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Schrödinger model and Stratonovich-Weyl correspondence for Heisenberg motion groups

Benjamin Cahen

Abstract. *We introduce a Schrödinger model for the unitary irreducible representations of a Heisenberg motion group and we show that the usual Weyl quantization then provides a Stratonovich-Weyl correspondence.*

Keywords: Stratonovich-Weyl correspondence, Berezin quantization, Berezin transform, Heisenberg motion group, reproducing kernel Hilbert space, coherent states, Schrödinger representation, Bargmann-Fock representation, Segal-Bargmann transform.
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1. Introduction

There are different ways to extend the usual Weyl correspondence between functions on $\mathbb{R}^{2n}$ and operators on $L^2(\mathbb{R}^n)$ to the general setting of a Lie group acting on a homogeneous space [1, 14, 31, 34]. Here we are concerned with Stratonovich-Weyl correspondences. The notion of Stratonovich-Weyl correspondence was introduced in [51] and its systematic study began with the work of J.M. Gracia-Bondìa, J.C. Vàrilly and their co-workers (see [26, 29, 32, 33] and also [12]). The following definition is taken from [32], see also [33].

Definition 1.1. *Let G be a Lie group and π be a unitary representation of G on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G-space and let μ be a G-invariant measure on M. Then a Stratonovich-Weyl correspondence for the triple (G, π, M) is an isomorphism $\mathcal{W}$ from a vector space of operators on $\mathcal{H}$ to a vector space of functions on M satisfying the following properties:*

1. *the function $\mathcal{W}(A^*)$ is the complex-conjugate of $\mathcal{W}(A)$;*
2. *Covariance: we have $\mathcal{W}(\pi(g)\, A\, \pi(g)^{-1})(x) = \mathcal{W}(A)(g^{-1}\cdot x)$;*
3. *Traciality: we have*

$$\int_M \mathcal{W}(A)(x)\mathcal{W}(B)(x)\, d\mu(x) = \mathrm{Tr}(AB).$$

Stratonovich-Weyl correspondences were constructed for various Lie group representations, see [26, 32]. In particular, in [20], Stratonovich-Weyl correspondences for the holomorphic representations of quasi-Hermitian Lie groups were obtained by taking the isometric part in the polar decomposition of the Berezin quantization map, see also [3, 4, 16, 17, 24, 29].

The basic example is the case when G is the $(2n+1)$-dimensional Heisenberg group acting on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ by translations. Each non-degenerate unitary irreducible representation of G has then two classical realizations: the Schrödinger model on $L^2(\mathbb{R}^n)$ and the Bargmann-Fock model on the Fock space [30], an intertwining operator between these realizations being the Segal-Bargmann transform [27, 30]. In this context, it is well-known that the usual Weyl correspondence provides a Stratonovich-Weyl correspondence for the Schrödinger realization [6, 49, 54]. It is also known that this Stratonovich-Weyl correspondence is connected by the Segal-Bargmann transform to the Stratonovich-Weyl correspondence for the Bargmann-Fock realization which was obtained by polarization of the Berezin quantization map [43, 44]. In [22], we obtained similar results for the $(2n+2)$-dimensional real diamond group. This group, also called oscillator group, is a semidirect product of the Heisenberg group by the real line.

The aim of the present paper is to extend the preceding results to the Heisenberg motion groups. An Heisenberg motion group is the semidirect product of the $(2n+1)$-dimensional Heisenberg group H_n by a compact subgroup K of the unitary group $U(n)$. Note that Heisenberg motion groups play an important role in the theory of Gelfand pairs, since the study of a Gelfand pair of the form (K_0, N) where K_0 is a compact Lie group acting by automorphisms on a nilpotent Lie group N can be reduced to that of the form (K_0, H_n), see [8, 9].

More precisely, we introduce a Schrödinger realization for the unitary irreducible representations of a Heisenberg motion group and we prove that we obtain a Stratonovich-Weyl correspondence by combining the usual Weyl correspondence and the unitary part of the Berezin calculus for K.

Let us briefly describe our construction. First notice that each Heisenberg motion group is, in particular, a quasi-Hermitian Lie group and that we can obtain its unitary irreducible representations as holomorphically induced representations on some generalized Fock space by the general method of [46], Chapter XII. Then we can get Schrödinger realizations for these representations by using, as in the case of the Heisenberg group, a (generalized) Bargmann-Fock transform. Hence we obtain a Stratonovich-Weyl correspondence for such a Schrödinger realization by introducing a generalization of the usual Weyl correspondence.

Note that, in [45], a Schrödinger model and a generalized Segal-Bargmann transform for the scalar highest weight representations of an Hermitian Lie group of tube type were introduced and studied. Let us also mentioned that

B. Hall has obtained some generalized Segal-Bargmann transforms in various situations by means of the heat kernel, see [36] and references therein. Then one can hope for futher generalizations of our results to quasi-Hermitian Lie groups.

This paper is organized as follows. In Section 2, we review some well-known facts about the Fock model and the Schrödinger model of the unitary irreducible representations of an Heisenberg group and about the corresponding Berezin calculus and Weyl correspondence. In Section 3, we introduce the Heisenberg motion groups and, in Section 4 and Section 5, we describe their unitary irreducible representations in the Fock model and the associated Berezin calculus. We introduce the (generalized) Segal-Bargmann transform and the Schrödinger model in Section 6. In Section 7, we show that the usual Weyl correspondence also gives a Stratonovich-Weyl correspondence for the Schrödinger model. Moreover, we compare it with the Stratonovich-Weyl correspondence for the Fock model which is directly obtained by polarization of the Berezin quantization map.

2. Heisenberg groups

In this section, we review some well-known results about the the Schrödinger model and the Fock model of the unitary irreducible (non-degenerated) representations of the Heisenberg group. We follow the presentation of [22] in a large extend.

Let G_0 be the Heisenberg group of dimension $2n+1$ and $\mathfrak{g}_0$ be the Lie algebra of G_0. Let $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n,\tilde{Z}\}$ be a basis of $\mathfrak{g}_0$ in which the only non trivial brackets are $[X_k\,,\,Y_k]=\tilde{Z}$, $1\leq k\leq n$ and let

$$\{X_1^*,\ldots,X_n^*,Y_1^*,\ldots,Y_n^*,\tilde{Z}^*\}$$

be the corresponding dual basis of $\mathfrak{g}_0^*$.

For $a=(a_1,a_2,\ldots,a_n)\in\mathbb{R}^n$, $b=(b_1,b_2,\ldots,b_n)\in\mathbb{R}^n$ and $c\in\mathbb{R}$, we denote by $[a,b,c]$ the element $\exp_{G_0}(\sum_{k=1}^n a_kX_k+\sum_{k=1}^n b_kY_k+c\tilde{Z})$ of G_0. Similarly, for $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\mathbb{R}^n$, $\beta=(\beta_1,\beta_2,\ldots,\beta_n)\in\mathbb{R}^n$ and $\gamma\in\mathbb{R}$, we denote by (α,β,γ) the element $\sum_{k=1}^n\alpha_kX_k^*+\sum_{k=1}^n\beta_kY_k^*+\gamma\tilde{Z}^*$ of $\mathfrak{g}_0^*$. The coadjoint action of G_0 is then given by

$$\mathrm{Ad}^*([a,b,c])\,(\alpha,\beta,\gamma)=(\alpha+\gamma\beta,\beta-\gamma\alpha,\gamma).$$

Now we fix a real number $\lambda>0$ and denote by $\mathcal{O}_\lambda$ the orbit of the element $\lambda\tilde{Z}^*$ of $\mathfrak{g}_0^*$ under the coadjoint action of G_0 (the case $\lambda<0$ can be treated similarly). By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) unitary irreducible representation of G_0 whose restriction to the center of G_0 is the character $[0,0,c]\rightarrow e^{i\lambda c}$ [7, 30]. Note that this representation is associated with the coadjoint orbit $\mathcal{O}_\lambda$ by the Kirillov-Kostant method of orbits [41, 42]. More precisely, if we choose the real polarization at $\lambda\tilde{Z}^*$ to be the space spanned by the elements Y_k for $1 \leq k \leq n$ and $\tilde{Z}$ then we obtain the Schrödinger representation σ_0 realized on $L^2(\mathbb{R}^n)$ as

$$\sigma_0([a,b,c])(f)(x) = e^{i\lambda(c-bx+\frac{1}{2}ab)} f(x-a),$$

see [30] for instance. Here we denote $xy := \sum_{k=1}^n x_k y_k$ for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$.

The differential of σ_0 is then given by

$$d\sigma_0(X_k)f(x) = -\partial_k f(x),\ d\sigma_0(Y_k)f(x) = -i\lambda x_k f(x),\ d\sigma_0(\tilde{Z})f(x) = i\lambda f(x)$$

where $k = 1, 2, \ldots, n$.

On the other hand, if we consider the complex polarization at $\lambda\tilde{Z}^*$ to be the space spanned by the elements $X_k + iY_k$ for $1 \leq k \leq n$ and $\tilde{Z}$ then the method of orbits leads to the Bargmann-Fock representation π_0 defined as follows [13].

Let $\mathcal{F}_0$ be the Hilbert space of holomorphic functions F on $\mathbb{C}^n$ such that

$$\|F\|^2_{\mathcal{F}_0} := \int_{\mathbb{C}^n} |F(z)|^2 \, e^{-|z|^2/2\lambda} \, d\mu_\lambda(z) < +\infty$$

where $d\mu_\lambda(z) := (2\pi\lambda)^{-n} \, dx \, dy$. Here $z = x + iy$ with x and y in $\mathbb{R}^n$.

Let us consider the action of G_0 on $\mathbb{C}^n$ defined by $g \cdot z := z + \lambda(b - ia)$ for $g = [a,b,c] \in G_0$ and $z \in \mathbb{C}^n$. Then π_0 is the representation of G_0 on $\mathcal{F}_0$ given by

$$\pi_0(g) \, F(z) = \alpha(g^{-1}, z) \, F(g^{-1} \cdot z)$$

where the map α is defined by

$$\alpha(g,z) := \exp\big(-ic\lambda + (1/4)(b + ai)(-2z + \lambda(-b + ai))\big)$$

for $g = [a,b,c] \in G_0$ and $z \in \mathbb{C}^n$.

The differential of π_0 is then given by

$$\begin{cases} d\pi_0(X_k)F(z) = \dfrac{1}{2} i z_k F(z) + \lambda i \dfrac{\partial F}{\partial z_k} \\ d\pi_0(Y_k)F(z) = \dfrac{1}{2} z_k F(z) - \lambda \dfrac{\partial F}{\partial z_k} \\ d\pi_0(\tilde{Z})F(z) = i\lambda F(z). \end{cases}$$

As in [35, Section 6] or [27, Section 1.3] we can verify by using the previous formulas for $d\pi_0$ and $d\sigma_0$ that the Segal-Bargmann transform $B_0 : L^2(\mathbb{R}^n) \to \mathcal{F}_0$ defined by

$$B_0(f)(z) = (\lambda/\pi)^{n/4} \int_{\mathbb{R}^n} e^{(1/4\lambda)z^2 + ixz - (\lambda/2)x^2} \, f(x) \, dx$$

is a (unitary) intertwining operator between σ_0 and π_0. The inverse Segal-Bargmann transform $B_0^{-1} = B_0^*$ is then given by

$$B_0^{-1}(F)(x) = (\lambda/\pi)^{n/4} \int_{\mathbb{C}^n} e^{(1/4\lambda)\bar{z}^2 - ix\bar{z} - (\lambda/2)x^2}\, F(z)\, e^{-|z|^2/2\lambda}\, d\mu_\lambda(z).$$

For $z \in \mathbb{C}^n$, consider the coherent state $e_z(w) = \exp(\bar{z}w/2\lambda)$. Then we have the reproducing property $F(z) = \langle F, e_z\rangle_{\mathcal{F}_0}$ for each $F \in \mathcal{F}_0$ where $\langle \cdot, \cdot\rangle_{\mathcal{F}_0}$ denotes the scalar product on $\mathcal{F}_0$.

Now, we introduce the Berezin quantization map and we review some of its properties. Let $\mathcal{C}_0$ be the space of all operators (not necessarily bounded) A on $\mathcal{F}_0$ whose domain contains e_z for each $z \in \mathbb{C}^n$. Then the Berezin symbol of $A \in \mathcal{C}_0$ is the function $S^0(A)$ defined on $\mathbb{C}^n$ by

$$S^0(A)(z) := \frac{\langle A\, e_z\, ,\, e_z\rangle_{\mathcal{F}_0}}{\langle e_z\, ,\, e_z\rangle_{\mathcal{F}_0}}.$$

We have the following result, see for instance [22].

Proposition 2.1. *1. Each $A \in \mathcal{C}_0$ is determined by $S^0(A)$;*

2. *For each $A \in \mathcal{C}_0$ and each $z \in \mathbb{C}^n$, we have $S^0(A^*)(z) = \overline{S^0(A)(z)}$;*
3. *For each $z \in \mathbb{C}^n$, we have $S^0(I_{\mathcal{F}_0})(z) = 1$. Here $I_{\mathcal{F}_0}$ denotes the identity operator of $\mathcal{F}_0$;*
4. *For each $A \in \mathcal{C}_0$, $g \in G_0$ and $z \in \mathbb{C}^n$, we have $\pi_0(g)^{-1}A\pi_0(g) \in \mathcal{C}_0$ and*
$$S^0(A)(g \cdot z) = S^0(\pi_0(g)^{-1}A\pi_0(g))(z);$$
5. *The map S^0 is a bounded operator from $\mathcal{L}_2(\mathcal{F}_0)$ (endowed with the Hilbert-Schmidt norm) to $L^2(\mathbb{C}^n, \mu_\lambda)$ which is one-to-one and has dense range.*

Proof. For 1 and 2, see [10] and [25]. Note that 4 follows from the following property: For each $g \in G_0$ and each $z \in \mathbb{C}^n$, we have $\pi_0(g)e_z = \overline{\alpha(g,z)}e_{g\cdot z}$, see [20]. Finally, 5 is a particular case of [52, Proposition 1.19]. □

Recall that the Berezin transform is then the operator $\mathcal{B}^0$ on $L^2(\mathbb{C}^n, \mu_\lambda)$ defined by $\mathcal{B}^0 = S^0(S^0)^*$. Thus we have the integral formula

$$\mathcal{B}^0(F)(z) = \int_{\mathbb{C}^n} F(w)\, e^{|z-w|^2/2\lambda}\, d\mu_\lambda(w),$$

see [10, 11, 48, 52] for instance. Recall also that we have $\mathcal{B}^0 = \exp(\lambda\Delta/2)$ where $\Delta = 4\sum_{k=1}^n \partial^2/\partial z_k \partial\bar{z}_k$, see [44, 52].

Note that Berezin transforms have been studied, in the general setting, by many authors, see in particular [28, 47, 48, 52, 56].

Note also that S^0 allows us to connect π_0 to $\mathcal{O}_\lambda$ as shown by the following proposition. Here we denote by $\mathfrak{g}_0^c$ the complexification of $\mathfrak{g}_0$.

PROPOSITION 2.2 ([22]). *Let Φ_λ be the map defined by*

$$\Phi_\lambda(z) := \sum_{k=1}^{n} (\operatorname{Re} z_k X_k^* + \operatorname{Im} z_k Y_k^*) + \lambda \tilde{Z}^*.$$

Then

1. *For each $X \in \mathfrak{g}_0^c$ and each $z \in \mathbb{C}^n$, we have*
$$S^0(d\pi_0(X))(z) = i\langle \Phi_\lambda(z), X\rangle.$$
2. *For each $g \in G_0$ and each $z \in \mathbb{C}^n$, we have $\Phi_\lambda(g \cdot z) = \operatorname{Ad}^*(g)\,\Phi_\lambda(z)$.*
3. *The map Φ_λ is a diffeomorphism from $\mathbb{C}^n$ onto $\mathcal{O}_\lambda$.*

Now we aim to transfer S^0 to operators on $L^2(\mathbb{R}^n)$. To this goal, we define $S^1(A) := S^0(B_0 A B_0^{-1})$ for A operator on $L^2(\mathbb{R}^n)$. Of course, the properties of S^0 give rise to similar properties of S^1. In particular, S^1 is a bounded operator from $\mathcal{L}_2(L^2(\mathbb{R}^n))$ to $L^2(\mathbb{C}^n, \mu_\lambda)$ and S^1 is G_0-covariant with respect to σ_0.

Moreover, denoting by I_{B_0} the (unitary) map from $\mathcal{L}_2(L^2(\mathbb{R}^n))$ onto $\mathcal{L}_2(\mathcal{F}_0)$ defined by $I_{B_0}(A) = B_0 A B_0^{-1}$, we have $S^1 = S^0 I_{B_0}$ then

$$S^1(S^1)^* = (S^0 I_{B_0})(S^0 I_{B_0})^* = S^0 I_{B_0} I_{B_0}^* (S^0)^* = S^0(S^0)^* = \mathcal{B}^0.$$

This shows that the Berezin transform corresponding to S^1 is the same as the Berezin transform corresponding to S^0. Then we can write the polar decompositions of S^0 and S^1 as $S^0 = (\mathcal{B}^0)^{1/2} U^0$ and $S^1 = (\mathcal{B}^0)^{1/2} U^1$ where the maps $U^0 : \mathcal{L}_2(\mathcal{F}_0) \to L^2(\mathbb{C}^n, \mu_\lambda)$ and $U^1 : \mathcal{L}_2(L^2(\mathbb{R}^n)) \to L^2(\mathbb{C}^n, \mu_\lambda)$ are unitary.

Moreover, as in the proof of [17], Proposition 3.1, we can verify that U^0 is a Stratonovich-Weyl correspondence for $(G_0, \pi_0, \mathbb{C}^n)$ and that U^1 is a Stratonovich-Weyl correspondence for $(G_0, \sigma_0, \mathbb{C}^n)$. Note that G_0-covariance of U^0 and U^1 immediately follows from G_0-covariance of S^0 and S^1. Note also that we have $U^1 = U^0 I_{B_0}$.

Now, we show how to use the usual Weyl correspondence in order to get another Stratonovich-Weyl correspondence for σ_0. The Weyl correspondence on $\mathbb{R}^{2n}$ is defined as follows. For each f in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, let $W_0(f)$ be the operator on $L^2(\mathbb{R}^n)$ defined by

$$W_0(f)\phi(p) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{isq}\, f(p + (1/2)s, q)\, \phi(p+s)\, ds\, dq.$$

The Weyl calculus can be extended to much larger classes of symbols (see for instance [38]). In particular, if $f(p,q) = u(p)q^\alpha$ where $u \in C^\infty(\mathbb{R}^n)$ then we have, see [53],

$$W_0(f)\varphi(p) = \left(i\frac{\partial}{\partial s} \right)^{\alpha} (u(p + (1/2)s)\, \phi(p+s)) \Big|_{s=0}.$$

From this, we can deduce the following proposition. Consider the action of G_0 on $\mathbb{R}^{2n}$ given by $g \cdot (p,q) := (p+a, q+\lambda b)$ where $g = [a,b,c]$.

Proposition 2.3 ([22]). *Let Ψ_λ be the map defined by*

$$\Psi_\lambda(p,q) := \sum_{k=1}^{n} (q_k X_k^* - \lambda p_k Y_k^*) + \lambda \tilde{Z}^*.$$

Then

1. *For each $X \in \mathfrak{g}_0^c$ and each $(p,q) \in \mathbb{R}^{2n}$, we have*

$$W_0^{-1}(d\sigma_0(X))(p,q) = i\langle \Psi_\lambda(p,q), X\rangle.$$

2. *For each $g \in G_0$ and $(p,q) \in \mathbb{R}^{2n}$, we have $\Psi_\lambda(g\cdot(p,q)) = \mathrm{Ad}^*(g)\,\Psi_\lambda(p,q)$.*

3. *The map Ψ_λ is a diffeomorphism from $\mathbb{R}^{2n}$ onto $\mathcal{O}_\lambda$.*

4. *For each $(p,q) \in \mathbb{R}^{2n}$, we have $\Phi_\lambda(q - \lambda p i) = \Psi_\lambda(p,q)$.*

Now, we assume that $\mathbb{R}^{2n}$ is equipped with the G_0-invariant measure $\tilde{\mu} := (2\pi)^{-n} dpdq$. Then one has the following result.

Proposition 2.4 ([22, 30]). *The map W_0^{-1} is a Stratonovich-Weyl correspondence for $(G_0, \sigma_0, \mathbb{R}^{2n})$.*

The following proposition asserts that if we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by the map $j : (p,q) \to q - \lambda p i$ then the unitary part in the polar decomposition of S^1 coincides with the inverse of the Weyl transform, see [44] and [48].

Proposition 2.5. *Let J be the map from $L^2(\mathbb{C}^n, \mu_\lambda)$ onto $L^2(\mathbb{R}^{2n})$ defined by $J(F) = F \circ j$. Then we have $U^1 = (W_0 J)^{-1}$.*

Finally, note that we can obtain Stratonovich-Weyl correspondences for $(G_0, \sigma_0, \mathcal{O}_\lambda)$ and $(G_0, \pi_0, \mathcal{O}_\lambda)$ by transferring W_0^{-1} and U^0 by using Φ_λ and Ψ_λ. More precisely, let ν_λ be the G_0-invariant measure on $\mathcal{O}_\lambda$ defined by $\nu_\lambda := (\Phi_\lambda^{-1})^*(\mu_\lambda) = (\Psi_\lambda^{-1})^*(\tilde{\mu})$. Then the maps $\tau_{\Phi_\lambda} : F \to F \circ \Phi_\lambda^{-1}$ from $L^2(\mathbb{C}^n, \mu_\lambda)$ onto $L^2(\mathcal{O}_\lambda, \nu_\lambda)$ and $\tau_{\Psi_\lambda} : F \to F \circ \Psi_\lambda^{-1}$ from $L^2(\mathbb{R}^{2n})$ onto $L^2(\mathcal{O}_\lambda, \nu_\lambda)$ are unitary and we have $\tau_{\Phi_\lambda} = \tau_{\Psi_\lambda} J$. Hence we can assert the following proposition.

Proposition 2.6. *The map $\mathcal{W}_1 := \tau_{\Psi_\lambda} W_0^{-1}$ is a Stratonovich-Weyl correspondence for $(G_0, \sigma_0, \mathcal{O}_\lambda)$, the map $\mathcal{W}_2 := \tau_{\Phi_\lambda} U^0$ is a Stratonovich-Weyl correspondence for $(G_0, \pi_0, \mathcal{O}_\lambda)$ and we have $\mathcal{W}_1 = \mathcal{W}_2 I_{B_0}$.*

3. Generalities on Heisenberg motion groups

In order to introduce the Heisenberg motion groups, it is convenient to write the elements of the Heisenberg group G_0 and its multiplication law as follows. For each $z \in \mathbb{C}^n$, $c \in \mathbb{R}$, we denote here by $(z, \bar{z}, c)$ the element G_0 which is denoted by $[\operatorname{Re} z, \operatorname{Im} z, c]$ in Section 2. Moreover, for each z, $w \in \mathbb{C}^n$, we denote $zw := \sum_{k=1}^n z_k w_k$ and we consider the symplectic form ω on $\mathbb{C}^{2n}$ defined by

$$\omega((z,w),(z',w')) = \frac{i}{2}(zw' - z'w).$$

for $z, w, z', w' \in \mathbb{C}^n$. Then the multiplication of G_0 is given by

$$((z,\bar{z}),c) \cdot ((z',\bar{z}'),c') = ((z+z', \bar{z}+\bar{z}'), c+c' + \tfrac{1}{2}\omega((z,\bar{z}),(z',\bar{z}'))), \qquad (1)$$

the complexification G_0^c of G_0 is $G_0^c = \{((z,w),c) \,:\, z, w \in \mathbb{C}^n, c \in \mathbb{C}\}$ and the multiplication of G_0^c is obtained by replacing $(z,\bar{z})$ by (z,w) and $(z',\bar{z}')$ by (z',w') in Eq. 1.

Now, let K be a closed subgroup of $U(n)$. Then K acts on G_0 by $k \cdot ((z,\bar{z}),c) = ((kz, \bar{kz}), c)$ and we can form the semidirect product $G := G_0 \rtimes K$ which is called a Heisenberg motion group. The elements of G can be written as $((z,\bar{z}),c,k)$ where $z \in \mathbb{C}^n$, $c \in \mathbb{R}$ and $k \in K$. The multiplication of G is then given by

$$\begin{aligned}&((z,\bar{z}),c,k) \cdot ((z',\bar{z}'),c',k') = \\ &\qquad ((z,\bar{z}) + (kz', \bar{kz'}), c+c' + \tfrac{1}{2}\omega((z,\bar{z}),(kz',\bar{kz'})), kk').\end{aligned}$$

We denote by K^c the complexification of K and we consider the action of K^c on $\mathbb{C}^n \times \mathbb{C}^n$ given by $k \cdot (z,w) = (kz, (k^t)^{-1}w)$ (here, the subscript t denotes transposition). The group G^c is then the semidirect product $G^c = G_0^c \rtimes K^c$. The elements of G^c can be written as $((z,w),c,k)$ where z, $w \in \mathbb{C}^n$, $c \in \mathbb{C}$ and $k \in K^c$ and the multiplication law of G^c is given by

$$\begin{aligned}&((z,w),c,k) \cdot ((z',w'),c',k') = \\ &\qquad ((z,w) + k\cdot(z',w'), c+c' + \tfrac{1}{2}\omega((z,w),k\cdot(z',w')), kk').\end{aligned}$$

We denote by $\mathfrak{k}$, $\mathfrak{k}^c$, $\mathfrak{g}$ and $\mathfrak{g}^c$ the Lie algebras of K, K^c, G and G^c. The derived action of $\mathfrak{k}^c$ on $\mathbb{C}^n \times \mathbb{C}^n$ is then $A \cdot (z,w) := (Az, -A^t w)$ and the Lie brackets of $\mathfrak{g}^c$ are given by

$$\begin{aligned}&[((z,w),c,A),((z',w'),c',A')] = \\ &\qquad (A\cdot(z',w') - A'\cdot(z,w), \omega((z,w),(z',w')), [A,A']).\end{aligned}$$

Let $\tilde{K}$ be the subgroup of G defined by $\tilde{K} := \{((0,0),c,k) \,:\, c \in \mathbb{R},\, k \in K\}$. Also, let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{k}$. Then the Lie algebra $\tilde{\mathfrak{k}}$ of $\tilde{K}$ is a maximal compactly embedded subalgebra of $\mathfrak{g}$ and the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ consisting of all elements of the form $((0,0),c,A)$ where $c \in \mathbb{R}$ and $A \in \mathfrak{h}_0$ is a compactly embedded Cartan subalgebra of $\mathfrak{g}$ [46], p. 250.

Following [46, Chapter XII.1], we set $\mathfrak{p}^+ = \{((z,0),0,0) \,:\, z \in \mathbb{C}^n\}$ and $\mathfrak{p}^- = \{((0,w),0,0) \,:\, w \in \mathbb{C}^n\}$ and we denote by P^+ and P^- the corresponding analytic subgroups of G^c, that is, $P^+ = \{((z,0),0,I_n) \,:\, z \in \mathbb{C}^n\}$ and $P^- = \{((0,w),0,I_n) \,:\, w \in \mathbb{C}^n\}$.

Note that G is a group of the Harish-Chandra type [46, p. 507] (see also [50] and [37, Chapter VIII]), that is, the following properties are satisfied:

1. $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \tilde{\mathfrak{k}}^c \oplus \mathfrak{p}^-$ is a direct sum of vector spaces, $(\mathfrak{p}^+)^* = \mathfrak{p}^-$ and $[\tilde{\mathfrak{k}}^c, \mathfrak{p}^{\pm}] \subset \mathfrak{p}^{\pm}$;
2. The multiplication map $P^+\tilde{K}^cP^- \to G^c$, $(z,k,y) \to zky$ is a biholomorphic diffeomorphism onto its open image;
3. $G \subset P^+\tilde{K}^cP^-$ and $G \cap \tilde{K}^cP^- = \tilde{K}$.

We denote by $p_{\mathfrak{p}^+}$, $p_{\tilde{\mathfrak{k}}^c}$ and $p_{\mathfrak{p}^-}$ the projections of $\mathfrak{g}^c$ onto $\mathfrak{p}^+$, $\tilde{\mathfrak{k}}^c$ and $\mathfrak{p}^-$ associated with the above direct decomposition.

We can easily verify that each $g = ((z_0,w_0),c_0,k) \in G^c$ has a $P^+\tilde{K}^cP^-$-decomposition given by

$$g = ((z_0,0),0,I_n) \cdot ((0,0),c,k) \cdot ((0,w_0),0,I_n)$$

where $c = c_0 - \frac{i}{4} z_0 w_0$. We denote by $\zeta : P^+\tilde{K}^cP^- \to P^+$, $\kappa : P^+\tilde{K}^cP^- \to K^c$ and $\eta : P^+\tilde{K}^cP^- \to P^-$ the projections onto P^+-, $\tilde{K}^c$- and P^--components.

We can introduce an action (defined almost everywhere) of G on $\mathfrak{p}^+$ as follows. For $Z \in \mathfrak{p}^+$ and $g \in G^c$, we define $g \cdot Z \in \mathfrak{p}^+$ by $g \cdot Z := \log \zeta(g \exp Z)$. From the above formula for the $P^+\tilde{K}^cP^-$-decomposition, we deduce that if $g = ((z_0,w_0),c_0,k) \in G$ and $Z = ((z,0),0,0) \in \mathfrak{p}^+$ then we have $g \cdot Z = \log \zeta(g \exp Z) = ((z_0 + kz, 0),0,0)$. Note that $\mathcal{D} := G \cdot 0 = \mathfrak{p}^+ \simeq \mathbb{C}^n$ here.

A useful section $Z \to g_Z$ for the action of G on $\mathcal{D}$ can be obtained by using [21, Proposition 4.5]. Here we get $g_Z = ((z,\bar{z}),0,I_n)$ for each $Z = ((z,0),0,0)$, $z \in \mathbb{C}^n$.

Now we compute the adjoint and coadjoint actions of G^c. Consider $g = (v_0,c_0,k_0) \in G^c$ where $v_0 \in \mathbb{C}^{2n}$, $c_0 \in \mathbb{C}$, $k_0 \in K^c$ and $X = (w,c,A) \in \mathfrak{g}^c$ where $w \in \mathbb{C}^{2n}$, $c \in \mathbb{C}$ and $A \in \mathfrak{k}^c$. We can easily verify that

$$\begin{aligned}\mathrm{Ad}(g)X = \frac{d}{dt}(g\exp(tX)g^{-1})|_{t=0} = \Big(&k_0 w - (\mathrm{Ad}(k_0)A)\cdot v_0, c \\ &+ \omega(v_0, k_0 w) - \tfrac{1}{2}\omega(v_0, (\mathrm{Ad}(k_0)A)\cdot v_0), \mathrm{Ad}(k_0)A\Big).\end{aligned}$$

Now, let us denote by $\xi = (u, d, \phi)$, where $u \in \mathbb{C}^{2n}$, $d \in \mathbb{C}$ and $\phi \in (\mathfrak{k}^c)^*$, the element of $(\mathfrak{g}^c)^*$ defined by

$$\langle \xi, (w, c, A) \rangle = \omega(u, w) + dc + \langle \phi, A \rangle.$$

Also, for $u, v \in \mathbb{C}^{2n}$, we denote by $v \times u$ the element of $(\mathfrak{k}^c)^*$ defined by $\langle v \times u, A \rangle := \omega(u, A \cdot v)$ for $A \in \mathfrak{k}^c$. Then, from the above formula for the adjoint action, we deduce that for each $\xi = (u, d, \phi) \in (\mathfrak{g}^c)^*$ and $g = (v_0, c_0, k_0) \in G^c$ we have

$$\mathrm{Ad}^*(g)\xi = \left(k_0 u - d v_0, d, \mathrm{Ad}^*(k_0)\phi + v_0 \times (k_0 u - \tfrac{d}{2} v_0)\right)$$

By restriction, we also get the analogous formula for the coadjoint action of G. From this, we see that if a coadjoint orbit of G contains a point (u, d, ϕ) with $d \neq 0$ then it also contains a point of the form $(0, d, \phi_0)$. Such an orbit is called *generic*.

4. Fock model for Heisenberg motion groups

In this section, we introduce the Fock model of the unitary irreducible representations of G by using the general method of [46, Chapter XII] that we describe here briefly.

Let ρ be a unitary irreducible representation of K on a (finite-dimensional) Hilbert space V and $\lambda \in \mathbb{R}$. Let $\tilde{\rho}$ be the representation of $\tilde{K}$ on V defined by $\tilde{\rho}((0,0), c, k) = e^{i\lambda c}\rho(k)$ for each $c \in \mathbb{R}$ and $k \in K$.

For each $Z, W \in \mathcal{D}$, let $K(Z, W) := \tilde{\rho}(\kappa(\exp W^* \exp Z))^{-1}$ and for each $g \in G$, $Z \in \mathcal{D}$, let $J(g, Z) := \tilde{\rho}(\kappa(g \exp Z))$, [46, Chapter XII.1]. Consider the Hilbert space $\tilde{\mathcal{F}}$ of all holomorphic functions on $\mathcal{D}$ with values in V such that

$$\|f\|^2_{\tilde{\mathcal{F}}} := \int_{\mathcal{D}} \langle K(Z, Z)^{-1} f(Z), f(Z) \rangle_V \, d\mu(Z) < +\infty$$

where μ denotes an invariant G-measure on $\mathcal{D}$. Then the equation

$$\tilde{\pi}(g) f(Z) = J(g^{-1}, Z)^{-1} \, f(g^{-1} \cdot Z)$$

defines a unitary representation of G on $\tilde{\mathcal{F}}$. This representation can be also obtained by holomorphic induction from $\tilde{\rho}$, that is, it corresponds to the natural action of G on the square-integrable holomorphic sections of the Hilbert G-bundle $G \times_{\tilde{\rho}} V$ over $G/K \cong \mathcal{D}$ [22]. Note also that $\tilde{\pi}$ is irreducible since $\tilde{\rho}$ is irreducible, [46, p. 515].

Here we can easily compute K and J. For each $Z = ((z, 0), 0, 0)$, $W = ((w, 0), 0, 0) \in \mathcal{D}$, we have $K(Z, W) = e^{\lambda z \bar{w}/2} I_V$. Moreover, for each $g = ((z_0, \bar{z}_0), c_0, k) \in G$ and $Z = ((z, 0), 0, 0) \in \mathcal{D}$, we have

$$J(g, Z) = \exp\left(i\lambda c_0 + \tfrac{\lambda}{2} \bar{z}_0 (kz) + \tfrac{\lambda}{4} |z_0|^2\right) \, \rho(k).$$

Note that μ can be taken to be the G-invariant measure on $\mathcal{D} \simeq \mathbb{C}^n$ defined by $d\mu(Z) := \lambda^n (2\pi)^{-n}\, dx\, dy$. Here $Z = ((z,0),0,0)$ and $z = x+iy$ with x and y in $\mathbb{R}^n$. From now on, we identify $Z = ((z,0),0,0) \in \mathcal{D}$ with $z \in \mathbb{C}^n$ and each function on $\mathcal{D}$ with the corresponding function on $\mathbb{C}^n$.

Consequently, the Hilbert product on $\tilde{\mathcal{F}}$ is given by

$$\langle f, g\rangle_{\tilde{\mathcal{F}}} = \int_{\mathbb{C}^n} \langle f(z), g(z)\rangle_V\, e^{-\lambda |z|^2/2} \left(\frac{\lambda}{2\pi}\right)^n dx\, dy$$

and we get the following formula for $\tilde{\pi}$:

$$(\tilde{\pi}(g)f)(z) = \exp\left(i\lambda c_0 + \tfrac{\lambda}{2}\bar{z}_0 z - \tfrac{\lambda}{4}|z_0|^2\right)\, \rho(k)\, f(k^{-1}(z - z_0))$$

where $g = ((z_0, \bar{z}_0), c_0, k) \in G$ and $z \in \mathbb{C}^n$.

In fact, in order to use the results of Section 2, it is convenient to replace $\tilde{\pi}$ by an equivalent representation π whose restriction to G_0 is precisely π_0. To this aim, we consider the Fock space $\mathcal{F}$ of all holomorphic functions $f : \mathbb{C}^n \to V$ such that

$$\|f\|^2_{\mathcal{F}} := \int_{\mathbb{C}^n} \|f(z)\|^2_V\, e^{-|z|^2/2\lambda}\, d\mu_\lambda(z) < +\infty.$$

Let $\mathcal{J} : \tilde{\mathcal{F}} \to \mathcal{F}$ be the unitary operator defined by $\mathcal{J}(f)(z) = f(i\lambda^{-1}z)$ and set $\pi(g) := \mathcal{J}\tilde{\pi}(g)\mathcal{J}^{-1}$ for each $g \in G$. Then we have

$$(\pi(g)f)(z) = \exp\left(i\lambda c_0 + \tfrac{1}{2} i\bar{z}_0 z - \tfrac{\lambda}{4}|z_0|^2\right)\, \rho(k)\, f(k^{-1}(z + i\lambda z_0))$$

where $g = ((z_0, \bar{z}_0), c_0, k) \in G$ and $z \in \mathbb{C}^n$.

We can easily compute the differential of π:

Proposition 4.1. *Let $X = ((a, \bar{a}), c, A) \in \mathfrak{g}$. Then, for each $f \in \mathcal{F}$ and each $z \in \mathbb{C}^n$, we have*

$$(d\pi(X)f)(z) = d\rho(A)f(z) + i(\lambda c + \tfrac{1}{2}\bar{a}z)f(z) + df_z(-Az + i\lambda a).$$

Clearly, one has $\mathcal{F} = \mathcal{F}_0 \otimes V$. For $f_0 \in \mathcal{F}_0$ and $v \in V$, we denote by $f_0 \otimes v$ the function $z \to f_0(z)v$. Moreover, if A_0 is an operator of $\mathcal{F}_0$ and A_1 is an operator of V then we denote by $A_0 \otimes A_1$ the operator of $\mathcal{F}$ defined by $(A_0 \otimes A_1)(f_0 \otimes v) = A_0 f_0 \otimes A_1 v$.

Let τ be the left-regular representation of K on $\mathcal{F}_0$, that is, $(\tau(k)f_0)(z) = f_0(k^{-1}z)$. Then we have

$$\pi((z_0, \bar{z}_0), c_0, k) = \pi_0((z_0, \bar{z}_0), c_0)\tau(k) \otimes \rho(k) \tag{2}$$

for each $z_0 \in \mathbb{C}^n$, $c_0 \in \mathbb{R}$ and $k \in K$. Note that this is precisely Formula (3.18) in [8].

5. Stratonovich-Weyl correspondence via Berezin quantization

In this section, we introduce the Berezin quantization map associated with π and the corresponding Stratonovich-Weyl correspondence. We consider first the Berezin quantization map associated with ρ [5, 15, 55].

Let us fix a positive root system of $\mathfrak{k}$ relative to $\mathfrak{h}_0$ and denote by $\Lambda \in (\mathfrak{h}_0^c)^*$ the highest weight of ρ and by $\mathfrak{k}^c = \mathfrak{n}^+ \oplus \mathfrak{h}_0^c \oplus \mathfrak{n}^-$ the corresponding triangular decomposition of $\mathfrak{k}^c$. Let $\tilde{\varphi_0}$ be the element of $(\mathfrak{k}^c)^*$ defined by $\tilde{\varphi_0} = -i\Lambda$ on $\mathfrak{h}_0$ and by $\tilde{\varphi_0} = 0$ on $\mathfrak{n}^{\pm}$. We denote by φ_0 the restriction of $\tilde{\varphi_0}$ to $\mathfrak{k}$. Then the orbit $o(\varphi_0)$ of φ_0 under the coadjoint action of K is said to be associated with ρ [14, 55].

Here we assume that φ_0 is regular in the sense that the stabilizer of φ_0 for the coadjoint action of K is precisely the connected subgroup H_0 of K with Lie algebra $\mathfrak{h}_0$ [15].

Note that a complex structure on $o(\varphi_0)$ is then defined by the diffeomorphism $o(\varphi_0) \simeq K/H_0 \simeq K^c/H_0^c N^-$ where H_0 is the connected subgroup of K with Lie algebra $\mathfrak{h}_0$ and N^- is the analytic subgroup of K^c with Lie algebra $\mathfrak{n}^-$.

Without loss of generality, we can assume that V is a space of holomorphic sections of a complex line bundle over $o(\varphi_0)$ as in [15]. For each $\varphi \in o(\varphi_0)$ there exists a unique function $e_\varphi \in V$ (a coherent state) such that $a(\varphi) = \langle a, e_\varphi \rangle_V$ for each $a \in V$. The Berezin calculus on $o(\varphi_0)$ associates with each operator B on V the complex-valued function $s(B)$ on $o(\varphi_0)$ defined by

$$s(B)(\varphi) = \frac{\langle Be_\varphi, e_\varphi \rangle_V}{\langle e_\varphi, e_\varphi \rangle_V}$$

which is called the symbol of B. We denote by $Sy(o(\varphi_0))$ the space of all such symbols. Then we have the following proposition, see [5, 15, 25].

PROPOSITION 5.1. *1. The map $B \to s(B)$ is injective.*

2. For each operator B on V, we have $s(B^*) = \overline{s(B)}$.

3. For each $\varphi \in o(\varphi_0)$, $k \in K$ and $B \in \mathrm{End}(V)$, we have

$$s(B)(\mathrm{Ad}^*(k)\varphi) = s(\rho(k)^{-1} B \rho(k))(\varphi).$$

4. For each $A \in \mathfrak{k}$ and $\varphi \in o(\varphi_0)$, we have $s(d\rho(A))(\varphi) = i\langle \varphi, A \rangle$.

In our papers [18, 19, 23], we developped a general method for constructing a Berezin quantization map associated with a unitary representation of a quasi-Hermitian Lie group which is holomorphically induced from a unitary irreducible representation of a maximal compactly embedded subgroup. This construction goes as follows.

The evaluation maps $K_z : \mathcal{H} \to V, f \to f(z)$ are continuous [46], p. 539. The vector coherent states of $\mathcal{F}$ are the maps $E_z = K_z^* : V \to \mathcal{F}$ defined by $\langle f(z), v\rangle_V = \langle f, E_z v\rangle_{\mathcal{F}}$ for $f \in \mathcal{F}$ and $v \in V$. Here we have that $E_z v = e_z \otimes v$, that is, we have $(E_z v)(w) = e^{\lambda \bar{z} w/2} v$.

Let $\mathcal{F}^s$ be the subspace of $\mathcal{F}$ generated by the functions $e_z \otimes v$ for $z \in \mathbb{C}^n$ and $v \in V$. Then $\mathcal{F}^s$ is a dense subspace of $\mathcal{F}$. Let $\mathcal{C}$ be the space consisting of all operators A on $\mathcal{F}$ such that the domain of A contains $\mathcal{F}^s$ and the domain of A^* also contains $\mathcal{F}^s$. Then, following an idea of [40] and [2], we first introduce the pre-symbol $S_0(A)$ of $A \in \mathcal{C}$ by

$$S_0(A)(z) = (E_z^* E_z)^{-1/2} E_z^* A E_z (E_z^* E_z)^{-1/2} = e^{-\lambda z \bar{z}/2} E_z^* A E_z.$$

The Berezin symbol $S(A)$ of A is thus defined as the complex-valued function on $\mathbb{C}^n \times o(\varphi_0)$ given by

$$S(A)(z, \varphi) = s(S_0(A)(z))(\varphi).$$

By applying [23, Proposition 4.4] we can see that S has the following properties.

Proposition 5.2. *1. Each $A \in \mathcal{C}$ is determined by $S(A)$.*

2. For each $A \in \mathcal{C}$, we have $S(A^*) = \overline{S(A)}$.

3. We have $S(I_{\mathcal{F}}) = 1$.

4. For each $A \in \mathcal{C}$, $g = ((z_0, \bar{z}_0), c, k) \in G$, $z \in \mathbb{C}^n$ and $\varphi \in o(\varphi_0)$, we have

$$S(A)(g \cdot z, \varphi) = S(\pi(g)^{-1} A \pi(g))(z, \mathrm{Ad}^*(k^{-1})\varphi).$$

Moreover, we can decompose S according to the decomposition $\mathcal{F} = \mathcal{F}_0 \otimes V$. Let f_0 be a complex-valued function on $\mathbb{C}^n$ and f_1 be a complex-valued function on $o(\varphi_0)$. Then we denote by $f_0 \otimes f_1$ the function on $\mathbb{C}^n \times o(\varphi_0)$ defined by $(f_0 \otimes f_1)(z, \varphi) = f_0(z) f_1(\varphi)$.

Proposition 5.3. *Let $A_0 \in \mathcal{C}_0$ and let A_1 be an operator on V. Then $A_0 \otimes A_1 \in \mathcal{C}$ and we have $S(A_0 \otimes A_1) = S^0(A_0) \otimes s(A_1)$.*

From this, we deduce the following result. We denote by φ^0 the restriction to $\mathfrak{g}$ of the extension of $\tilde{\varphi}_0 \in (\mathfrak{k}^c)^*$ to $\mathfrak{g}^c$ which vanishes on $\mathfrak{p}^{\pm}$. We also denote by $\mathcal{O}(\varphi^0)$ the orbit of φ^0 for the coadjoint action of G.

Proposition 5.4 ([23]). *1. Let $g = ((z_0, \bar{z}_0), c_0, k) \in G$. For each $z \in \mathbb{C}^n$ and $\varphi \in o(\varphi_0)$, we have*

$$S(\pi(g))(z, \varphi) = \exp\left(i\lambda c_0 + \tfrac{1}{2} i \bar{z}_0 z - \tfrac{\lambda}{4}|z_0|^2 - \tfrac{\lambda}{2}|z|^2 + \tfrac{\lambda}{2}\bar{z} k^{-1}(z + i\lambda z_0)\right) \times s(\rho(k))(\varphi).$$

2. For each $X=((a,\bar{a}),c,A)\in\mathfrak{g}$, $z\in\mathbb{C}^n$ and $\varphi\in o(\varphi_0)$, we have

$$S(d\pi(X))(z,\varphi)=i\lambda c+\frac{i}{2}\left(\bar{a}z+\lambda^2 a\bar{z}\right)-\frac{\lambda}{2}\bar{z}(Az)+s(d\rho(A))(\varphi).$$

3. For each $X=((a,\bar{a}),c,A)\in\mathfrak{g}$, $z\in\mathbb{C}^n$ and $\varphi\in o(\varphi_0)$, we have

$$S(d\pi(X))(z,\varphi)=i\langle\Phi(z,\varphi),X\rangle$$

where the map $\Phi:\mathbb{C}^n\times o(\varphi_0)\to\mathfrak{g}^*$ is defined by

$$\Phi(z,\varphi)=\left(i(-z,\lambda^2\bar{z}),\lambda,\varphi-\tfrac{\lambda}{2}(z,\bar{z})\ \times(z,\bar{z})\right).$$

Moreover Φ is a diffeomorphism from $\mathbb{C}^n\times o(\varphi_0)$ onto $\mathcal{O}(\varphi^0)$.

Consider now the Berezin transforms $\mathcal{B}:=SS^*$, $\mathcal{B}^0:=S^0(S^0)^*$, $b:=ss^*$ and the corresponding maps $U:=\mathcal{B}^{-1/2}S$, $U^0:=(\mathcal{B}^0)^{-1/2}S^0$ and $w:=b^{-1/2}s$. We fix a K-invariant measure ν on $o(\varphi_0)$ and we endow $\mathbb{C}^n\times o(\varphi_0)$ with the measure $\mu_\lambda\otimes\nu$. Also, we consider the action of G on $\mathbb{C}^n\times o(\varphi_0)$ given by

$$g\cdot(z,\varphi):=(g\cdot z,\mathrm{Ad}^*(k)\varphi)$$

for $g=((z_0,\bar{z}_0),c_0,k)\in G$. Then we have the following results.

Proposition 5.5 ([23]). *The map U is a Stratonovich-Weyl correspondence for $(G,\pi,\mathbb{C}^n\times o(\varphi_0))$.*

Proposition 5.6 ([23]). *For each $f\in L^2(\mathbb{C}^n\times o(\varphi_0),\mu_\lambda\otimes\nu)$, we have*

$$\mathcal{B}(f)(z,\psi)=\int_{\mathbb{C}^n\times o(\varphi_0)}k_{\mathcal{B}}(z,w,\psi,\varphi)\,f(w,\varphi)\,d\mu_\lambda(w)d\nu(\varphi)$$

where

$$k_{\mathcal{B}}(z,w,\psi,\varphi):=e^{-\lambda|z-w|^2/2}\,\frac{|\langle e_\psi,e_\varphi\rangle_V|^2}{\langle e_\varphi,e_\varphi\rangle_V\langle e_\psi,e_\psi\rangle_V}.$$

In particular, for each $f_0\in L^2(\mathbb{C}^n)$ and $f_1\in Sy(o(\varphi_0))$, we have $B(f_0\otimes f_1)=B_0(f_0)\otimes b(f_1)$. Moreover for each A_0 operator on $\mathcal{F}_0$ and A_1 operator on V, we have $U(A_0\otimes A_1)=U^0(A_0)\otimes w(A_1)$.

Note that it is well-known that if $\Delta_0:=4\sum_{k=1}^n(\partial_{z_k}\partial_{\bar{z}_k})$ is the Laplace operator then we have $\mathcal{B}^0=\exp(\Delta_0/2\gamma)$, see [44]. Thus we get

$$U^0=\exp(-\Delta_0/4\gamma)S^0.$$

Hence, by applying Proposition 5.4 and Proposition 5.6, we obtain the following result.

Proposition 5.7 ([23]). *For each $X=((a,\bar{a}),c,A)\in\mathfrak{g}$, $z\in\mathbb{C}^n$ and $\varphi\in o(\varphi_0)$, we have*

$$U(d\pi(X))(z,\varphi)=ic\lambda+w(d\rho(A))(\varphi)+\frac{1}{2}\,\mathrm{Tr}(A)+\frac{i}{2}\left(\bar{a}z+\lambda^2a\bar{z}\right)-\frac{\lambda}{2}\bar{z}(Az).$$

6. Schrödinger model for Heisenberg motion groups

Here we introduce the Schrödinger representations of G by using a Segal-Bargmann transform which is obtained by a slight modification of B_0. More precisely, let us define the map B from $L^2(\mathbb{R}^n, V) \cong L^2(\mathbb{R}^n) \otimes V$ to $\mathcal{F} \cong \mathcal{F}_0 \otimes V$ by $B := B_0 \otimes I_V$ or, equivalently, by the integral formula

$$B(f)(z) = (\lambda/\pi)^{n/4} \int_{\mathbb{R}^n} e^{(1/4\lambda)z^2 + ixz - (\lambda/2)x^2} f(x)\, dx$$

for each $f \in L^2(\mathbb{R}^n, V)$.

Now, by analogy with the case of the Heisenberg group, we define the Schrödinger representation σ of G on $L^2(\mathbb{R}^n, V)$ by $\sigma(g) := B^{-1}\pi(g)B$. Similarly, recalling that τ is the representation of K on $\mathcal{F}_0$ given by $(\tau(k)F)(z) = F(k^{-1}z)$, we define the representation $\tilde{\tau}$ of K on $L^2(\mathbb{R}^n)$ by $\tilde{\tau}(k) := B_0^{-1}\tau(k)B_0$. Then we have the following proposition.

Proposition 6.1. *Let $g_0 \in G_0$, $k \in K$ and $g = (g_0, k) \in G$. Then we have $\sigma(g) = \sigma_0(g_0)\tilde{\tau}(k) \otimes \rho(k)$.*

Proof. Let $f_0 \in L^2(\mathbb{R}^n)$ and $v \in V$. Then by Eq. 2 we have

$$\begin{aligned}\sigma(g)(f_0 \otimes v) &= (B_0^{-1} \otimes I_V)(\pi_0(g_0)\tau(k) \otimes \rho(k))(B_0 \otimes I_V)(f_0 \otimes v)\\ &= (B_0^{-1}\pi_0(g_0)\tau(k)B_0)f_0 \otimes \rho(k)v\\ &= \sigma_0(g_0)(B_0^{-1}\tau(k)B_0)f_0 \otimes \rho(k)v,\end{aligned}$$

hence the result. □

The following proposition gives an explicit expression for $d\sigma(X)$ when X is of the form $((0,0), 0, A)$ where $A \in \mathfrak{k}$.

Proposition 6.2. *1. For each $A = (a_{kl}) \in \mathfrak{k}$, we have*

$$\begin{aligned}d\tilde{\tau}(A) =& \frac{1}{2\lambda}\sum_{k,l} a_{kl}\frac{\partial^2}{\partial x_k \partial x_l} + \frac{1}{2}\sum_{k,l} a_{kl}\left(x_k\frac{\partial}{\partial x_l} - x_l\frac{\partial}{\partial x_k}\right)\\ &- \frac{\lambda}{2}x(Ax) + \frac{1}{2}\operatorname{Tr}(A).\end{aligned}$$

2. For each $X = ((0,0), 0, A)$ with $A \in \mathfrak{k}$, we have

$$d\sigma(X) = d\tilde{\tau}(A) \otimes I_V + I_{\mathcal{F}_0} \otimes d\rho(A)$$

where $d\tilde{\tau}(A)$ is as in 1.

Proof. In order to prove the first statement, first note that for each $A \in \mathfrak{k}$ and $F^0 \in \mathcal{F}_0$ we have

$$(d\tau(A)F^0)(z) = -(dF^0)_z(Az) = -\sum_k \frac{\partial F^0}{\partial z_k}(z)(e_k(Az)).$$

To simplify the notation we denote by $k_{B_0}(z,x)$ the kernel of B_0, that is,

$$k_{B_0}(z,x) := \left(\frac{\lambda}{\pi}\right)^{n/4} e^{(1/4\lambda)z^2 + ixz - (\lambda/2)x^2}.$$

Then, for each $f_0 \in \mathcal{S}(\mathbb{R}^n)$ we have

$$(d\tau(A)B_0 f_0)(z) = -\int_{\mathbb{R}^n} \left(\frac{1}{2\lambda} z(Az) + ix(Az)\right) k_{B_0}(z,x) f_0(x)dx.$$

Thus writing $z(Az) = \sum_{k,l} a_{kl} z_k z_l$ and integrating by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^n} & z(Az) k_{B_0}(z,x) f_0(x)dx \\ &= -\left(\frac{\lambda}{\pi}\right)^{n/4} \sum_{k,l} a_{kl} \int_{\mathbb{R}^n} e^{(1/4\lambda)z^2 + ixz} \frac{\partial^2}{\partial x_k \partial x_l}(e^{-(\lambda/2)x^2} f_0(x))dx\end{aligned}$$

and, similarly,

$$\begin{aligned}\int_{\mathbb{R}^n} & ix(Az) k_{B_0}(z,x) f_0(x)dx \\ &= -\left(\frac{\lambda}{\pi}\right)^{n/4} \sum_{k,l} a_{kl} \int_{\mathbb{R}^n} e^{(1/4\lambda)z^2 + ixz} \frac{\partial}{\partial x_l}(e^{-(\lambda/2)x^2} x_k f_0(x))dx.\end{aligned}$$

The first statement hence follows. The second statement is an immediate consequence of Proposition 6.1 . □

Note that σ is completely determined by the fact that $\sigma(g_0, I_n) = \sigma_0(g_0) \otimes I_V$ and by Proposition 6.2.

7. Stratonovich-Weyl correspondence via Weyl calculus

In this section we first introduce a slight modification of the usual Weyl correspondence in the spirit of our previous works, see for instance [14].

Recall that the Berezin calculus s associates with each operator B on V a complex-valued function $s(B)$ on $o(\varphi_0)$ which is called the symbol of B and that the space of all such symbols is denoted by $Sy(o(\varphi_0))$, see Section 5. Then the unitary part w of s is an isomorphism from $\mathrm{End}(V)$ onto $Sy(o(\varphi_0))$.

Now we say that a complex-valued smooth function $f : (p, q, \varphi) \to f(p, q, \varphi)$ is a symbol on $\mathbb{R}^{2n} \times o(\varphi_0)$ if for each $(p, q) \in \mathbb{R}^{2n}$ the function $f(p, q, \cdot) : \varphi \to f(p, q, \varphi)$ is an element of $Sy(o(\varphi_0))$. In that case, we denote $\hat{f}(p, q) := w^{-1}(f(p, q, \cdot))$. A symbol f on $\mathbb{R}^{2n} \times o(\varphi_0)$ is called an $\mathcal{S}$-symbol if the function $\hat{f}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^{2n}, \mathrm{End}(V))$ of rapidly decreasing smooth functions on $\mathbb{R}^{2n}$ with values in $\mathrm{End}(V)$. For each $\mathcal{S}$-symbol on $\mathbb{R}^{2n} \times o(\varphi_0)$, we define the operator $W(f)$ on the Hilbert space $L^2(\mathbb{R}^n, V)$ by

$$W(f)\phi(p) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{isq}\, \hat{f}(p + (1/2)s, q)\, \phi(p + s)\, ds\, dq.$$

Of course, W can be extended to much larger classes of symbols as the usual Weyl calculus, see Section 2. As an immediate consequence of the definition of W, we have the following proposition.

Proposition 7.1. *1. The map W is a unitary operator from $L^2(\mathbb{R}^{2n}, V)$ onto $\mathcal{L}_2(L^2(\mathbb{R}^n, V))$;*

2. For each $f_0 \in \mathcal{S}(\mathbb{R}^n)$ and $f_1 \in Sy(o(\varphi_0))$, we have

$$W(f_0 \otimes f_1) = W_0(f_0) \otimes w^{-1}(f_1).$$

In order to compare W and U, it is convenient to transfer U to operators on $L^2(\mathbb{R}^n, V)$ in the spirit of Proposition 2.5. First, for any operator A on $L^2(\mathbb{R}^n, V)$, we define $S_1(A) := S(BAB^{-1})$. Clearly, one has $S_1 S_1^* = SS^* = \mathcal{B}$. Then the unitary part U_1 of S_1 is given by $U_1(A) := U(BAB^{-1})$ for any operator A on $L^2(\mathbb{R}^n, V)$. Moreover, we have

$$\begin{aligned} U_1 = \mathcal{B}^{-1/2} S_1 &= \Big((\mathcal{B}^0)^{-1/2} \otimes b^{-1/2}\Big)\big(S^1 \otimes s\big) \\ &= (\mathcal{B}^0)^{-1/2} S^1 \otimes b^{-1/2} s = U^1 \otimes w \end{aligned}$$

with obvious notation. Hence we are in position to extend Proposition 2.5 to Heisenberg motion groups.

Proposition 7.2. *We have $U_1 = (J^{-1} \otimes I_{Sy(o(\varphi_0))}) W^{-1}$.*

Proof. By using Proposition 7.1 and Proposition 2.5, we get

$$\begin{aligned} (J^{-1} \otimes I_{Sy(o(\varphi_0))}) W^{-1} &= (J^{-1} \otimes I_{Sy(o(\varphi_0))})(W_0^{-1} \otimes w) \\ &= (J^{-1} W_0^{-1}) \otimes w = U^1 \otimes w = U_1. \end{aligned}$$

This is the desired result. □

Now consider the action of G on $\mathbb{R}^{2n}\times o(\varphi_0)$ given by

$$g\cdot(p,q,\varphi):=(j^{-1}(g\cdot j(p,q)),\mathrm{Ad}^*(k)\varphi)$$

for $g=((z_0,\bar z_0),c_0,k)\in G$. Then we have the following result.

PROPOSITION 7.3. *1. The map W^{-1} is a Stratonovich-Weyl correspondence for $(G,\sigma,\mathbb{R}^{2n}\times o(\varphi_0))$.*

2. For each $X=((a,\bar a),c,A)\in\mathfrak{g}$, $p,q\in\mathbb{R}^n$ and $\varphi\in o(\varphi_0)$, we have

$$\begin{aligned}W^{-1}(d\sigma(X))(p,q,\varphi)=i\lambda c+\frac{1}{2}\,\mathrm{Tr}(A)+\frac{i}{2}\left(\bar a j(p,q)+\lambda^2 a\overline{j(p,q)}\right)\\ -\frac{\lambda}{2}\overline{j(p,q)}(Aj(p,q))+w(d\rho(A))(\varphi).\end{aligned}$$

Proof. 1. For each $g=((z_0,\bar z_0),c_0,k)\in G$ let us denote by L_g the operator of $L^2(\mathbb{C}^n\times o(\varphi_0),\mu_\lambda\otimes\nu)$ defined by

$$(L_gF)(z,\varphi)=F(g\cdot z,\mathrm{Ad}^*(k)\varphi).$$

Then the covariance property for U can be rewritten as

$$L_gU(A)=U(\pi(g)^{-1}A\pi(g))$$

for each $g\in G$ and $A\in\mathcal{L}_2(\mathcal{F})$. This gives the following covariance property for U_1:

$$L_gU_1(A)=U_1(\sigma(g)^{-1}A\sigma(g))$$

for each $g\in G$ and $A\in\mathcal{L}_2(L^2(\mathbb{R}^n,V))$. But by Proposition 7.2 we have $U_1=(J^{-1}\otimes I_{Sy(o(\varphi_0))})W^{-1}$. Thus we get

$$(J\otimes I_{Sy(o(\varphi_0))})L_g(J^{-1}\otimes I_{Sy(o(\varphi_0))})W^{-1}(A)=W^{-1}(\sigma(g)^{-1}A\sigma(g))$$

for each $g\in G$ and $A\in\mathcal{L}_2(L^2(\mathbb{R}^n,V))$.

Now let

$$(\tilde L_gf)(p,q,\varphi):=f(j^{-1}(g\cdot j(p,q)),\mathrm{Ad}^*(k)\varphi)$$

for each $g=((z_0,\bar z_0),c_0,k)\in G$ and $(p,q,\varphi)\in\mathbb{R}^{2n}\times o(\varphi_0)$. Since it is clear that for each $g\in G$ we have

$$\tilde L_g=(J\otimes I_{Sy(o(\varphi_0))})L_g(J^{-1}\otimes I_{Sy(o(\varphi_0))}),$$

we see that

$$\tilde L_gW^{-1}(A)=W^{-1}(\sigma(g)^{-1}A\sigma(g))$$

for each $g\in G$ and $A\in\mathcal{L}_2(L^2(\mathbb{R}^n,V))$. Hence W^{-1} is G-covariant. The other properties of a Stratonovich-Weyl correspondence can be easily verified.

2. For each $X \in \mathfrak{g}^c$, we have

$$U(d\pi(X)) = U_1(d\sigma(X)) = ((J^{-1} \otimes I_{Sy(o(\varphi_0))})W^{-1}(d\sigma(X))$$

hence the result follows from Proposition 5.7. □

Finally, we can obtain Stratonovich-Weyl correspondences for $(G, \pi, \mathcal{O}(\varphi^0))$ and for $(G, \sigma, \mathcal{O}(\varphi^0))$ by transferring U and W^{-1} by means of Φ. Let

$$\Psi := \Phi \circ (j \otimes 1) : \mathbb{R}^{2n} \times o(\varphi_0) \to \mathcal{O}(\varphi^0)$$

and let $\tilde{\nu}$ be the G-invariant measure on $\mathcal{O}(\varphi^0)$ defined by

$$\tilde{\nu} := (\Phi^{-1})^*(\mu_\lambda \otimes \nu) = (\Psi^{-1})^*(\tilde{\mu} \otimes \nu).$$

Consider also the unitary maps $\tau_\Phi : F \to F \circ \Phi^{-1}$ from $L^2(\mathbb{C}^n \times o(\varphi_0), \mu_\lambda \otimes \nu)$ onto $L^2(\mathcal{O}(\varphi^0), \tilde{\nu})$ and $\tau_\Psi : F \to F \circ \Psi^{-1}$ from $L^2(\mathbb{R}^{2n} \times o(\varphi_0), \tilde{\mu} \otimes \nu)$ onto $L^2(\mathcal{O}(\varphi^0), \tilde{\nu})$. Then we have the following proposition.

Proposition 7.4. *The map $\mathcal{W}_1' := \tau_\Psi W^{-1}$ is a Stratonovich-Weyl correspondence for $(G, \sigma, \mathcal{O}(\varphi^0))$, the map $\mathcal{W}_2' := \tau_\Phi U$ is a Stratonovich-Weyl correspondence for $(G, \pi, \mathcal{O}(\varphi^0))$ and we have $\mathcal{W}_1' = \mathcal{W}_2' I_B$.*

Proof. The first and the second assertions immediately follow from Proposition 5.5 and Proposition 7.3. To prove the third assertion, note that we have $\tau_\Psi(J \otimes I_{Sy(o(\varphi_0))}) = \tau_\Phi$. Then, by Proposition 7.2, we can write

$$\mathcal{W}_2' I_B = \tau_\Phi U I_B = \tau_\Phi U_1 = \tau_\Phi (J^{-1} \otimes I_{Sy(o(\varphi_0))}) W^{-1} = \tau_\Psi W^{-1} = \mathcal{W}_1',$$

hence the result. □

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Author's address:

Benjamin Cahen
Université de Lorraine, Site de Metz, UFR-MIM
Département de Mathématiques
Bâtiment A, Ile du Saulcy, CS 50128, F-57045, Metz cedex 01, France
E-mail: benjamin.cahen@univ-lorraine.fr

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