On the boundary behaviour
of solutions to parabolic equations
of $p-$Laplacian type

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Dedicated to Giovanni Alessandrini for his 60th Birthday

Abstract. *We describe some recent results on the boundary behavior of non-negative solutions to a class of degenerate/singular parabolic equations, whose prototype is the parabolic p-Laplacian. More precisely we focus on Carleson-type estimates and boundary Harnack principles.*

1. Introduction

Carleson and boundary Harnack estimates are among the most important tools in the study of the boundary behaviour of solutions to elliptic and parabolic equation. Carleson estimates apply to nonnegative solutions u continuously *vanishing* on some distinguished part S of the boundary with the goal of showing that nearby, u is controlled above in a non-tangential fashion. More precisely, this means that an inequality of the following type

$$u \le \gamma\, u(P_\rho) \tag{1}$$

holds in a box ψ_ρ of size ρ, based on S, where P_ρ approaches S in a *non-tangential* fashion as $\rho \to 0$ and γ depends only on the dimension and the structure of the equation. The first results of this kind are due to Carleson [13], for harmonic functions in sawtooth regions and to Kemper [38] for solutions of the heat equation in parabolic $C^{1,1/2}$ domains. Since then, an inequality like (1) is known as a Carleson estimate.

A *Boundary Comparison Principle* or *Boundary Harnack Inequality* is a relation of the type

$$u/v \approx u(P_r)/v(P_r) \text{ in } \psi_\rho. \tag{2a}$$

where both u and v are nonnegative solutions vanishing on S. It implies that u and v vanish at the same speed approaching S. For linear equations, it also implies the Hölder continuity up to the boundary of the quotient u/v.

Both (1) and (2a) have been generalized to more general contexts and operators. In the elliptic context we mention [37] for the Laplace operator in non-tangentially accessible domains, [11], and [3, 7, 26] for elliptic operators in divergence and non-divergence form, respectively, [44, 46], for the p-Laplace operator, [16, 15] for the Kolmogorov operator.

Actually, for uniformly elliptic linear equations, the Carleson estimate has been proved to be equivalent to the boundary Harnack principle as shown in [1]. It would be interesting to explore this connection between the two inequalities also in the nonlinear setting.

For parabolic operators, we quote [28, 29, 35, 50] for cylindrical domains, and [27] for parabolic Lipschitz domains.

A classical application of the two inequalities is to Fatou-type theorems, but even more remarkable is their use in the regularity theory of two-phase free boundary problems, as shown in the two seminal papers [9, 10], where a general strategy to attack the regularity of the free boundary governed by the Laplace operator has been set up.

This technique has been subsequently extended to stationary problems governed by variable coefficients linear and semilinear operators [14, 32], to fully nonlinear operators [30, 31], and to the p-Laplace operator [45, 47].

The free boundary regularity theory for two-phase parabolic problems is less developed. For Stefan type problems we mention [12, 17, 33, 34] and the references therein.

In this brief review we describe and comment recent results concerning a class of singular/degenerate equations whose prototype is the parabolic p-Laplace equation

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0, \tag{1$_o$}$$

where Dw denotes the gradient of w with respect to the space variables. Precisely, let Ω be an open set in $\mathbb{R}^N$ and for $T > 0$ let Ω_T denote the cylindrical domain $\Omega \times (0, T]$. Moreover let

$$S_T = \partial\Omega \times (0, T), \qquad \partial_P\Omega_T = S_T \cup (\Omega \times \{0\})$$

denote the lateral, and the parabolic boundary respectively.

We shall consider quasi-linear, parabolic partial differential equations of the form

$$u_t - \operatorname{div}\mathbf{A}(x, t, u, Du) = 0 \quad \text{weakly in } \Omega_T \tag{3}$$

where the function $\mathbf{A} : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, t, u, \xi) \cdot \xi \geq C_o|\xi|^p \\ |\mathbf{A}(x, t, u, \xi)| \leq C_1|\xi|^{p-1} \end{cases} \quad \text{a.e. } (x, t) \in \Omega_T, \, \forall\, u \in \mathbb{R}, \, \forall \xi \in \mathbb{R}^N \tag{4}$$

where C_o and C_1 are given positive constants, and $p > 1$. We refer to the parameters N, p, C_o, C_1 as our structural data. We say that a constant is *universal* if it depends only on the structural data and on the Lipschitz (or C^k, if it is the case) character of the domain Ω.

A function

$$u \in C([0, T]; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)) \tag{5}$$

is a weak sub(super)-solution to (3)–(4) if for every sub-interval $[t_1, t_2] \subset (0, T]$

$$\int_\Omega uv dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega [-uv_t + \mathbf{A}(x, t, u, Du) \cdot Dv] dx dt \leq (\geq) \, 0 \tag{6}$$

for all non-negative test functions

$$v \in W^{1,2}(0, T; L^2(\Omega)) \cap L^p(0, T; W_o^{1,p}(\Omega)).$$

Under the conditions (4), equation (3) is degenerate when $p > 2$ and singular when $1 < p < 2$, since the modulus of ellipticity $|Du|^{p-2}$ respectively tends to 0 or to $+\infty$ as $|Du| \to 0$. In the latter case, we further distinguish between singular *super-critical* range (when $\frac{2N}{N+1} < p < 2$), and singular *critical and sub-critical* range (when $1 < p \leq \frac{2N}{N+1}$).

Let us first focus on Carleson's estimate and, in particular, on the approach developed for linear elliptic equations in [11] and for linear parabolic equations in [50]. Two are the main tools: the Harnack inequality and the geometric decay of the oscillation of u up to the boundary. Let us sketch the main strategy. Consider a non-negative solution u in a cylinder, and assume further that the solution vanishes on a part of the lateral boundary, which we assume to be a part of the hyperplane $\{x_N = 0\}$, containing the origin. One wants to show that

$$u(P) \leq \gamma, \tag{7}$$

γ universal, for all $P \in \Psi_1$, where

$$\Psi_r = \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}, \ |x'| \leq r, 0 < x_N < r \right\} \times \left(-2r^2, -r^2 \right).$$

Observe that, if $\mathrm{dist}(P, \partial\Omega_T) \approx 2^{-k}$, by the Harnack inequality we infer $u(P_1) \leq H^k$. Suppose now that (7) is not true. Then, given an integer h, there must exist $P_1 \in \Psi_1$, such that $u(P_1) > H^h$, which forces $\mathrm{dist}(P_1, \partial\Omega_T) < 2^{-h}$. By the geometric decay of the oscillation of u up to the boundary, one deduces the existence of P_2 such that $u(P_2) > H^{h+1}$, and $\mathrm{dist}(P_2, \partial\Omega_T) < 2^{-(h+1)}$.

If h is chosen large enough, an iteration of this procedure yields a sequence of points $\{P_j\}_{j=1}^\infty$ all belonging to $\Psi_{2/3}$ (say) and approaching the boundary, whereas the sequence $\{u(P_j)\}_{j=1}^\infty$ blows up. This contradicts the assumption that u vanishes continuously on the boundary, and we conclude

$$\sup_{\Psi_1} u \leq H^h \equiv \gamma.$$

Due to recent development in the field of Harnack inequalities for the above class of equations [20, 21, 23, 39], it is possible to prove suitable versions of Carleson estimate for non-negative solutions to (3)–(4) both in cylindrical Lipschitz domains and in time-independent NTA-cylinders (non-tangentially accessible domain). For more particulars on these sets, we refer the reader to [12, § 12.3].

According to the theory developed in the above papers, a Carleson type estimate makes sense only for $p > 2N/(N+1)$.

Indeed, in the critical and sub-critical range, explicit counterexamples rule out the possibility of a Harnack inequality. Only so-called Harnack-type estimates are possible, where, however, the ratio of infimum over supremum in proper space-time cylinders depends on the solution itself (for more details, see [24, Chapter 6, § 11–15]).

Although the overall strategy in the nonlinear setting follows the same kind of arguments of the linear case, its implementation presents a difficulty due to the lack of homogeneity of the equations. Also there is a striking difference between the singular and the degenerate case; this is already reflected in the intrinsic character of the interior Harnack inequality, and it is amplified when approaching the boundary through dyadically shrinking intrinsic cylinders. Concerning the Carleson estimate, its statement in the degenerate case can be considered as the intrinsic version of the analogous statement in the linear uniformly parabolic case. Things are different in the singular super-critical case, where, in general, one can only prove a somewhat weaker estimate, due to the possibility for a solution to extinguish in finite time. Indeed, we exhibit some counterexamples which show that one cannot do any better, unless some control of the interior oscillation of the solution is available.

The difference between the two cases, degenerate and singular super-critical, becomes more evident when one considers the validity of a boundary Harnack principle, even in smooth cylinders. In the singular case, for C^2 cylinders, the existence of suitable barriers provides a linear behavior. Together with Carleson's estimate, this fact implies almost immediately a Hopf principle and the boundary Harnack inequality. The extension of the boundary Harnack principle to Lipschitz cylinders remains an open question.

On the other hand, solutions to the parabolic p-Laplace equations can vanish arbitrarily fast in the degenerate case $p > 2$, so that no possibility exists to prove a boundary Harnack principle in its generality. Indeed, when $p > 2$, two explicit solutions to the parabolic p-Laplacian in the half space $\{x_N \geq 0\}$, that vanish at $x_N = 0$, are given by

$$u_1(x,t) = x_N, \ u_2(x,t) = \left(\frac{p-2}{p^{\frac{p-1}{p-2}}}\right)(T-t)^{-\frac{1}{p-2}}x_N^{\frac{p}{p-2}}. \tag{8}$$

The power–like behavior, as exhibited in the second one of (8), is not the

"worst" possible case. Indeed, let $\Omega = \{-1 \le x_i \le 1, 0 \le x_N \le \frac{1}{4}\}$, and consider the following Cauchy-Dirichlet Problem in $\Omega \times [0, T[$:

$$\begin{cases} u_t - \operatorname{div}(|Du|^{p-2}Du) = 0 \\ u(x,0) = C\,T^{-\frac{1}{p-2}} \exp(-\frac{1}{x_N}) \\ u(x',0,t) = 0 \\ u(x',\frac{1}{4},t) = C(T-t)^{-\frac{1}{p-2}} e^{-4} \\ u(x,t) = C(T-t)^{-\frac{1}{p-2}} \exp(-\frac{1}{x_N}), \qquad x \in \partial\Omega \cap \{0 < x_N < \frac{1}{4}\}, \end{cases} \tag{9}$$

where

$$C = \frac{1}{2(p-1)(p-2)} \left(\frac{\Omega(p-2)}{2p} \right)^{\frac{2p}{p-2}}.$$

It is easy to check that the function

$$u_3 = C(T-t)^{-\frac{1}{p-2}} \exp(-\frac{1}{x_N}), \qquad x_N > 0 \tag{10}$$

is a super-solution to such a problem. Therefore, the *solution* to the same problem (which is obviously positive) lies below u_3 and approaches the zero boundary value at $x_N = 0$ at least with exponential speed.

There is more. Let $\gamma \in (0,1)$, $\Omega = \{x_N > 0\}$, $T = \frac{2}{\gamma} - 1$: then

$$u(x,t) = \left[\frac{p-2}{p-1} \gamma^{\frac{1}{p-1}}(t+1)\left(\gamma + \frac{x_N - 2}{t+1} \right)_+ \right]^{\frac{p-1}{p-2}} \tag{11}$$

is a solution to $(1)_o$ in Ω_T, and vanishes not only on the boundary $\{x_N = 0\}$, but also in the set $\{0 < x_N < 2 - \gamma(t+1), \;\; 0 < t < T\}$, which has positive measure.

Therefore, if one wants to prove an estimate like (2a), one needs to be able to rule out examples like the ones we have just discussed.

2. The Degenerate Case $p > 2$

2.1. Harnack inequality and Harnack chains

As we mentioned in the Introduction, our results are strongly based on the interior Harnack inequalities proved in [20, 21, 22, 39], that we recall below.

First we need to introduce further notations. $D'w$ stands for the gradient of w with respect to x'.

For $y \in \mathbb{R}^N$ and $\rho > 0$, $K_\rho(y)$ denotes the cube of edge 2ρ, centered at y with faces parallel to the coordinate planes. When y is the origin of $\mathbb{R}^N$ we simply write K_ρ; $K'_\rho(y')$ denotes the $(N-1)$-dimensional cube $\{(x' : |x_i - y_i| < \rho, i = 1, 2, ..., N-1\}$; we write for short $\{|x_i - y_i| < \rho\}$.

For $\theta > 0$ we also define

$$Q_\rho^-(\theta) = K_\rho \times (-\theta\rho^p, 0], \qquad Q_\rho^+(\theta) = K_\rho \times (0, \theta\rho^p]$$

and for $(y, s) \in \mathbb{R}^N \times \mathbb{R}$,

$$(y, s) + Q_\rho^-(\theta) = K_\rho(y) \times (s - \theta\rho^p, s], \quad (y, s) + Q_\rho^+(\theta) = K_\rho(y) \times (s, s + \theta\rho^p].$$

Now fix $(x_o, t_o) \in \Omega_T$ such that $u(x_o, t_o) > 0$ and construct the cylinders

$$(x_o, t_o) + Q_\rho^\pm(\theta) \quad \text{where} \quad \theta = \left(\frac{c}{u(x_o, t_o)} \right)^{p-2}, \tag{12}$$

and c is a given positive constant. These cylinders are "intrinsic" to the solution, since their height is determined by the value of u at (x_o, t_o). Cylindrical domains of the form $K_\rho \times (0, \rho^p]$ reflect the natural, parabolic space-time dilations that leave the homogeneous, prototype equation $(1)_o$ invariant. The latter however is not homogeneous with respect to the solution u. The time dilation by a factor $u(x_o, t_o)^{2-p}$ is intended to restore the homogeneity. Most of the results we describe in this paper hold in such geometry.

Here is the Harnack inequality.

THEOREM 2.1. *Let u be a non-negative, weak solution to (3)–(4) in Ω_T for $p > 2$, $(x_o, t_o) \in \Omega_T$ such that $u(x_o, t_o) > 0$. There exist positive universal constants c and γ, such that for all intrinsic cylinders $(x_o, t_o) + Q_{2\rho}^\pm(\theta)$ as in (12), contained in Ω_T,*

$$\gamma^{-1} \sup_{K_\rho(x_o)} u(\cdot, t_o - \theta\rho^p) \le u(x_o, t_o) \le \gamma \inf_{K_\rho(x_o)} u(\cdot, t_o + \theta\rho^p). \tag{13}$$

The constants γ and c deteriorate as $p \to \infty$ in the sense that $\gamma(p)$, $c(p) \to \infty$ as $p \to \infty$; however, they are stable as $p \to 2$.

Some comments are in order. It could be interesting to examine the existence of a so-called Harnack chain allowing the control of the value of $u(x, t)$ by the value of $u(x_o, t_o)$ with $t < t_o$, thanks to the repeated application of the Harnack inequality. A Harnack chain argument is indeed one of the usual tools for proving a Carleson estimate.

In [21], the authors show that such a result actually holds for solutions defined in $\mathbb{R}^N \times (0, T)$, and not in a smaller domain Ω_T. Although the correct form of the Harnack chain for solutions defined in Ω_T, when $\Omega \subset \mathbb{R}^N$, can be given, nevertheless, such a result is of no use in the proof of Carleson's estimates, as there are two different, but equally important obstructions.

First of all u can vanish and hence prevent any further application of the Harnack inequality. Indeed, let us consider the following two examples.

Let $\gamma \in (0,1)$; the function

$$u(x,t) = \left[\frac{p-2}{p-1} \gamma^{\frac{1}{p-1}} (t+1) \left(\gamma + \frac{x_N - 2}{t+1} \right)_+ \right]^{\frac{p-1}{p-2}}$$
$$+ \left[\frac{p-2}{p-1} \gamma^{\frac{1}{p-1}} (t+1) \left(\gamma - \frac{x_N + 2}{t+1} \right)_+ \right]^{\frac{p-1}{p-2}}$$

is a solution to the parabolic p-Laplacian in the set $\mathbb{R}^N \times (0, \frac{2}{\gamma} - 1)$ and vanishes in the cone

$$\begin{cases} 0 < t < \frac{2}{\gamma} - 1 \\ -(2 - \gamma(t+1)) < x_N < 2 - (\gamma(t+1)). \end{cases}$$

If we take (x,t) and (x_o, t_o) with $t < t_o$ on opposite sides of the cone, there is no way to build a Harnack chain that connects the two points.

Let $\gamma_p = \left(\frac{1}{\lambda} \right)^{\frac{1}{p-1}} \frac{p-2}{p}$, with $\lambda = N(p-2) + p$, consider the cylinder $\{x_N > 0\} \times (0, (2\gamma_p)^\lambda)$ and let $x_1 = (0,0,\ldots,2)$, $x_2 = (0,0,\ldots,6)$. The function

$$u(x,t) = t^{-\frac{N}{\lambda}} \left[1 - \gamma_p \left(\frac{|x - x_1|}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}} \right]_+^{\frac{p-1}{p-2}} + t^{-\frac{N}{\lambda}} \left[1 - \gamma_p \left(\frac{|x - x_2|}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}} \right]_+^{\frac{p-1}{p-2}}$$

is a solution to the parabolic p-Laplacian in the indicated cylinder and vanishes on its parabolic boundary. Notice that such a solution is the sum of two Barenblatt functions with poles respectively at x_1 and x_2 and masses $M_1 = M_2 = 1$: in the interval $0 < t < (2\gamma_p)^\lambda$ the support of u is given by two disjoint regions R_1 and R_2, and only at time $T = (2\gamma_p)^\lambda$ the support of u finally becomes a simply connected set. Once more, taking (x,t) and (x_o, t_o) respectively in R_1 and R_2, there is no way to connect them with a Harnack chain. As a matter of fact, before the two supports touch, each Barenblatt function does not *feel* in any way the presence of the other one. In particular, we can change the mass of the two Barenblatt functions: this will modify the time T the two supports touch, but up to T, there is no way one Barenblatt component can detect the change performed on the other one.

On the other hand, one could think that if we have a solution vanishing on a flat piece of the boundary and strictly positive everywhere in the interior, then one could build a Harnack chain extending arbitrarily close to the boundary. However, this is not the case, as clearly shown by the following example.

Let us consider a domain $\Omega \subset \mathbb{R}^N$, which has a part of its boundary that coincides with the hyperplane $\{x_N = 0\}$, and let $\Gamma = \partial\Omega \cap \{x_N = 0\}$. Let $\bar{T} > 0$, be given and consider a non-negative solution u to

$$\begin{cases} u_t - \operatorname{div}(|Du|^{p-2} Du) = 0, & \text{in } \Omega_{\bar{T}} \\ u > 0, & \text{in } \Omega_{\bar{T}} \\ u = 0, & \text{on } \Gamma \times (0, \bar{T}]. \end{cases}$$

Let u be such that its value is bounded above by the distance to the flat boundary piece raised to some given power $a > 0$, i.e.

$$u(x,t) \leq \gamma \, \mathrm{dist}(x,\Gamma)^a, \quad a > 0, \ (x,t) \in \Omega_{\bar{T}}, \tag{14}$$

where $\gamma > 0$ is a proper parameter.

Let $(x_o, t_o) = (x_o', x_{o,N}, t_o) \in \Omega_T$ be such that $\gamma(x_o, \Gamma) = 1$. The goal is to form a Harnack chain of dyadic non-tangential cylinders approaching the boundary, while the chain stays inside $\Omega_{\bar{T}}$: we want to control the size of the time interval, which we need to span in order to complete the chain. Let

$$
\begin{aligned}
u_o &= u(x_o, t_o) \\
r_k &= 2^{-k} \\
x_k &= (\hat{x}_o', 2^{-k}) \\
t_k &= t_o - c^{p-2} \sum_{i=0}^{k-1} u_i^{2-p} r_i^p \\
u_k &= u(x_k, t_k) \approx (2^{-k})^a
\end{aligned}
$$

for $k = 1, \ldots$ Assuming that at each step one can use Harnack's inequality, we get an estimate on the size of t_k from above

$$t_k \leq t_o - c^{p-2} \sum_{i=0}^{k-1} (2^{-ai})^{2-p} 2^{-ip} \leq t_o - c^{p-2} \sum_{i=0}^{k-1} 2^{ai(p-2)-ip}$$

which diverges to $-\infty$ as $k \to \infty$ and $x_k \to \Gamma$, if $a \geq p/(p-2)$. Considering the solution u_2 from (8), we see that the above dyadic Harnack chain would diverge for such a solution as $a = \frac{p}{p-2}$.

The infinite length of the time interval needed to reach the boundary, is just one face (i. e. consequence) of the finite speed of propagation when $p > 2$. Points (x,t) that lie *inside* a proper p-paraboloid centered at (x_o, t_o) can be reached, starting from (x_o, t_o): if u_o is very small, and therefore the p-paraboloid is very narrow, with small values of r one ends up with very large values of t. On the other hand, points (x,t) that lie *outside* the same p-paraboloid centered at (x_o, t_o) cannot be reached.

These difficulties have been recently overcome in [6], where a sequence of Harnack chain estimates has been proved. The authors develop Harnack chains based on the weak Harnack inequality of [39], valid for supersolutions to the p-parabolic equation. As truncations of solutions are supersolutions, the authors achieve a finer control of the waiting times (for further details, see § 3 of [6]).

2.2. The Carleson estimates

We need to introduce some further notation. Let Ω_T be a Lipschitz cylinder and fix $(x_o, t_o) \in S_T$; in a neighbourhood of such a point, the cross section

is represented by the graph $\{(x', x_N) : x_N = \Phi(x')\}$, where Φ is a Lipschitz function with Lipschitz constant L. Without loss of generality, from here on we assume $\Phi(x'_o) = 0$ and $L \geq 1$.

For $\rho \in (0, r_o)$, let $x_\rho = (x'_o, 2L\rho)$, $P_\rho = P_\rho(x_o, t_o) = (x'_o, 2L\rho, t_o) \in \Omega_T$ such that $u(P_\rho) > 0$. Note that $\text{dist}(x_\rho, \partial\Omega)$ is of order ρ. Set

$$\Psi_\rho^-(x_o, t_o) = \Omega_T \cap \left\{ |x_i - x_{o,i}| < \tfrac{\rho}{4}, |x_N| < 2L\rho, t \in (t_o - \tfrac{\alpha+\beta}{2}\theta\rho^p, t_o - \beta\theta\rho^p \right\}$$

where $\theta = \left[\frac{c}{u(P_\rho)} \right]^{p-2}$, with c given in Theorem 2.1, and $\alpha > \beta$ are two positive parameters. We are now ready to state our main result in the degenerate case $p > 2$ (see [5]).

THEOREM 2.2. (Carleson's Estimate, $p > 2$) *Let u be a non-negative, weak solution to* (3)–(4) *in Ω_T. Assume that*

$$(t_o - \theta(4\rho)^p, t_o + \theta(4\rho)^p] \subset (0, T]$$

and that u vanishes continuously on

$$\partial\Omega \cap \{|x_i - x_{o,i}| < 2\rho,\ |x_N| < 8L\rho\} \times (t_o - \theta(4\rho)^p, t_o + \theta(4\rho)^p).$$

Then there exist two universal positive parameters $\alpha > \beta$, and a constant $\tilde{\gamma} > 0$, such that

$$u(x,t) \leq \tilde{\gamma}\, u(P_\rho) \qquad \text{for every } \ (x,t) \in \Psi_\rho^-(x_o, t_o). \tag{15}$$

Without going too much into details here, let us point out that for the prototype equation $(1)_o$, estimate (15) have been extended in [4] from Lipschitz cylinders to a wider class of cylinders Ω_T, whose cross section Ω is a NTA domain.

Weak solutions to (3) with zero Dirichlet boundary conditions on a Lipschitz domain are Hölder continuous up to the boundary (see, for example, [19, Chapter III, Theorem 1.2]). Combining this result with the previous Carleson estimate, yields a quantitative estimate on the decay of u at the boundary, invariant by the intrinsic rescaling

$$x = x_o + \rho y, \qquad t = t_o + \frac{\rho^p}{u(P_\rho)^{p-2}} \tau.$$

COROLLARY 2.3. *Under the same assumption of Theorem 2.2, we have*

$$0 \leq u(x,t) \leq \gamma \left(\frac{\text{dist}(x, \partial\Omega)}{\rho} \right)^\mu u(P_\rho),$$

for every $(x,t) \in \Psi_{\frac{\rho}{2}}^-(x_o, t_o)$, where $\mu \in (0,1)$ is universal.

If we restrict our attention to solutions to the model equation $(1)_o$, the result of Corollary 2.3 was strengthened for C^2 cylinders in [5].

THEOREM 2.4. (Lipschitz Decay) *Let* Ω_T *be a* C^2 *cylinder and* u *a non-negative, weak solution to* $(1)_o$ *in* Ω_T. *Let the other assumptions of Theorem 2.2 hold. Then there exist two positive parameters* $\alpha > \beta$, *and a constant* $\gamma > 0$, *depending only on* p, N, *and the* C^2–*constant* M_2 *of* Ω, *such that*

$$0 \leq u(x,t) \leq \gamma \left(\frac{\text{dist}(x, \partial\Omega)}{\rho} \right) u(P_\rho), \tag{16}$$

for every (x,t) *in the set*

$$\Omega_T \cap \left\{ |x_i - x_{o,i}| < \frac{\rho}{4}, \ 0 < x_N < 2M_2\rho \right\} \times \left(t_o - \frac{\alpha + 3\beta}{4}\theta\rho^p, t_o - \beta\theta\rho^p \right].$$

Following Definition 2.2 of [6], let us recall that for a bounded domain $\Omega \subset \mathbb{R}^N$, we say that it satisfies the ball condition with radius $r_o > 0$, if for each point $y \in \partial\Omega$ there exist points $x^+ \in \Omega$ and $x^- \in \Omega^c$ such that $B_{r_o}(x^+) \subset \Omega$, $B_{r_o}(x^-) \subset \Omega^c$, $\partial B_{r_o}(x^+) \cap \partial\Omega = \{y\} = \partial B_{r_o}(x^-) \cap \partial\Omega$, and $x^+(y)$, $x^-(y)$, and y are collinear for each $y \in \partial\Omega$; the previous result has been further extended to $C^{1,1}$ domains satisfying the ball condition with radius r_o: in such a case it is shown that u has a linear decay at the boundary (see Theorem 9.3 of [6]), giving proper decay estimates both from above and from below.

Relying on the recent papers [8, 40, 41, 42], these results can be extended both to a wider class of degenerate equations with *differentiable* principal part which have the same structure of the p-Laplacian.

2.3. The Boundary Harnack Inequality

For $x_o \in \partial\Omega$, let $a_r(x_o) := x_o + \frac{r}{2}\frac{x^+ - x_o}{|x^+ - x_o|}$. In [6], the following result is proven.

THEOREM 2.5. *Let* u *and* v *be two non-negative, weak solutions to* $(1)_o$ *in* Ω_T, *where* Ω *is a* $C^{1,1}$ *domain satisfying the ball condition with radius* r_o. *Let* $x_o \in \partial\Omega$, $t_o \in (0, T)$, *and* $r \in (0, r_o)$ *be fixed. Let* $A_- = (a_r(x_o), t_o)$, *and assume that* $u(A_-) = v(A_-)$. *There exist constants* c_4, c_5, c_6, *which depend only on the data, which satisfy the following. Let* $\theta_- = u(A_-)^{2-p}$, *and assume*

$$\theta_- r^p < t_o, \quad \text{and} \quad t_o + 2c_4\theta_- r^p < T.$$

Set

$$A_+ = (a_r(x_o), t_o + 2c_4\theta_- r^p), \qquad \theta_{+,u} = c_6^{-1}u(A_+)^{2-p}.$$

Assume that $v(A_+) \geq u(A_+)$. Then there exists a time t_+^*, depending on v, satisfying

$$t_+^* \in (t_o + (2c_4\theta_- - \theta_{+,u})r^p, t_o + 2c_4\theta_- r^p)$$
$$A_+^* = (a_r(x_o), t_+^*), \qquad \theta_{+,v}^* = c_6^{-1}v(A_+^*)^{2-p},$$

such that the following holds. If both u and v vanish continuously on

$$S_T \cap (B_r(x_o) \times (t_o + [2c_4\theta_- - 5\theta_{+,u}]r^p, t_o + [2c_4\theta_- - \theta_{+,u}]r^p)),$$

then

$$\frac{1}{c_5}\frac{u(A_-)}{v(A_+^*)} \leq \frac{u(x,t)}{v(x,t)} \leq c_5 \frac{u(A_+)}{v(A_-)},$$

whenever (x,t) belongs to the set

$$(B_r(x_o) \cap \Omega) \times \left(t_o + [2c_4\theta_- - (\theta_{+,v}^* + \theta_{+,u})]r^p, t_o + [2c_4\theta_- - \theta_{+,u}]r^p\right).$$

It is important to notice that t_+^* cannot be precisely controlled, and the only information at disposal is the interval it lies in. Moreover, the previous theorem reduces to the classical Boundary Harnack inequality for linear parabolic equations, whenever $p = 2$.

Finally, in [6] a global Harnack inequality is established as well; we refer the interested reader to § 8 of this work.

3. The Singular Super-critical Case $\frac{2N}{N+1} < p < 2$

3.1. The Harnack inequality

As already mentioned in the introduction, in the singular case, Harnack inequality exhibits different features with respect the degenerate case. The following theorem is proved in [23] (see also [24] for a thorough presentation).

For fixed $(x_o, t_o) \in \Omega_T$ and $\rho > 0$, set $\mathcal{M} = \sup_{K_\rho(x_o)} u(x, t_o)$, and require that

$$K_{8\rho}(x_o) \times I(t_o, 8\rho, \mathcal{M}^{2-p}) \subset \Omega_T. \tag{17}$$

THEOREM 3.1. (Harnack Inequality) *Let u be a non-negative, weak solution to (3)–(4), in Ω_T for $p \in (\frac{2N}{N+1}, 2)$. There exist universal constants $\bar{\epsilon} \in (0,1)$ and $\bar{\gamma} > 1$ such that for all intrinsic cylinders $(x_o, t_o) + Q_{8\rho}^{\pm}(\theta)$ for which (17) holds,*

$$\bar{\gamma}^{-1} \sup_{K_\rho(x_o)} u(\cdot, \sigma) \leq u(x_o, t_o) \leq \bar{\gamma} \inf_{K_\rho(x_o)} u(\cdot, \tau) \tag{18}$$

for any pair of time levels σ, τ in the range

$$t_o - \bar{\epsilon}\, u(x_o, t_o)^{2-p}\rho^p \leq \sigma, \tau \leq t_o + \bar{\epsilon}\, u(x_o, t_o)^{2-p}\rho^p. \tag{19}$$

The constants $\bar{\epsilon}$ and $\bar{\gamma}^{-1}$ tend to zero as either $p \to 2$ or as $p \to \frac{2N}{N+1}$.

With respect to the degenerate case, we now have $c = 1$ for the size of the intrinsic cylinders. The upper bound $\mathcal{M}$ has only the qualitative role to insure that $(x_o, t_o) + Q_{8\rho}^{\pm}(\mathcal{M})$ are contained within the domain of definition of u.

3.2. A Weak Carleson Estimate

Relying on the above Harnack inequality, one can first prove a weak form of Carleson estimate. Let Ω_T, u, (x_o, t_o), ρ, x_ρ, P_ρ be as in Theorem 2.2 and set

$$I(t_o, \rho, h) = (t_o - h\rho^p, t_o + h\rho^p).$$

Moreover, let u be a weak solution to (3)–(4) such that

$$0 < u \le M \qquad \text{in} \quad \Omega_T, \tag{20}$$

and assume that

$$I(t_o, 9\rho, M^{2-p}) \subset (0, T]. \tag{21}$$

Then we define

$$\tilde{\Psi}_\rho = \Omega_T \cap \left\{ (x, t) : |x_i - x_{o,i}| < 2\rho, |x_N < 4L\rho|, t \in I(t_o, 9\rho, \eta_\rho^{2-p}) \right\}$$

$$\bar{\Psi}_\rho = \Omega_T \cap \left\{ (x, t) : |x_i - x_{o,i}| < \tfrac{\rho}{4}, |x_N < 2L\rho|, t \in I(t_o, \rho, \eta_\rho^{2-p}) \right\}$$

where η_ρ is the first root of the equation

$$\max_{\tilde{\Psi}_\rho(x_o, t_o)} u = \eta_\rho. \tag{22}$$

Notice that both the functions $y_1(\eta_\rho) = \max_{\tilde{\Psi}_\rho(x_o,t_o)} u$, $y_2(\eta_\rho) = \eta_\rho$ are monotone increasing. Moreover

$$\begin{cases} y_1(0) \ge u(P_\rho) > 0, \\ y_2(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} y_1(M) \le M, \\ y_2(M) = M. \end{cases}$$

Therefore, it is immediate to conclude that at least one root of (22) actually exists. Moreover, by (21) $\tilde{\Psi}_\rho(x_o, t_o) \subset \Omega_T$.

A weak form of the Carleson estimate, is expressed by the following theorem (see [5]).

THEOREM 3.2. (Carleson-type Estimate, weak form, $\frac{2N}{N+1} < p < 2$). *Let u be a weak solution to (3)–(4), that satisfies (20). Assume that (21) holds true and u vanishes continuously on*

$$\partial\Omega \cap \{|x_i - x_{o,i}| < 2\rho, |x_N| < 8L\rho\} \times I(t_o, 9\rho, M^{2-p}).$$

Then there exist universal constants $\gamma > 0$ and $\alpha \in (0,1)$, such that

$$u(x,t) \leq \gamma \left(\frac{\operatorname{dist}(x, \partial\Omega)}{\rho} \right)^\alpha \times \sup_{\tau \in I(t_o, \rho, 2\eta_\rho^{2-p})} u(x_\rho, \tau),$$

for every $(x,t) \in \bar{\Psi}_\rho(x_o, t_o)$.

If we let

$$\Psi_{\rho,M}(x_o, t_o) = \Omega_T \cap \left\{ (x,t) : |x_i - x_{o,i}| < \frac{\rho}{4}, \, |x_N| < 2L\rho, t \in I(t_o, \rho, M^{2-p}) \right\},$$

we have a second statement.

COROLLARY 3.3. *Under the same assumptions of Theorem 3.2, we have*

$$u(x,t) \leq \gamma \left(\frac{\operatorname{dist}(x, \partial\Omega)}{\rho} \right)^\alpha \times \sup_{\tau \in I(t_o, \rho, 2M^{2-p})} u(x_\rho, \tau),$$

for every $(x,t) \in \Psi_{\rho,M}(x_o, t_o)$.

The quantity η_ρ is known only qualitatively through (22), whereas M is a datum. Therefore, Corollary 3.3 can be viewed as a quantitative version of a purely qualitative statement. On the other hand, since η_ρ could be attained in P_ρ, Theorem 3.2 gives the sharpest possible statement, and is genuinely intrinsic.

Moreover, with respect to Theorem 2.2 and Corollary 2.3, Theorem 3.2 combines two distinct statements in a single one (mainly for simplicity), and presents two fundamental differences: when $p > 2$, the value of u at a point above controls the values of u below, whereas when $\frac{2N}{N+1} < p < 2$, the **maximum** of u over a proper time interval centered at t_o controls the values of u **both above and below** the time level t_o. These are consequences of the different statements of the Harnack inequality in the two cases.

Can we improve the result of Theorem 3.2, namely can we substitute the supremum of u on $I(t_o, \rho, 2\eta_\rho^{2-p})$ with the pointwise value $u(P_\rho)$? This would certainly be possible, if there existed a universal constant γ such that

$$\forall t \in I(t_o, \rho, 2\eta_\rho^{2-p}) \qquad u(x_\rho, t) \leq \gamma \, u(P_\rho).$$

Under a geometrical point of view, this amounts to building a Harnack chain connecting (x_ρ, t) and P_ρ, for all $t \in I(t_o, \rho, 2\eta_\rho^{2-p})$. In general, without further assumptions on u, this is not possible, as the following counterexample shows.

Let u be the unique non-negative solution to

$$\begin{cases} u \in C(\mathbb{R}_+; L^2(\Omega)) \cap L^p(\mathbb{R}_+; W_o^{1,p}(\Omega)) \\ u_t - \operatorname{div}(|Du|^{p-2}Du) = 0 \quad \text{in } \Omega_T \\ u(\cdot, 0) = u_o \in C^o(\overline{\Omega}), \end{cases}$$

with $u_o > 0$ in Ω, and $u_o = 0$ on $\partial\Omega$.

By Proposition 2.1, Chapter VII of [19], there exists a finite time T_*, depending only on N, p, u_o, such that $u(\cdot, t) \equiv 0$ for all $t \geq T_*$. By the results of [19, Chapter IV], $u \in C^o(\overline{\Omega \times (0, T_*)})$. Suppose now that at time $t = T_* + 1$, we modify the boundary value and for any $t > T_* + 1$ we let $u(\cdot, t) = g(\cdot, t)$ on $\partial\Omega$, where g is continuous and strictly positive. It is immediate to verify that u becomes strictly positive for any $t > T_* + 1$. Therefore, the positivity set for u is not a connected set, $u(x, t) \equiv 0$ for all $\forall (x, t) \in \overline{\Omega \times (T_*, T_* + 1)}$, and if (x_ρ, t) and P_ρ lie on opposite sides of the vanishing layer for u, by the intrinsic nature of Theorem 3.1, there is no way to connect them with a Harnack chain.

The previous counterexample allows u to vanish identically for t in a proper interval, but by suitably modifying the boundary values, it is clear that we can have u strictly positive, and as close to zero as we want. Therefore, the impossibility of connecting two arbitrary points by a Harnack chain, does not depend on the vanishing of u, but it is a general property of solutions to (3)–(4), whenever $\Omega \neq \mathbb{R}^N$. Moreover, by properly adjusting the boundary value, one can even create an arbitrary number of oscillations for u between positivity and null regions.

We considered solutions to the p-Laplacian just for the sake of simplicity, but everything continues to hold, if we consider the same boundary value problem for (3)–(4).

Notice that if we deal with weak solutions to (3)–(4) in $\mathbb{R}^N \times (0, T]$, then we do not have boundary values any more, the situation previously discussed cannot occur, and therefore any two points (x, t) and (x_o, t_o) can always be connected by a Harnack chain, provided both $u(x, t)$ and $u(x_o, t_o)$ are strictly positive, and $0 < t - t_o < \frac{\epsilon}{8^p} t_o$, as discussed in [24, Chapter 7, Proposition 4.1]. The sub potential lower bound discussed there is then a property of weak solutions given in the whole $\mathbb{R}^N \times (0, T)$.

The Harnack inequality given in Theorem 3.1 is time-insensitive, and its constants are not stable as $p \to 2$. A different statement, analogous to the one given in Theorem 2.1, could be given, and in such a case the constants would be stable (see [24, Chapter 6] for a thorough discussion of the two possible forms). However, the eventual result is the same, and independently of the kind of Harnack inequality one considers, two points (x, t) and (x_o, t_o) of positivity for u, cannot be connected by a Harnack chain.

Notice that we have a sort of dual situation: when $1 < p < 2$ the support of u can be disconnected in time, when $p > 2$, the support can be disconnected in space.

Strictly speaking, the previous counterexample only shows that we cannot replace the line with a point, but per se it does not rule out the possibility for a strong form of Carleson's estimate to hold true all the same. However, if one tries to adapt to the singular super-critical context the standard proof based

on the Harnack inequality and the boundary Hölder continuity (as we did, for example, in the degenerate context), then one quickly realizes that, one needs to know in advance the oscillation of u: this suggests that only a control in terms of the supremum taken in a proper set can be feasible.

3.3. A Strong Carleson Estimate

With respect to the statement of Theorem 3.2, a stronger form is indeed possible, provided we allow the parameter γ to depend not only on the data, but also on the oscillation of u. Let Ω_T, u, (x_o, t_o), ρ, P_ρ be as in Theorem 2.2, and for $k = 0, 1, 2, \ldots$ set

$$\rho_k = \left(\frac{7}{8}\right)^k \rho, \qquad \sigma_k = \frac{\rho_k}{\gamma^{k\frac{2-p}{p}}},$$

$$x_{\rho_k} = (x'_o, 2L\rho_k), \qquad P_{\rho_k} = (x'_o, 2L\rho_k, t_o),$$

$$\Psi_{\rho_k, M}(x_o, t_o)$$
$$= E_T \cap \left\{(x, t) : |x_i - x_{o,i}| < \frac{\rho_k}{4}, \ |x_N| < 2L\rho_k, \ t \in I(t_o, \sigma_k, M^{2-p})\right\},$$

$$m_o = \inf_{\tau \in I(t_o, \rho, 2M^{2-p})} u(x_\rho, \tau), \quad M_o = \sup_{\tau \in I(t_o, \rho, 2M^{2-p})} u(x_\rho, \tau).$$

COROLLARY 3.4. (Carleson-type Estimate, strong form, $\frac{2N}{N+1} < p < 2$). *Let u be a weak solution to* (3)–(4) *such that $0 < u \leq M$ in Ω_T. Assume that $I(t_o, 9\rho, M^{2-p}) \subset (0, T]$ and that u vanishes continuously on*

$$\partial\Omega \cap \{|x_i - x_{o,i}| < 2\rho, |x_N| < 8L\rho\} \times I(t_o, 9\rho, M^{2-p}).$$

Then there exists a constant γ, depending only on ρ, N, C_o, C_1, L, and $\frac{M}{m_o}$, such that

$$u(x, t) \leq \gamma\, u(P_{\rho_k}), \tag{23}$$

for every $(x, t) \in \Psi_{\rho_k, M}(x_o, t_o)$, for all $k = 0, 1, 2, \ldots$.

The strong form of the Carleson-type estimate is derived from Corollary 3.3. An analogous statement can be derived from Theorem 3.2.

Estimate (23) has the same structure as the backward Harnack inequality for caloric functions that vanish just on a disk at the boundary (see [12, Theorem 13.7, page 234]). This is not surprising, because (23) is indeed a backward Harnack inequality, due to the specific nature of the Harnack inequality for the singular case. However, it is worth mentioning that things are not completely equivalent; indeed, the constants we have in the time-insensitive Harnack inequality (18)–(19) are not stable (and cannot be stabilized), and therefore, the result for caloric functions cannot be recovered from the singular case, by simply letting $p \to 2$ (as it is instead the case for many other results).

3.3.1. Hopf Principle and Boundary Harnack inequality

Another striking difference with respect to the degenerate case appears when we consider $C^{1,1}$ cylinders and (mainly for simplicity) the prototype equation $(1)_o$. In this case, indeed, weak solutions vanishing on the lateral part enjoy a linear behavior at the boundary with implications expressed in the following result. Note that the role of L in the definition of $\Psi_{\rho,M}$ is now played by $C^{1,1}$ constant $M_{1,1}$ of Ω.

THEOREM 3.5. *Let $\frac{2N}{N+1} < p < 2$. Assume Ω_T is a $C^{1,1}$ cylinder, and (x_o, t_o), ρ, P_ρ are as in Theorem 2.2. Let u, v be two weak solutions to $(1)_o$ in Ω_T, satisfying the hypotheses of Theorem 3.2, $0 < u, v \le M$ in Ω_T. Then there exist positive constants $\bar{s}$, γ, β, $0 < \beta \le 1$, depending only on N, p, and $M_{1,1}$, and ρ_o, $c_o > 0$, depending also on the oscillation of u, such that the following properties hold.*

(a) *Hopf Principle:*
$$|Du| \ge c_o \qquad in \qquad \Psi_{\rho_o, M}(x_o, t_o). \tag{24}$$

(b) *Boundary Harnack Inequality:*
$$\gamma^{-1} \frac{\displaystyle\inf_{\tau \in I(t_o, \rho, 2M^{2-p})} u(x_\rho, \tau)}{\displaystyle\sup_{\tau \in I(t_o, \rho, 2M^{2-p})} v(x_\rho, \tau)} \le \frac{u(x, t)}{v(x, t)} \le \gamma \frac{\displaystyle\sup_{\tau \in I(t_o, \rho, 2M^{2-p})} u(x_\rho, \tau)}{\displaystyle\inf_{\tau \in I(t_o, \rho, 2M^{2-p})} v(x_\rho, \tau)}, \tag{25}$$

for all $(x, t) \in \{x \in K_{\bar{s}\frac{\rho}{4}}(x_o) \cap \Omega : \operatorname{dist}(x, \partial\Omega) < \bar{s}\frac{\rho}{8}\} \times I(t_o, \rho, \frac{1}{2}M^{2-p})$, with $\rho < \rho_o$.

(c) *The quotient u/v is Hölder continuous with exponent β in $\Psi_{\frac{\rho_o}{2}, M}(x_o, t_o)$*

Since
$$\frac{\displaystyle\sup_{\tau \in I(t_o, \rho, 2M^{2-p})} u(x_\rho, \tau)}{\displaystyle\inf_{\tau \in I(t_o, \rho, 2M^{2-p})} v(x_\rho, \tau)} \le \frac{M_{o,u} u(P_\rho)}{m_{o,u}} \frac{M_{o,v}}{m_{o,v} v(P_\rho)},$$

$$\frac{\displaystyle\inf_{\tau \in I(t_o, \rho, 2M^{2-p})} u(x_\rho, \tau)}{\displaystyle\sup_{\tau \in I(t_o, \rho, 2M^{2-p})} v(x_\rho, \tau)} \le \frac{m_{o,u} u(P_\rho)}{M_{o,u}} \frac{m_{o,v}}{M_{o,v} v(P_\rho)},$$

the Boundary Harnack Inequality (25) can be rewritten as

$$\tilde{\gamma}^{-1} \frac{u(P_\rho)}{v(P_\rho)} \le \frac{u(x, t)}{v(x, t)} \le \tilde{\gamma} \frac{u(P_\rho)}{v(P_\rho)}$$

where now $\tilde{\gamma}$ depends not only on N, p, $M_{1,1}$, but also on $M_{o,u}/m_{o,u}$ and $M_{o,v}/m_{o,v}$.

Note that **(a)** implies that near a part of the lateral boundary, where a non-negative solution vanishes, the parabolic p-Laplace operator is uniformly elliptic. Since we do not have an estimate at the boundary of the type

$$|Du(x,t)| \geq c \frac{u(x,t)}{\text{dist}(x,\partial\Omega)},$$

(a) and **(c)** hold only in a small neighbourhood of S_T, whose size depends on the solution, as both c_o and the oscillation of the gradient Du depend on the oscillation of u: this is precisely the meaning of ρ_o.

The proof relies on proper estimates from above and below, which were originally proved in [25, § 4] for solutions to the singular porous medium equations in C^2 domains by building explicit barriers.

We recast these estimates in the lemma below, in a form tailored to our purposes. Indeed, the Hopf Principle and a weak version of the Boundary Harnack Inequality follow easily from these estimates. Our improvement lies in the use of the Carleson estimates, that allow a more precise bound for $\frac{u(x,t)}{v(x,t)}$ in terms of $\frac{u(P_\rho)}{v(P_\rho)}$. The restriction to $\frac{2N}{N+1} < p < 2$ comes into play only in this last step.

Thus, let $\partial\Omega$ be of class $C^{1,1}$ and u be a non-negative, weak solution to $(1)_o$ in Ω_T, for $1 < p < 2$. Assume that $u \leq M$ in Ω_T. For $x \in \mathbb{R}^N$, set $d(x) = \text{dist}(x, \partial\Omega)$, and for $s > 0$, let

$$\Omega^s = \{x \in \Omega : \quad \frac{s}{2} \leq d(x) \leq 2s\}.$$

LEMMA 3.6. *Let $\tau \in (0, T)$ and fix $x_o \in \partial\Omega$. Assume that u vanishes on*

$$\partial\Omega \cap K_{2\rho}(x_o) \times (\tau, T).$$

For every $\nu > 0$, there exist positive constants γ_1, γ_2, and $0 < \bar{s} < \frac{1}{2}$, depending only on N, p, ν, and $M_{1,1}$, such that for all $\tau + \nu M^{2-p}\rho^p < t < T$, and for all $x \in \Omega \cap K_{2\bar{s}\rho}(x_o)$ with $d(x) < \bar{s}\rho$,

$$\gamma_2 \left(\frac{d(x)}{\rho}\right) \inf_{K_{2\rho}(x_o) \cap \Omega^{\bar{s}\rho} \times (\tau, T)} u \leq u(x, t) \leq \gamma_1 \left(\frac{d(x)}{\rho}\right) \sup_{\Omega \cap K_{2\rho}(x_o) \times (\tau, T)} u.$$

Relying on the above lemma, the proof of Theorem 3.5 follows rather easily.

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