension* 2.

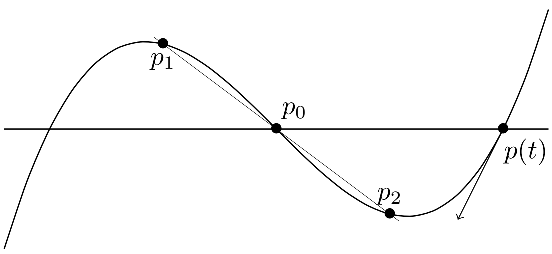
Remark 5.2. *Notice that in the algebraic category a modification of the argument above would lead to a local triviality of M' over N_c only in étale topology. This would not affect however our conclusions.*

5.3. Connectedness

As shown in 4.3, every fibre of M'_0 over N is (path-)connected. Therefore, since N is (path-)connected, M'_0 is (path-)connected. Since M'_1, which is isomorphic to the universal singular locus of plane quartic curves, is (path-)connected, it remains to connect M'_1 with M'_0.

The latter can be done, for example, as follows. Let C be a quartic curve with a simple double point singularity $p_0 \in C$. Fix a line through p_0 that is not a component of C and intersects C at 3 different points p_0, p_1, p_2.

Consider a degeneration $Z_t = \{p_0, p_1, p(t)\}$ of a configuration of 3 non-collinear points on C to the configuration $Z_0 = \{p_0, p_1, p_2\}$, i. e., $p(t) \to p_2$, $t \to 0$.



This gives a degeneration of the twisted ideal sheaf $\mathcal{E}_t = \mathcal{I}_{Z_t}(1)$ of Z_t in C to the twisted ideal sheaf $\mathcal{I}_{Z_0}(1)$ of Z_0.

Notice that $\mathcal{E}_0 = \mathcal{I}_{Z_0}(1)$ is a non-trivial extension (2) with $p = p_0$. Therefore, $\mathcal{E}_0$ defines a point in M_1. Since p_0 is a singular point of C, as mentioned in 4.1.1, $\mathcal{E}$ must be a singular sheaf. On the other hand, as we shall show in Proposition 6.1, $\mathcal{E}_t$ is a singular sheaf for $t \neq 0$ as p_0 is a singular point of C. This gives a path connecting M'_0 with M'_1.

6. Singular sheaves and singularities of their support

Let $[\mathcal{E}] \in M_{00} = \mathbb{B}_0$. Let $C = \operatorname{Supp} \mathcal{E}$ be its support, which is a quartic curve in $\mathbb{P}_2$. As $\mathcal{E}$ is a part of an exact sequence (5), it is a subsheaf of $\mathcal{O}_C(1)$, hence a torsion free sheaf on C. Since torsion free sheaves on smooth curves are locally free (see e.g. [10, Lemma 5.2.1]), we conclude that $\mathcal{E}$ is non-singular if C is smooth at all points of Z. So $\mathcal{E}$ can only be singular if C is singular at some points of Z. This demonstrates that the image of $M'_{00} = M' \cap M_{00}$ under (9) is included in the subvariety of pairs (C, Z) such that Z contains a singular point of C. We shall demonstrate that M'_{00} generically coincides with this variety. More precisely, the image of $M'_c = M' \cap M_c$ under the morphism

$$M_c \xrightarrow{\mu \times \nu} \mathbb{P}S^4V^* \times H_0$$

consists of the pairs (C, Z), $Z \subseteq C$, such that C is a singular plane curve of degree 4 whose singular locus contains at least one of the points of Z.

Proposition 6.1. *Let $[\mathcal{E}]$ as above belong to M_c, then*

1) $\mathcal{E}$ is singular if and only if $\operatorname{Sing} C \cap Z \neq \emptyset$*;*

2) the fibre of M'_0 over $Z \in H_c$, $Z = \{\mathrm{pt}_0, \mathrm{pt}_1, \mathrm{pt}_2\}$, under the morphism $M'_0 \xrightarrow{\nu} N_c \cong H_c$ corresponds to the variety of plane quartic curves through Z such that one of the points of Z is a singular point of C;

3) for each $i = 0, 1, 2$, the variety of quartics through Z such that pt_i is a singular point of C coincides with one of three different irreducible components of the fibre.

Proof. Follows from the computations given in A.1. □

Remark 6.2. *Since $\mathcal{E}$ is a twisted ideal sheaf of* 3 *different points on a quartic curve (cf.* (5)*), the statement 1) of Proposition 6.1 immediately follows from Lemma 6.3 below.*

6.1. An observation from commutative algebra

Let $R = \mathcal{O}_{C,p}$ be a local $\mathbb{k}$-algebra of a curve C at point $p \in C$. Let $\mathfrak{m} = \mathfrak{m}_{C,p}$ be its maximal ideal and let $\mathbb{k}_p = R/\mathfrak{m}$ be the local ring of the structure sheaf of the one point subscheme $\{p\} \subseteq C$. An R-module homomorphism $R \xrightarrow{\varphi} \mathbb{k}_p$ is uniquely defined by $\varphi(1) = \lambda \in \mathbb{k}_p$. Then $\varphi(s) = \bar{s} \cdot \lambda$. If φ is different from zero, then the kernel of φ coincides with $\mathfrak{m}$.

LEMMA 6.3. *Consider an exact sequence of R-modules.*

$$0 \to M \to R \to \Bbbk_p \to 0$$

with a non-zero R-module M. Then M is free if and only if R is regular.

Proof. If M is free, then $M \cong R$ (otherwise $M \to R$ would not be injective) and we obtain an exact sequence of R-modules

$$0 \to R \to R \to \Bbbk_p \to 0,$$

which means that the maximal ideal $\mathfrak{m}$ of p is generated by one element. Therefore, R is regular in this case.

Vice versa, assume R is regular. Notice that M is always a torsion free R-module as a submodule of R. Therefore, if R is regular, M is free as a torsion free module over a regular one-dimensional local ring. □

REMARK 6.4. *Notice that Proposition 6.1 does not hold over $N_0 \setminus N_c$. Indeed, take $[\mathcal{E}_A] \in M_{00} \setminus M_c$ with*

$$A = \begin{pmatrix} x_2^2 & 0 & x_1^2 \\ x_0 & x_1 & 0 \\ 0 & x_0 & x_2 \end{pmatrix}.$$

Then the support C of $\mathcal{E}_A$ is given by $x_1(x_2^3 + x_0^2 x_1) = 0$, one obtains an exact sequence

$$0 \to \mathcal{E}_A \to \mathcal{O}_C(1) \to \mathcal{O}_Z \to 0$$

such that Z consists of the simple point $\langle 0,0,1 \rangle$ and the double point $\langle 0,1,0 \rangle$. In this case $\mathcal{E}_A$ is a non-singular sheaf but $\langle 0,1,0 \rangle \in Z \cap \operatorname{Sing} C$.

In A.2 we compute that every matrix A as in 4.3.2 with $a_3 = 0$, $a_5 \neq 0$ defines a non-singular sheaf, however the intersection of Z with the singular locus of the supporting curve C is non-empty.

A. Computations of the fibres of M_0' over N with SINGULAR

A.1. Fibres of type (1)

```
t1.sng   1> LIB "elim.lib";
t1.sng   2> ring r=0, (x(0..2), a(0..5), b(0..5), c(0..5)), dp;
t1.sng   3> ideal maxm=x(0..2);
t1.sng   4> poly X=x(0)*x(1)*x(2);
t1.sng   5> poly q(0..2);
t1.sng   6> q(0) = a(0)*x(0)^2 + a(1)*x(0)*x(1) + a(2)*x(0)*x(2) + a(3)*x(1)^2 + a(4)*x(1)*x(2) + a(5)*x(2)^2;
t1.sng   7> q(1) = b(0)*x(0)^2 + b(1)*x(0)*x(1) + b(2)*x(0)*x(2) + b(3)*x(1)^2 + b(4)*x(1)*x(2) + b(5)*x(2)^2;
t1.sng   8> q(2) = c(0)*x(0)^2 + c(1)*x(0)*x(1) + c(2)*x(0)*x(2) + c(3)*x(1)^2 + c(4)*x(1)*x(2) + c(5)*x(2)^2;
t1.sng   9> q(1)=subst(q(1), x(1), 0);
t1.sng  10> q(2)=subst(q(2), x(2), 0);
t1.sng  11> // general form of matrices representing the points in the fibre
t1.sng  12. matrix A[3][3] = q(0), q(1), q(2), x(0), x(1), 0, x(0), 0, x(2);
```

```
t1.sng  13> print(A);
A[1,1],A[1,2],A[1,3],
x(0),  x(1),  0,
x(0),  0,     x(2)
t1.sng  14> // the Kronecker module corresponding to 3 non-collinear points
t1.sng  15. matrix Phi=submat(A, 2..3, 1..3);
t1.sng  16> print(Phi);
x(0),x(1),0,
x(0),0,   x(2)
t1.sng  17> // the ideal of 2x2 minors of A
t1.sng  18. ideal minm = minor(A, 2);
t1.sng  19> minm = sat(minm, maxm)[1]; // compute its saturation
t1.sng  20> minm = elim(minm, X); // eliminate the variables x(0), x(1), x(2)
t1.sng  21> print(minm);
b(5)*c(0)*c(3),
a(5)*c(0)*c(3),
b(0)*b(5)*c(3),
a(5)*b(0)*c(3),
a(3)*b(5)*c(0),
a(3)*a(5)*c(0),
a(3)*b(0)*b(5),
a(3)*a(5)*b(0)
t1.sng  22> // look at the primary decomposition of the result
t1.sng  23. primdecGTZ(minm);
[1]:
   [1]:
      _[1]=c(3)
      _[2]=a(3)
   [2]:
      _[1]=c(3)
      _[2]=a(3)
[2]:
   [1]:
      _[1]=c(0)
      _[2]=b(0)
   [2]:
      _[1]=c(0)
      _[2]=b(0)
[3]:
   [1]:
      _[1]=b(5)
      _[2]=a(5)
   [2]:
      _[1]=b(5)
      _[2]=a(5)
t1.sng  24> // let us establish a link between the singular sheaves
t1.sng  25. // and the singularities of their support
t1.sng  26. poly f = det(A); // determinant of A
t1.sng  27> // ideal of partial derivatives of f
t1.sng  28. // with respect to x(0), x(1), x(2)
t1.sng  29. // together with the 2x2-minors of Phi,
t1.sng  30. // its zeroes are exactly the singular points of C contained in Z
t1.sng  31. ideal D = diff(f, x(0)), diff(f, x(1)), diff(f, x(2)), minor(Phi, 2);
t1.sng  32> // look at the equations of the subvariety of the fibre defining such sheaves
t1.sng  33. D = sat(D, maxm)[1];
t1.sng  34> D = elim(D, X);
t1.sng  35> // the result coincides with the ideal for singular sheaves
t1.sng  36. primdecGTZ(D);
[1]:
   [1]:
      _[1]=c(3)
      _[2]=a(3)
   [2]:
      _[1]=c(3)
      _[2]=a(3)
[2]:
   [1]:
      _[1]=c(0)
      _[2]=b(0)
   [2]:
      _[1]=c(0)
      _[2]=b(0)
[3]:
   [1]:
      _[1]=b(5)
      _[2]=a(5)
   [2]:
      _[1]=b(5)
      _[2]=a(5)
t1.sng  37> // the ideal of the intersection of pt0 with the singular locus of C
t1.sng  38. D = diff(f, x(0)), diff(f, x(1)), diff(f, x(2)), x(1), x(2);
t1.sng  39> D = sat(D, maxm)[1];
t1.sng  40> D = elim(D, X);
t1.sng  41> // the result coincides with one of the components
```

```
t1.sng  42. // of the fibre of singular sheaves computed above
t1.sng  43. primdecGTZ(D);
[1]:
   [1]:
      _[1]=c(0)
      _[2]=b(0)
   [2]:
      _[1]=c(0)
      _[2]=b(0)
t1.sng  44> // the ideal of the intersection of pt0 with the singular locus of C
t1.sng  45. D = diff(f, x(0)), diff(f, x(1)), diff(f, x(2)), x(0), x(2);
t1.sng  46> D = sat(D, maxm)[1];
t1.sng  47> D = elim(D, X);
t1.sng  48> // the result coincides with one of the components
t1.sng  49. // of the fibre of singular sheaves computed above
t1.sng  50. primdecGTZ(D);
[1]:
   [1]:
      _[1]=c(3)
      _[2]=a(3)
   [2]:
      _[1]=c(3)
      _[2]=a(3)
t1.sng  51> // the ideal of the intersection of pt0 with the singular locus of C
t1.sng  52. D = diff(f, x(0)), diff(f, x(1)), diff(f, x(2)), x(0), x(1);
t1.sng  53> D = sat(D, maxm)[1];
t1.sng  54> D = elim(D, X);
t1.sng  55> // the result coincides with one of the components
t1.sng  56. // of the fibre of singular sheaves computed above
t1.sng  57. primdecGTZ(D);
[1]:
   [1]:
      _[1]=b(5)
      _[2]=a(5)
   [2]:
      _[1]=b(5)
      _[2]=a(5)
t1.sng  58> $
```

A.2. Fibres of type (2)

```
t2.sng   1> LIB "elim.lib";
t2.sng   2> ring r = 0, (x(0..2), a(0..5), b(0..5), c(0..5)), dp;
t2.sng   3> ideal maxm = x(0..2);
t2.sng   4> poly X = x(0)*x(1)*x(2);
t2.sng   5> poly q(0..2);
t2.sng   6> q(0) = a(0)*x(0)^2 + a(1)*x(0)*x(1) + a(2)*x(0)*x(2) + a(3)*x(1)^2 + a(4)*x(1)*x(2) + a(5)*x(2)^2;
t2.sng   7> q(1) = b(0)*x(0)^2 + b(1)*x(0)*x(1) + b(2)*x(0)*x(2) + b(3)*x(1)^2 + b(4)*x(1)*x(2) + b(5)*x(2)^2;
t2.sng   8> q(2) = c(0)*x(0)^2 + c(1)*x(0)*x(1) + c(2)*x(0)*x(2) + c(3)*x(1)^2 + c(4)*x(1)*x(2) + c(5)*x(2)^2;
t2.sng   9> q(1) = subst(q(1), x(1), 0);
t2.sng  10> q(2) = subst(q(2), x(2), 0);
t2.sng  11> // general form of matrices representing the points in the fibre
t2.sng  12. matrix A[3][3] = q(0), q(1), q(2), x(0), x(1), 0, 0, x(0), x(2);
t2.sng  13> print(A);
A[1,1],A[1,2],A[1,3],
x(0),  x(1),  0,
0,     x(0),  x(2)
t2.sng  14> // the Kronecker module corresponding to 3 non-collinear points
t2.sng  15. matrix Phi = submat(A, 2..3,1..3);
t2.sng  16> print(Phi);
x(0),x(1),0,
0,   x(0),x(2)
t2.sng  17> // the ideal of 2x2 minors
t2.sng  18. ideal minm = minor(A, 2);
t2.sng  19> minm = sat(minm, maxm)[1]; // compute its saturation
t2.sng  20> minm = elim(minm, X); // eliminate the variables x(0), x(1), x(2)
t2.sng  21> minm;
minm[1]=b(5)*c(3)^2
minm[2]=a(5)*c(3)^2
minm[3]=a(3)*b(5)*c(3)
minm[4]=a(3)*a(5)*c(3)
minm[5]=a(3)*b(5)*c(1)-a(1)*b(5)*c(3)
minm[6]=a(3)*a(5)*c(1)-a(1)*a(5)*c(3)
minm[7]=a(3)^2*b(5)
minm[8]=a(3)^2*a(5)
t2.sng  22> // look at the primary decomposition of the result
t2.sng  23. primdecGTZ(minm);
[1]:
   [1]:
      _[1]=c(3)^2
      _[2]=a(3)*c(3)
```

```
      _[3]=a(3)^2
      _[4]=-a(3)*c(1)+a(1)*c(3)
   [2]:
      _[1]=c(3)
      _[2]=a(3)
[2]:
   [1]:
      _[1]=b(5)
      _[2]=a(5)
   [2]:
      _[1]=b(5)
      _[2]=a(5)
t2.sng  24> // polynomial defining the quartic curve C
t2.sng  25. poly f=det(A);
t2.sng  26> // ideal of singularities of the curve C lying on Z
t2.sng  27. ideal D = diff(f, x(0)), diff(f, x(1)), diff(f, x(2)), minor(Phi, 2);
t2.sng  28> // compute the equations of the subvariety of the corresponding sheaves
t2.sng  29. D = sat(D, maxm)[1];
t2.sng  30> D = elim(D, X);
t2.sng  31> D;
D[1]=a(3)*b(5)*c(3)
D[2]=a(3)*a(5)*c(3)
D[3]=a(3)^2*b(5)
D[4]=a(3)^2*a(5)
t2.sng  32> // look at its primary decomposition
t2.sng  33. // the corresponding variety has an extra component
t2.sng  34. // whose points do not define singular sheaves
t2.sng  35. primdecGTZ(D);
[1]:
   [1]:
      _[1]=a(3)
   [2]:
      _[1]=a(3)
[2]:
   [1]:
      _[1]=b(5)
      _[2]=a(5)
   [2]:
      _[1]=b(5)
      _[2]=a(5)
[3]:
   [1]:
      _[1]=c(3)
      _[2]=a(3)^2
   [2]:
      _[1]=c(3)
      _[2]=a(3)
t2.sng  36> $
```

A.3. Fibres of type (3)

```
t3.sng   1> LIB "elim.lib";
t3.sng   2> ring r = (0,s, t), (x(0..2), a(0..5), b(0..5), c(0..5)), dp;
t3.sng   3> ideal I = x(2)^3, x(1)*x(0)-s*x(2)*x(0)-(1/t)*x(2)^2;
t3.sng   4> I=sat(I, x(0))[1];
t3.sng   5> I;
I[1]=x(1)*x(2)+(-s)*x(2)^2
I[2]=x(1)^2+(-2*s)*x(1)*x(2)+(s^2)*x(2)^2
I[3]=(t)*x(0)*x(1)+(-s*t)*x(0)*x(2)-x(2)^2
I[4]=x(2)^3
t3.sng   6> ideal J = I[1..3];
t3.sng   7> std(J);
_[1]=x(1)*x(2)+(-s)*x(2)^2
_[2]=x(1)^2+(-2*s)*x(1)*x(2)+(s^2)*x(2)^2
_[3]=(t)*x(0)*x(1)+(-s*t)*x(0)*x(2)-x(2)^2
_[4]=x(2)^3
t3.sng   8> // thus I = J
t3.sng   9. I = J;
t3.sng  10> matrix S[2][3] = (2*s*t)*x(0)+x(2), x(1)-(s)*x(2), (t)*x(0), x(1)+(s)*x(2), 0 ,x(2);
t3.sng  11> print(S);
(2*s*t)*x(0)+x(2),x(1)+(-s)*x(2),(t)*x(0),
x(1)+(s)*x(2),    0,             x(2)
t3.sng  12> // the ideal of maximal minors coincides with I
t3.sng  13. minor(S,2);
_[1]=x(1)*x(2)+(-s)*x(2)^2
_[2]=(-t)*x(0)*x(1)+(s*t)*x(0)*x(2)+x(2)^2
_[3]=-x(1)^2+(s^2)*x(2)^2
t3.sng  14> ideal maxm=x(0..2);
t3.sng  15> poly X=x(0)*x(1)*x(2);
t3.sng  16> poly q(0..2);
t3.sng  17> q(0) = a(0)*x(0)^2 + a(1)*x(0)*x(1) + a(2)*x(0)*x(2) + a(3)*x(1)^2 + a(4)*x(1)*x(2) + a(5)*x(2)^2;
```

```
t3.sng  18> q(1) = b(0)*x(0)^2 + b(1)*x(0)*x(1) + b(2)*x(0)*x(2) + b(3)*x(1)^2 + b(4)*x(1)*x(2) + b(5)*x(2)^2;
t3.sng  19> q(2) = c(0)*x(0)^2 + c(1)*x(0)*x(1) + c(2)*x(0)*x(2) + c(3)*x(1)^2 + c(4)*x(1)*x(2) + c(5)*x(2)^2;
t3.sng  20> q(1) = subst(q(1), x(1), 0);
t3.sng  21> q(2) = subst(q(2), x(2), 0);
t3.sng  22> // the linear part is S
t3.sng  23. matrix A[3][3];
t3.sng  24> A = q(0), q(1), q(2), (2*s*t)*x(0)+x(2), x(1)-(s)*x(2), (t)*x(0), x(1)+(s)*x(2), 0, x(2);
t3.sng  25> print(A);
A[1,1],           A[1,2],        A[1,3],
(2*s*t)*x(0)+x(2),x(1)+(-s)*x(2),(t)*x(0),
x(1)+(s)*x(2),    0,             x(2)
t3.sng  26> // the linear part Phi
t3.sng  27. matrix Phi = submat(A, 2..3, 1..3);
t3.sng  28> //ideal of 2x2 minor sof A
t3.sng  29. ideal minm = minor(A, 2);
t3.sng  30> // compute the ideal of the subvariety of singular sheaves in the fibre
t3.sng  31. minm = sat(minm, maxm)[1];
t3.sng  32> minm = elim(minm, X);
t3.sng  33> // look at its primary decomposition
t3.sng  34. list PD = primdecGTZ(minm);
t3.sng  35> // it has only one component
t3.sng  36. size(primdecGTZ(minm));
1
t3.sng  37> // the corresponding prime ideal is
t3.sng  38. PD[1][2];
_[1]=b(0)
_[2]=a(0)+(-2*s)*c(0)
t3.sng  39> $
```

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