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Notes on a combinatorial identity

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Abstract. *We present a short and simple proof by induction for*

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q \frac{1}{(1-q^{k+m})^2} = \frac{q^m}{1-q^m}\prod_{j=1}^{n}\frac{1-q^j}{1-q^{j+m}}\left(2+\sum_{j=0}^{n}\frac{q^{j+m}}{1-q^{j+m}}\right),$$

where $n \geq 1$ is an integer and $m \neq 0, -1, \ldots, -n$ is a complex number. This is a q-analogue of a combinatorial identity obtained by Kirschenhofer (1996) and Larcombe, Fennessey, and Koepf (2004). Moreover, we show that the alternating q-binomial sum is completely monotonic with respect to m, if $m > 0$ and $q \in (0, 1)$. The general case where the exponent 2 is replaced by a positive integer d is dealt with using the elementary technique of partial fraction decomposition.

Keywords: Combinatorial identity, q-binomial coefficient, completely monotonic, partial fraction decomposition.
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1. Introduction

The work on this note has been inspired by an interesting research paper published in 1996 by Kirschenhofer [11], who performed manipulations of generating functions to find identities for the alternating binomial sum

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k). \tag{1}$$

A well-known approach to study sums of the type (1) is attributed to Rice, who made use of Complex Analysis. The Rice method is based upon the formula

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k) = -\frac{1}{2\pi i}\int_{\mathscr{C}} B(n+1,-z)dz,$$

where $B(x, y)$ is Euler's beta function, $\mathscr{C}$ is a positively oriented closed curve surrounding $0, 1, 2, \ldots, n$ and f is an analytic function with no poles inside the region surrounded by $\mathscr{C}$.

The main reason for the interest in alternating binomial sums is that they have remarkable applications in Computer Science and the Theory of Algorithms. For more information on this subject we refer to [7, 8, 16].

Kirschenhofer proved that the sum

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(k+m)^d} \quad (d\in\mathbb{N}) \tag{2}$$

can be expressed in terms of Bell polynomials and harmonic numbers, whereas Coffey [5] showed that this sum can be written as an infinite series involving Stirling numbers. As a special case Kirschenhofer found the elegant identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(k+m)^2}=\frac{h_{m,n}^{(1)}}{m\binom{m+n}{m}}, \tag{3}$$

where

$$h_{m,n}^{(j)}=\sum_{k=m}^{m+n}\frac{1}{k^j}=H_{m+n}^{(j)}-H_{m-1}^{(j)}.$$

Here, $H_n^{(j)}$ denotes the n-th harmonic number of order j.

In 2004, Larcombe et al. [13] presented a new method to find identities for (2). They used an integration technique to offer proofs for (3) and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(k+m)^3}=\frac{(h_{m,n}^{(1)})^2+h_{m,n}^{(2)}}{2m\binom{m+n}{m}}. \tag{4}$$

Moreover, they demonstrated that (3) and (4) as well as corresponding identities for the sum in (2) with $d\geq 4$ can be obtained by differentiation with respect to m, starting with

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{k+m}=\frac{1}{m\binom{m+n}{n}}.$$

Larcombe et al. [14] provided a recursive equation for a sum closely related to (2) and used their result to find new proofs for (3), (4) and similar identities. Other methods to deal with the sums in question are described in [10, 12, 15].

In this paper we demonstrate that the identity (3) can be proved easily by using induction. More precisely, in the next section we present a short and elementary proof for a q-analogue of (3). Furthermore, as an application we prove a monotonicity property of the alternating q-binomial sum. In a final section, we show how to deal with the general case in a completely elementary fashion, using not more than partial fraction decomposition from elementary calculus.

2. The identity

The q-binomial coefficients (also known as Gaussian binomial coefficients) are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=1}^{k} \frac{1-q^{n+1-j}}{1-q^j} \quad \text{if} \quad 0 \le k \le n \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = 0 \quad \text{otherwise.}$$

If $q \to 1$, then $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ tends to $\binom{n}{k}$. A collection of the most important properties of q-binomial coefficients can be found, for instance, in [4].

The following q-version of (3) holds.

THEOREM 2.1. *Let $n \ge 1$ be an integer and let m be a complex number with $m \ne 0, -1, \dots, -n$. Then,*

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{(1-q^{k+m})^2} = \frac{q^m}{1-q^m} \prod_{j=1}^{n} \frac{1-q^j}{1-q^{j+m}} \left(2 + \sum_{j=0}^{n} \frac{q^{j+m}}{1-q^{j+m}}\right). \quad (5)$$

Throughout, we denote the sum on the left-hand side of (5) by $S(m,n,q)$.

Proof. We use induction on n to prove (5). If $n = 1$, then both sides of (5) are equal to

$$\frac{(1-q)q^m(2-q^m-q^{m+1})}{(1-q^m)^2(1-q^{m+1})^2}.$$

We set

$$T(m,n,q) = 2 + \sum_{j=0}^{n} \frac{q^{j+m}}{1-q^{j+m}}.$$

Applying the recurrence formula

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

and the induction hypothesis yields

$$\begin{aligned} S(m,n+1,q) &= S(m,n,q) - q^n S(m+1,n,q) \\ &= \frac{q^m}{1-q^m} \prod_{j=1}^{n+1} \frac{1-q^j}{1-q^{j+m}} \\ &\qquad \times \left(\frac{1-q^{m+n+1}}{1-q^{n+1}} T(m,n,q) - \frac{q^{n+1}(1-q^m)}{1-q^{n+1}} T(m+1,n,q) \right) \\ &= \frac{q^m}{1-q^m} \prod_{j=1}^{n+1} \frac{1-q^j}{1-q^{j+m}} T(m,n+1,q). \end{aligned}$$

This gives (2.1) with $n+1$ instead of n. □

Remark 2.2. (i) If we multiply both sides of (5) by $(1-q)^2$ and let $q \to 1$, then we obtain

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(k+m)^2}=\frac{n!\Gamma(m)}{\Gamma(m+n+1)}\big[\psi(m+n+1)-\psi(m)\big],$$

where $\psi=\Gamma'/\Gamma$ denotes the digamma function. This identity is given in [13]. The special case that m is a natural number yields (3).

(ii) If we differentiate both sides of (5) with respect to m, then we obtain the following q-analogue of (4):

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{1}{(1-q^{k+m})^3}$$
$$=\frac{1}{1-q^m}\prod_{j=1}^{n}\frac{1-q^j}{1-q^{j+m}}\Big[(1+\sigma_1)\Big(1+\frac{1}{2}\sigma_1\Big)+\sigma_2\Big], \quad (6)$$

where

$$\sigma_k=\sigma_k(m,n,q)=\frac{1}{k}\sum_{j=0}^{n}\frac{q^{j+m}}{(1-q^{j+m})^k}.$$

Identity (4) follows easily from (6). Indeed, if we multiply both sides of (6) by $(1-q)^3$ and let $q\to 1$, then we arrive at (4).

We recall that a function $f:(0,\infty)\to\mathbb{R}$ is said to be completely monotonic, if

$$(-1)^N f^{(N)}(x)\ge 0 \qquad (N=0,1,2,\ldots;\ x>0).$$

These functions have interesting applications, for instance, in probability theory, numerical and asymptotic analysis. In numerous papers it was proved that various functions which are defined in terms of gamma, polygamma and other special functions are completely monotonic. We refer to [2, 3] and the references therein. See also [17] for background information.

An application of Theorem 2.1 reveals that the alternating q-binomial sum $S(m,n,q)$ satisfies the following monotonicity property.

Corollary 2.3. *Let $n\ge 1$ be an integer and $q\in(0,1)$ be a real number. The function $m\mapsto S(m,n,q)$ is completely monotonic on $(0,\infty)$.*

Proof. Since a nonnegative constant function is completely monotonic and the sum and the product of completely monotonic functions are also completely monotonic, we conclude from (5) that in order to show that $S(m,n,q)$ is completely monotonic with respect to m it suffices to show that the functions

$$f_1(m)=q^m \quad\text{and}\quad f_2(m)=\frac{1}{1-q^{j+m}} \quad (j\ge 0)$$

are completely monotonic. This follows from

$$(-1)^N f_1^{(N)}(m) = (-\log q)^N q^m > 0$$

and

$$(-1)^N f_2^{(N)}(m) = (-\log q)^N \sum_{k=0}^{\infty} q^{k(j+m)} k^N > 0$$

which hold for all integers $N \geq 0$. □

Remark 2.4. (i) Fink [6] proved that a completely monotonic function is not only convex but even log-convex. This means that Corollary 2.3 leads to the inequality

$$S\Big(\frac{a+b}{2}, n, q\Big) \leq \sqrt{S(a,n,q)\, S(b,n,q)}. \tag{7}$$

(ii) A theorem of Hardy et al. [9, p. 97] states that if a function ϕ is twice differentiable and convex on $(0,\infty)$, then so is $x \mapsto x\phi(1/x)$. Using this result with $\phi = \log S$ we obtain

$$S\Big(\frac{2}{1/a + 1/b}, n, q\Big)^{a+b} \leq S(a,n,q)^b\, S(b,n,q)^a. \tag{8}$$

The inequalities (7) and (8) are valid for all $a, b > 0$, $n \geq 1$ and $q \in (0,1)$.

(iii) We have shown that identity (5) can be applied to prove a monotonicity property of $S(m,n,p)$. It might be of interest to present series, product or integral representations for other binomial sums in order to find similar results. An example is given in [1].

3. The general case

Let, as usual, $(z;q)_n = (1-z)(1-zq)\dots(1-zq^{n-1})$ and set

$$F(z) = \frac{(q;q)_n}{(z;q)_{n+1}} \frac{z^d}{(z-q^m)^d}.$$

This rational function has poles at $q^0, q^{-1}, \dots, q^{-n}$, and at q^m. We construct the partial fraction decomposition:

$$F(z) = \sum_{k=0}^{n} (-1)^{k-1} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{(1-q^{k+m})^d} \frac{1}{z-q^{-k}} + \frac{A_d}{(z-q^m)^d} + \frac{A_{d-1}}{(z-q^m)^{d-1}} + \dots + \frac{A_1}{(z-q^m)^1}.$$

Multiplying this relation by z, and then letting $z \to \infty$, we get for $n \geq 1$:

$$0 = \sum_{k=0}^{n} (-1)^{k-1} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{(1-q^{k+m})^d} + A_1.$$

So

$$\begin{aligned}
A_1 &= \sum_{k=0}^{n} (-1)^{k} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{(1-q^{k+m})^d} \\
&= [(z-q^m)^{-1}]F(z) = [(z-q^m)^{d-1}] \frac{(q;q)_n z^d}{(z;q)_{n+1}} = [w^{d-1}] \frac{(q;q)_n (w+q^m)^d}{(w+q^m;q)_{n+1}} \\
&= [w^{d-1}] \frac{(q;q)_n (w+q^m)^d}{(1-w-q^m)(1-wq-q^{m+1})\dots(1-wq^n-q^{m+n})} \\
&= [w^{d-1}] \frac{(q;q)_n (w+q^m)^d}{(1-q^m)\dots(1-q^{m+n})(1-\frac{w}{1-q^m})\dots(1-\frac{wq^n}{1-q^{n+m}})} \\
&= [w^{d-1}] \frac{(q;q)_n (q;q)_{m-1}}{(q;q)_{m+n}(1-\frac{w}{1-q^m})\dots(1-\frac{wq^n}{1-q^{n+m}})} \sum_{j=0}^{d} \binom{d}{j} w^{d-j} q^{mj} \\
&= \frac{(q;q)_n (q;q)_{m-1}}{(q;q)_{m+n}} \sum_{j=0}^{d-1} \binom{d}{j+1} q^{m(j+1)} [w^j] \frac{1}{(1-\frac{w}{1-q^m})\dots(1-\frac{wq^n}{1-q^{n+m}})}.
\end{aligned}$$

We continue with the computation of

$$\begin{aligned}
&[w^j] \frac{1}{(1-\frac{w}{1-q^m})\dots(1-\frac{wq^n}{1-q^{n+m}})} \\
&\quad = [w^j] \exp\left\{ \log \frac{1}{1-\frac{w}{1-q^m}} + \dots + \log \frac{1}{1-\frac{wq^n}{1-q^{m+n}}} \right\} \\
&\quad = [w^j] \exp\left\{ \sum_{k \geq 1} \frac{1}{k} \frac{w^k}{(1-q^m)^k} + \dots + \sum_{k \geq 1} \frac{1}{k} \frac{w^k q^{nk}}{(1-q^{m+n})^k} \right\} \\
&\quad = [w^j] \exp\left\{ \sum_{k \geq 1} \tau_k w^k \right\},
\end{aligned}$$

with

$$\tau_k = \tau_k(m,n,q) = \frac{1}{k} \sum_{j=0}^{n} \Big(\frac{q^j}{1-q^{j+m}} \Big)^k.$$

Furthermore,

$$[w^j]\exp\Big\{\sum_{k\ge1}\tau_k w^k\Big\} = [w^j]e^{\tau_1 w}e^{\tau_2 w^2}e^{\tau_3 w^3}\dots$$
$$= \sum_{k_1+2k_2+3k_3+\dots=j}\frac{\tau_1^{k_1}\tau_2^{k_2}\tau_3^{k_3}\dots}{k_1!k_2!k_3!\dots}$$

which leads to the final formula:

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{1}{(1-q^{k+m})^d}$$
$$=\frac{(q;q)_n(q;q)_{m-1}}{(q;q)_{m+n}}\sum_{j=0}^{d-1}\binom{d}{j+1}q^{m(j+1)}\sum_{k_1+2k_2+3k_3+\dots=j}\frac{\tau_1^{k_1}\tau_2^{k_2}\tau_3^{k_3}\dots}{k_1!k_2!k_3!\dots}.$$

The special case $d=2$ gives (5) and for $d=3$ we obtain the following counterpart of (6):

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{1}{(1-q^{k+m})^3}$$
$$=\frac{q^m}{1-q^m}\prod_{j=1}^{n}\frac{1-q^j}{1-q^{j+m}}\Big[3+3q^m\tau_1+q^{2m}\Big(\frac{1}{2}\tau_1^2+\tau_2\Big)\Big].$$

Remark 3.1. If m is a positive integer, then

$$\tau_k=\frac{q^{-mk}}{k}\sum_{j=m}^{m+n}\Big(\frac{q^j}{1-q^j}\Big)^k=\frac{q^{-mk}}{k}\Big(\mathscr{H}_{m+n}^{(k)}-\mathscr{H}_{m-1}^{(k)}\Big),$$

where $\mathscr{H}_n^{(k)}$ denotes the q-analogue of the n-th harmonic number of order k.

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