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On the improved Massera's theorem for the unique existence of the limit cycle for a Liénard equation

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Abstract. *We further generalize a recent improvement obtained by G. Villari of the classical Massera's theorem about the unique existence of the limit cycle of a Liénard equation.*

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1. Background for the improved Massera's theorem

In this paper, we consider the well-known Liénard equation

$$\ddot{x} + f(x)\dot{x} + x = 0.$$

Throughout, we assume for the above equation that the function $f(x)$ satisfies smoothness conditions in order to guarantee the uniqueness of solutions of initial value problems. This equation has been widely investigated in the literature (for instance see [9]). We are interested in the unique existence of the limit cycle of the equation under the following **Property (A)** (see [8]):

$f(x)$ is continuous and there exist $a < 0 < b$ such that $f(x) < 0$ for $a < x < b$, $f(x) > 0$ for $x \leq a$ or $x \geq b$; moreover, $xF(x) > 0$ for $|x|$ large, where $F(x) = \int_0^x f(t)dt$.

Note that $F(x)$ has three zeros at $\alpha < 0$, 0, $\beta > 0$ and is monotone increasing for $x < \alpha$ and for $x > \beta$.

It is well-known that the Liénard equation is equivalent to the Liénard system

$$\dot{x} = y - F(x), \qquad \dot{y} = -x. \tag{L}$$

First, we recall some previous results for system (L). Levinson-Smith [3] in 1942 and Sansone [5] in 1949 (see also the paper of Villari [7] in 1985) have proved the following

Proposition 1.1. *Under the property* (A) *a limit cycle intersecting both the lines $x = \alpha$ and $x = \beta$ is at most one.*

Afterwards Massera [4] in 1954 improved a result of Sansone [6] in 1951 by using the phase-plane analysis as follows.

Proposition 1.2. (Massera's Theorem) *System* (L) *has at most one limit cycle which is stable if $f(x)$ is monotone decreasing for $x < 0$ and $f(x)$ is monotone increasing for $x > 0$.*

We remark that the existence of a limit cycle is not guaranteed in the above theorem.

Recently, Villari [8] in 2012, on these bases, has presented the following

Proposition 1.3. *Under the property* (A) *system* (L) *has exactly one limit cycle, which is stable, provided that*

- *if $|\alpha| > \beta$, then $f(x)$ is monotone decreasing for $\alpha < x < 0$, $f(x)$ is monotone increasing for $0 < x < \delta$,*
- *if $|\alpha| < \beta$, then $f(x)$ is monotone decreasing for $\delta_1 < x < 0$, $f(x)$ is monotone increasing for $0 < x < \beta$,*

where $\delta = \sqrt{\left(1 + F(a) + \dfrac{\alpha^2}{2}\right)^2 + \beta^2}$ *and* $\delta_1 = -\sqrt{\left(1 - F(b) + \dfrac{\beta^2}{2}\right)^2 + \alpha^2}$.

Our aim is to give a new criterion for the unique existence of the limit cycle of system (L) by combining Proposition 1.3 with our result [2] in 2000 below.

Proposition 1.4. *Assume that $f(x)$ is continuous, $f(a) = f(b) = 0$ for $a < 0 < b$, $f(0) < 0$ and $xF(x) > 0$ for $|x|$ large. System* (L) *has exactly one limit cycle, which is stable, provided that*

(i) $|\alpha| = \beta$ *and* $f(x) > 0$ *for* $|x| \geq \beta$,

(ii) $|a| \leq \beta < |\alpha|$ *and* $f(x) > 0$ *for* $|x| \geq \beta$,

(iii) $b \leq |\alpha| < \beta$ *and* $f(x) > 0$ *for* $|x| \geq |\alpha|$.

We produce the proof of the above proposition in the Appendix.

2. Main results

We show in this section that our method yields an improvement of the result of Villari [8]. Instead of the Property(A), assume the following **Property (B)**:

$f(x)$ is continuously differentiable and $F(0) = F(\alpha) = F(\beta) = 0$, $\dfrac{F(x)}{x} < 0$ for $\alpha < 0 < \beta$, $f(x) > 0$ for $x \leq p$ and $x \geq \beta$, or $x \leq \alpha$ and $x \geq q$, where

$$p = \min\{x \in (\alpha, 0)|\ F^{'}(x) = 0,\ F^{''}(x) \neq 0\}$$

and

$$q = \max\{x \in (0, \beta)|\ F^{'}(x) = 0,\ F^{''}(x) \neq 0\}.$$

Remark that Property (B) includes Property (A). We now state our result concerning the unique existence of limit cycles of system (L).

THEOREM 2.1. *Under the property* (B)*, if system* (L) *satisfies one of the conditions :*

(1) $|\alpha| = \beta$ *and* $f(x) > 0$ *for* $|x| \geq \beta$*,*

(2) $|p| \leq \beta < |\alpha|$ *and* $f(x) > 0$ *for* $|x| \geq \beta$*,*

(3) $q \leq |\alpha| < \beta$ *and* $f(x) > 0$ *for* $|x| \geq |\alpha|$*,*

(4) $|\alpha| > \beta$ *and* $\beta < |p|$*,* $f(x) > 0$ *for* $x \leq p$ *and* $x \geq \beta$*,* $f(x)$ *is monotone decreasing for* $p \leq x < 0$*,* $f(x)$ *is monotone increasing for* $0 < x < \delta^*$*, where* $\delta^* = \sqrt{\left(1 + F(a^*) + \frac{p^2}{2}\right)^2 + \beta^2}$ *for* $a^* = \min\{x|\ \max_{x \in (\alpha, 0)} F(x)\}$*,*

(5) $|\alpha| < \beta$ *and* $|\alpha| < q$*,* $f(x) > 0$ *for* $x \leq \alpha$ *and* $x \geq q$*,* $f(x)$ *is monotone decreasing for* $\delta_1^* < x < 0$*,* $f(x)$ *is monotone increasing for* $0 < x \leq q$*, where* $\delta_1^* = -\sqrt{\left(1 - F(b^*) + \frac{q^2}{2}\right)^2 + \alpha^2}$ *for* $b^* = \max\{x|\ \min_{x \in (0, \beta)} F(x)\}$*,*

then it has a unique stable limit cycle.

REMARK 2.2. In [8] the case of $p = a = a^*$ or $q = b = b^*$ is treated.

REMARK 2.3. In Theorem 2.1 the unique limit cycle intersects the lines $x = \pm\beta$ in the case (1) or (2). In the case (3) it intersects the lines $x = \pm\alpha$, in the case (4) $x = p$ and $x = \beta$, in the case (5) $x = \alpha$ and $x = q$.

We now apply Theorem 2.1 to the Liénard equation with a positive parameter λ:

$$\ddot{x} + \lambda f(x)\dot{x} + x = 0.$$

It is equivalent to the Liénard system

$$\dot{x} = y - \lambda F(x), \qquad \dot{y} = -x. \tag{L$_\lambda$}$$

Theorem 2.4. *Under each condition in Theorem* 2.1 *system* (L_λ) *satisfies the following:*

$(1)'$ *if* $|\alpha| = \beta$, *then it has a unique stable limit cycle intersecting the lines* $x = \alpha$ *and* $x = \beta$, *for all* $\lambda > 0$,

$(2)'$ *if* $|p| \leq \beta < |\alpha|$, *then it has a unique stable limit cycle intersecting the lines* $x = \pm\beta$, *for all* $\lambda > 0$.

$(3)'$ *if* $q \leq |\alpha| < \beta$, *then it has a unique stable limit cycle intersecting the lines* $x = \pm\alpha$, *for all* $\lambda > 0$.

$(4)'$ *if* $|\alpha| > \beta$ *and* $\beta < |p|$, *then it has a unique stable limit cycle intersecting the lines* $x = p$ *and* $x = \beta$, *for all* $\lambda > \tilde{\lambda}_1 = \sqrt{\dfrac{p^2 - \beta^2}{F^2(b^*)}}$.

$(5)'$ *if* $|\alpha| < \beta$ *and* $|\alpha| < q$, *then it has a unique stable limit cycle intersecting the lines* $x = \alpha$ *and* $x = q$, *for all* $\lambda > \tilde{\lambda}_2 = \sqrt{\dfrac{q^2 - \alpha^2}{F^2(a^*)}}$.

3. Proofs of theorems

Proof of Theorem 2.1. First, the cases of (1), (2) and (3) follow from [1] and [2]. So we omit the details. Next, we prove the case (4). By the Property (B), the existence of the limit cycle for system (L) is guaranteed. From [2] system (L) has at most one limit cycle intersecting the lines $x = p$ and $x = \beta$. Further it is stable. On the other hand, the limit cycle of system (L) contained in the region $D = \{(x, y) \mid p \leq x \leq \delta^*,\ y \in \mathbb{R}\}$ is at most one, by the monotonicity condition on the function $f(x)$, and is stable (see [8]). Thus we conclude from the stability of the limit cycle that system (L) has exactly one limit cycle, either intersecting the lines $x = p$ and $x = \beta$, or in D. Similarly, we can prove the case (5). □ □

Proof of Theorem 2.2. The case $(1)'$ is well-known from [1] or [8]. In the case $(2)'$ or $(3)'$ the result in [2] applies. So we consider the case $(4)'$. Any positive semitrajectory which starts from the point $(\beta, \lambda F(b^*))$ must intersect the line $x = p$ for the positive number λ such that

$$\sqrt{\lambda^2 F^2(b^*) + \beta^2} \geq |p|,$$

namely, for all $\lambda > \tilde{\lambda}_1$. Then, as was mentioned in Theorem 2.1, the unique limit cycle intersecting $x = p$ and $x = \beta$ exists. Further δ^* is given by

$$\delta^* = \sqrt{\left(1 + \lambda F(a^*) + \frac{p^2}{2}\right)^2 + \beta^2}$$

for each λ satisfying $\lambda > \tilde{\lambda}_1$. Similarly, the case $(5)'$ is discussed, where

$$\delta_1^* = -\sqrt{\Big(1 - \lambda F(b^*) + \frac{q^2}{2}\Big)^2 + \alpha^2}$$

for all $\lambda > \tilde{\lambda}_2$. □

4. An example

We shall apply our results to some polynomial system.

EXAMPLE 4.1. Consider the function

$$F(x) = \begin{cases} \dfrac{1}{3}x^3 + \dfrac{3}{2}x^2 - 4x & \text{for } x \le -4,\ x \ge 0 \\ -\dfrac{1}{2}x^2 - 4x & \text{for } \ -4 < x < 0 \end{cases}$$

for system (L). This system has a unique stable limit cycle. Indeed, we have $\alpha = (-9 - \sqrt{273})/4 < p(= a^*) = -4 < b = 1 < \beta = (-9 + \sqrt{273})/4$ and all conditions of the case (4) in Theorem 2.1 hold. For instance we have that $F'(x)$ is monotone decreasing for $-4 < x < 0$ and $F'(x)$ is monotone increasing for $x > 0$.

5. Appendix

We give the outline of the proof of Theorem 2 in our result in [2]. This is a special case of Theorem 1 in [2]. It is well-known from the Poincaré-Bendixson's theorem that if System (L) satisfies the conditions that $f(0) < 0$ and $xF(x) > 0$ for $|x|$ large, then it has at least one limit cycles.

We consider the case of $|a| \le \beta \le |\alpha|$ and $f(x) = F'(x) > 0$ for $|x| \ge |\beta|$. The other case can be discussed similarly. Letting $G(x) = (1/2)x^2$, there exists a negative number $-\beta \in [\alpha, 0)$ such that $G(-\beta) = G(\beta)$. Then System (L) has no limit cycles in the strip domain $\Omega = \{(x, y) \mid |x| \le \beta,\ y \in \mathbb{R}\}$ because of $xF(x) < 0$ for $|x| < \beta$ (for instance see [1]). Thus, we know that there is a closed orbit which C surrounds the origin and meets Ω^c.

We show its uniqueness. Without loss of generality we can assume that $\tilde{C}$ is outside C. We define Lyapunov-type functions by

$$V(x, y, t) = \begin{cases} V_1(x, y) = (1/2)y^2 + G(x) \quad \text{if } x \ge \beta, \\ V_2(x, y, t) = (1/2)y^2 + G(x) + \gamma_1 t \quad \text{if } |x| < \beta \text{ and } y < F(x), \\ V_3(x, y) = (1/2)(y - F(a))^2 + G(x) \quad \text{if } x \le -\beta, \\ V_4(x, y, t) = (1/2)y^2 + G(x) + \gamma_2 t \quad \text{if } |x| < \beta \text{ and } y > F(x). \end{cases}$$

We use the same notations as in [2]. Let $(x(t), y(t))$ be a periodic solution which starts from a point on the positive half of the vertical line $x = \beta$, $T > 0$ be its smallest period and

$$A = y(T_2) - y(T_3) - \delta_1 \quad \text{and} \quad \tilde{A} = \tilde{y}(\tilde{T}_2) - \tilde{y}(\tilde{T}_3) - \delta_2$$

for some constants δ_1 and δ_2.

We assume $M = (T - T_3)(\tilde{T}_2 - \tilde{T}_1) - (\tilde{T} - \tilde{T}_3)(T_2 - T_1) > 0$. Then the constants γ_1 and γ_2 are defined by

$$\gamma_1 = \frac{F(a)\{(\tilde{T} - \tilde{T}_3)A - (T - T_3)\tilde{A}\}}{M}$$

and

$$\gamma_2 = \frac{F(a)\{(\tilde{T}_2 - \tilde{T}_1)A - (T_2 - T_1)\tilde{A}\}}{M}.$$

Since $\tilde{y}(\tilde{T}_2) - \tilde{y}(\tilde{T}_3) < y(T_2) - y(T_3) < 0$ and $F(a) > 0$, we can take the numbers δ_1 and δ_2 such that $\gamma_1 > 0$, $\gamma_2 > 0$ and $\delta_1 \leq \delta_2$.

Then it follows from the same calculations as in [2] that $I_i = \int_{C_i} dV_i > \tilde{I}_i = \int_{\tilde{C}_i} dV_i$ for $i = 1, \ldots, 4$. Hence we have $I = \sum_{i=1}^{4} I_i > \tilde{I} = \sum_{i=1}^{4} \tilde{I}$.

On the other hand, we have from the choice of δ_1 and δ_2 that

$$\begin{aligned} I = \oint_C dV &= F(a)\{y(T_2) - y(T_3)\} + \gamma_1(T_2 - T_1) - \gamma_2(T - T_3) \\ &= F(a)(A + \delta_1) + \gamma_1(T_2 - T_1) - \gamma_2(T - T_3) = F(a)\delta_1. \end{aligned}$$

Similarly we have

$$\tilde{I} = F(a)(\tilde{A} + \delta_2) + \gamma_1(\tilde{T}_2 - \tilde{T}_1) - \gamma_2(\tilde{T} - \tilde{T}_3) = F(a)\delta_2.$$

Thus we have $I \leq \tilde{I}$. This contradicts $I > \tilde{I}$.

In the case $M < 0$, by replacing with $V_2(x, y, t) = (1/2)y^2 + G(x) - \gamma_1 t$ and $V_4(x, y, t) = (1/2)y^2 + G(x) - \gamma_2 t$, we can take the numbers δ_1 and δ_2 satisfying $\gamma_1 < 0$, $\gamma_2 < 0$ and $\delta_1 \leq \delta_2$. In the case $M = 0$, we have by taking $\delta_1 = \delta_2$ that $I = \tilde{I}$ for some numbers $\gamma_1 > 0$ and $\gamma_2 > 0$. These contradict $I > \tilde{I}$ too.

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