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Stability analysis of the inverse inclusion problem

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Dedicated to Professor Giovanni Alessandrini on the occasion of his 60th birthday

Abstract. *We review some results concerning the determination of an inclusion within a body. In particular we show stability estimates, that is the continuous dependance of the inclusion from the boundary measurements. We present the cases of an electrical conductor, an elastic body and a thermal conductor.*

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1. Introduction

In this note we consider the inverse problem of determining an inclusion D contained in a domain Ω. More precisely we aim to locate a region of a specimen whose physical properties are different from the properties of the surrounding material. For instance, if we consider an electrical conductor Ω of constant conductivity 1, the inclusion D has a conductivity equals to some unknown constant k, different from 1.

Prescribing a voltage $f \in H^{1/2}(\partial\Omega)$ on the boundary of Ω, the induced potential $u \in H^1(\Omega)$ is the solution of the problem

$$\begin{cases} \operatorname{div}((1+(k-1)\chi_D)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where χ_D denotes the characteristic function of the set D.

The normal derivative of the solution u on the boundary $\frac{\partial u}{\partial \nu}_{|\partial\Omega}$ corresponds to the current density measured. The pair of Cauchy data $\left\{f, \frac{\partial u}{\partial \nu}_{|\partial\Omega}\right\}$ represents the electrostatic measurements performed on the boundary. We define the so called Dirichlet–to–Neumann map Λ_D as

$$\begin{array}{cccc} \Lambda_D & : & H^{1/2}(\partial\Omega) & \to & H^{-1/2}(\partial\Omega) \\ & & f & \to & \frac{\partial u}{\partial \nu}_{|\partial\Omega}. \end{array}$$

Its knowledge corresponds to performing infinitely many boundary measurements.

The inverse problem we are addressing to is to recover information on the inclusion D from a knowledge of the map Λ_D.

This problem is a special instance of the well-known Calderon's inverse conductivity problem [11]. Uniqueness was established in 1988 by Isakov [21], whose approach makes use of the Runge approximation Theorem and solutions of the equation with Green's function type singularities.

In 2005 Alessandrini and Di Cristo [4] have studied the stability issue, that is the continuous dependance of the solution D from the given data Λ_D. Converting Isakov's argument in a quantitative form, the authors prove that under mild a priori assumptions on the regularity and the topology of the inclusion, the modulus of continuity is of logarithmic type. Though such a modulus of continuity is weak, in [14] it is shown that, keeping as minimal as possible, the a priori information on the solution, it turns out to be optimal. To improve this rate of continuity, more a priori information on the inclusion are needed (see for instance [8]).

The argument proposed in [4] is very flexible and it can be extended to other problems like locating a scattered object by the knowledge of the near field data [13] or an inclusion in an elastic body by measuring the displacement and the traction on the boundary [5] or in a thermal conductor from the knowledge of the temperature and the heat flux on the boundary [15].

Let us mention here that in all these papers a crucial role is played by the explicit representation of the fundamental solution of the operator $\mathrm{div}(1 + (k-1)\chi^+\nabla\cdot))$, where χ^+ is the characteristic function of the half space. It would be interesting generalize such argument when different information on the fundamental solution are available. Some ideas in this direction can be found in the parabolic case (see Section 4) but still it is not clear what kind of analysis is needed.

In this review note we illustrate the main step to get stability in the impedance tomography case (Section 2). Then in the subsequent Section 3 we analyze the elastic body context, emphasizing the main differences and the new tools needed. We conclude in the last Section 4 with the parabolic case.

2. Electrical Conductors

Let us first premise some notations and definitions we will use later on. In places we denote a point $x \in \mathbf{R}^n$ by $x = (x', x_n)$, where $x' \in \mathbf{R}^{n-1}$, $x_n \in \mathbf{R}$.

DEFINITION 2.1. *Let Ω be a bounded domain in $\mathbf{R}^n$. Given α, $0 < \alpha \leq 1$, we shall say that a portion S of $\partial\Omega$ is of class $C^{1,\alpha}$ with constants r_0, $M_0 > 0$ if, for any $P \in S$, there exists a rigid transformation of coordinates under which* *we have* $P = 0$ *and*

$$\Omega \cap B_{r_0}(0) = \{x \in B_{r_0} \, : \, x_n > \varphi(x')\},$$

where φ *is a* $C^{1,\alpha}$ *function on* $B_{r_0}(0) \subset \mathbf{R}^{n-1}$ *satisfying* $\varphi(0) = |\nabla\varphi(0)| = 0$ *and* $\|\varphi\|_{C^{1,\alpha}(B_{r_0}(0))} \leq M_0 r_0$.

Definition 2.2. *We shall say that a portion* S *of* $\partial\Omega$ *is of Lipschitz class with constants* r_0, $M_0 > 0$ *if for any* $P \in S$, *there exists a rigid transformation of coordinates under which we have* $P = 0$ *and*

$$\Omega \cap B_{r_0}(0) = \{x \in B_{\overline{r}} \, : \, x_n > \varphi(x')\},$$

where φ *is a Lipschitz continuous function on* $B_{r_0}(0) \subset \mathbf{R}^{n-1}$ *satisfying* $\varphi(0) = 0$ *and* $\|\varphi\|_{C^{0,1}(B_{r_0}(0))} \leq M_0 r_0$.

Assumptions on the domain
Given $r_0, M_0, M_1 > 0$ and $0 < \alpha < 1$ as constants, we assume that $\Omega \subset \mathbf{R}^n$ is of class $C^{1,\alpha}$ class with constants r_0, M_0 such that

$$|\Omega| \leq M_1 r_0^n,$$

where $|\cdot|$ denotes the Lebesgue measure of Ω.

Assumptions on the inclusion
Let D be a domain contained in Ω such that $\mathbf{R}^n \setminus D$ is connected, ∂D is of $C^{1,\alpha}$ class with constants r_0, M_0 and, for a given $\delta_0 > 0$, $\mathrm{dist}(D, \partial\Omega) \geq \delta_0$.

In what follows we will refer to constants $k, n, r_0, M_0, M_1, \alpha, \delta_0$ as to the a priori data. We recall that $n \geq 2$ is the dimension and k is the conductivity inside the inclusion.

We denote by D_1 and D_2 two possible inclusions in Ω both satisfying the aforementioned properties and by Λ_{D_1} and Λ_{D_2} the corresponding Dirichlet–to–Neumann maps.

Remark 2.3. As it is well known, the Dirichlet–to–Neumann map Λ_D associated to problem (1) is defined by

$$< \Lambda_D u, v >= \int_\Omega (1 + (k-1)\chi_D)\nabla u \cdot \nabla v,$$

for every $u \in H^1(\Omega)$ solution to (1) and $v \in H^1(\Omega)$. Here $< \cdot, \cdot >$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. With a slight abuse of notation, we will write

$$< g, f >= \int_{\partial\Omega} g f d\sigma,$$

for any $f \in H^{1/2}(\partial\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$.

THEOREM 2.4. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be as above, $k > 0$, $k \neq 1$ be given and D_1 and D_2 be two inclusions in Ω as above. If, given $\varepsilon > 0$, we have*

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} \leq \varepsilon, \tag{2}$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq \omega(\varepsilon), \tag{3}$$

where ω is an increasing function on $[0, +\infty)$, which satisfies

$$\omega(t) \leq C|\log t|^{-\eta}, \qquad \text{for every} \quad 0 < t < 1$$

and C, η, $C > 0$, $0 < \eta \leq 1$, are constants only depending on the a priori data.

Here $d_{\mathcal{H}}$ denotes the Hausdorff distance between bounded closed sets of $\mathbf{R}^n$ and $\|\cdot\|_{\mathcal{L}(H^{1/2}H^{-1/2})}$ denotes the operator norm on the space of bounded linear operators between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. Let us also stress here that this theorem holds in any dimension $n \geq 2$ as the proof is based on singular solutions arguments that are not related to the dimension.

REMARK 2.5. For the sake of simplicity we have chosen to present the theorem in the case of piecewise constant conductivity with the knowledge of the full Dirichlet–to–Neumann map- It is possible to consider a slightly more general case with conductivities of the form

$$\gamma(x) = a(x) + b(x)\chi_D,$$

where $a \in C^{0,1}(\overline{\Omega})$ and $b \in C^{\alpha}(\Omega)$, and when only a portion of the boundary $\partial\Omega$ is available to perform measurements. We refer to [12] for a detailed study of this problem.

Let us sketch the argument to prove this theorem. For the reader convenience we divide it into several steps.

Step 1: modified distance.
Let $\mathcal{G}$ be the connected component of $\mathbf{R}^n \setminus (\overline{D_1 \cup D_2})$ which contains $\mathbf{R}^n \setminus \overline{\Omega}$ and let us denote $\Omega_D = \mathbf{R}^n \setminus \overline{\mathcal{G}}$. As we shall see later, one of the key ingredients of the stability proof consists in propagating the smallness appearing in the measurements (2) from the boundary $\partial\Omega$ inside Ω. Since the value $d_H(\partial D_1, \partial D_2)$ may be attained at some point not belonging to $\overline{\mathcal{G}}$ and, therefore, not reachable from the exterior, it is necessary to introduce a *modified distance* following the ideas developed in [4]. Precisely, let us introduce the *modified distance* between D_1 and D_2

$$d_\mu(D_1, D_2) = \max\left\{ \max_{x \in \partial D_1 \cap \partial\Omega_D} \operatorname{dist}(x, D_2), \max_{x \in \partial D_2 \cap \partial\Omega_D} \operatorname{dist}(x, D_1) \right\}. \tag{4}$$

We remark here that d_μ is not a metric and, in general, it does not dominate the Hausdorff distance. However, under our a priori assumptions on the inclusion, the following lemma holds true.

LEMMA 2.6. *Under the assumptions of Theorem 2.4, there exists a constant $c_0 \geq 1$ only depending on M_0 and α such that*

$$d_H(\partial D_1, \partial D_2) \leq c_0 d_\mu(D_1, D_2). \tag{5}$$

Proof. See [4, Proposition 3.3]. □

It is easy to verify that

$$\max_{x\in\partial D_1\cap\partial\Omega_D} \mathrm{dist}(x, D_2) = \max_{x\in\partial D_1\cap\partial\Omega_D} \mathrm{dist}(x, \partial D_2)$$

$$\max_{x\in\partial D_2\cap\partial\Omega_D} \mathrm{dist}(x, D_1) = \max_{x\in\partial D_2\cap\partial\Omega_D} \mathrm{dist}(x, \partial D_1),$$

so that $d_\mu(D_1, D_2) \leq d_H(\partial D_1, \partial D_2)$, and therefore, in view of Lemma 2.6, these two quantities are comparable.

Another obstacle comes out from the fact that the propagation of smallness arguments are based on an iterated application of the three-spheres inequality for solutions of the equation over chains of balls contained in $\mathcal{G}$ and, in this step, it is crucial to control from below the radii of these balls. In the following Lemma 2.7 we treat the case of points of $\partial\Omega_D$ that are not reachable by such chains of balls. This problem was originally considered by [7] in the context of cracks detection in electrical conductors and was underestimated in the papers [4, 12, 13, 15, 16]. The procedure developed here enables to fill the possible gaps in the proofs.

Let us premise some notation. Given $O = (0, \dots, 0)$ the origin, v a unit vector, $h > 0$ and $\vartheta \in \left(0, \frac{\pi}{2}\right)$, we denote

$$C(O, v, h, \vartheta) = \{x \in \mathbf{R}^n |\ |x - (x \cdot v)v| \leq \sin\vartheta |x|,\ 0 \leq x \cdot v \leq h\} \tag{6}$$

the closed truncated cone with vertex at O, axis along the direction v, height h and aperture 2ϑ. Given R, d, $0 < R < d$ and $Q = -de_n$, where $e_n = (0, \dots, 0, 1)$, let us consider the cone $C\left(O, -e_n, \frac{d^2-R^2}{d}, \arcsin\frac{R}{d}\right)$.

From now on, for simplicity, we assume that

$$d_\mu(D_1, D_2) = \max_{x\in\partial D_1\cap\partial\Omega_D} \mathrm{dist}(x, \partial D_2) \tag{7}$$

and we write $d_\mu = d_\mu(D_1, D_2)$.

Let us define

$$S_{2\rho_0} = \left\{x \in \mathbf{R}^n \ |\rho_0 < \mathrm{dist}(x, \overline{\Omega}) < 2\rho_0\right\}. \tag{8}$$

We shall make use of paths connecting points in order that appropriate tubular neighborhoods of such paths still remain within $\mathbf{R}^n \setminus \Omega_D$.

Let us pick a point $P \in \partial D_1 \cap \partial\Omega_D$, let ν be the outer unit normal to ∂D_1 at P and let $d > 0$ be such that the segment $[(P + d\nu), P]$ is contained in $\mathbf{R}^n \setminus \Omega_D$. Given $P_0 \in \mathbf{R}^n \setminus \Omega_D$, let γ be a path in $\mathbf{R}^n \setminus \Omega_D$ joining P_0 to $P + d\nu$. We consider the following neighborhood of $\gamma \cup [(P + d\nu), P] \setminus \{P\}$ formed by a tubular neighborhood of γ attached to a cone with vertex at P and axis along ν

$$V(\gamma) = \bigcup_{S\in\gamma} B_R(S) \cup C\left(P, \nu, \frac{d^2 - R^2}{d}, \arcsin \frac{R}{d}\right). \tag{9}$$

Note that two significant parameters are associated to such a set, the radius R of the tubular neighborhood of γ, $\cup_{S\in\gamma} B_R(S)$, and the half-aperture $\arcsin \frac{R}{d}$ of the cone $C\left(P, \nu, \frac{d^2-R^2}{d}, \arcsin \frac{R}{d}\right)$. In other terms, $V(\gamma)$ depends on γ and also on the parameters R and d. At each of the following steps, such two parameters shall be appropriately chosen and shall be accurately specified. For the sake of simplicity we convene to maintain the notation $V(\gamma)$ also when different values of R, d are introduced.

Also we warn the reader that it will be convenient at various stages to use a reference frame such that $P = O = (0, \dots, 0)$ and $\nu = -e_n$.

Lemma 2.7. *Under the above notation, there exist positive constants $\overline{d}$, c_1, where $\frac{\overline{d}}{\rho_0}$ only depends on M_0 and α, and c_1 only depends on M_0, α, M_1, and there exists a point $P \in \partial D_1$ satisfying*

$$c_1 d_\mu \le dist(P, D_2), \tag{10}$$

and such that, giving any point $P_0 \in S_{2\rho_0}$, there exists a path $\gamma \subset (\overline{\Omega^{\rho_0}} \cup S_{2\rho_0}) \setminus \overline{\Omega_D}$ joining P_0 to $P + \overline{d}\nu$, where ν is the unit outer normal to D_1 at P, such that, choosing a coordinate system with origin O at P and axis $e_n = -\nu$, the set $V(\gamma)$ introduced in (9) *satisfies*

$$V(\gamma) \subset \mathbf{R}^n \setminus \Omega_D, \tag{11}$$

provided $R = \frac{\overline{d}}{\sqrt{1+L_0^2}}$, where L_0, $0 < L_0 \le M_0$, is a constant only depending on M_0 and α.

In order to prove Lemma 2.7, we shall use the following results.

Lemma 2.8 (Lemma 5.5 in [6]). *Let U be a Lipschitz domain in $\mathbf{R}^n$ with constants ρ_0, M_0. There exists h_0, $0 < h_0 < 1$, only depending on M_0, such that*

$$U_{h\rho_0} \text{ is connected for every } h,\ 0 < h \le h_0. \tag{12}$$

THEOREM 2.9 (Theorem 3.6 in [3]). *There exist positive constants d_0, r_0, L_0, $L_0 \leq M_0$, with $\frac{d_0}{\rho_0}$, $\frac{r_0}{\rho_0}$ only depending on M_0 and L_0 only depending on α and M_0, such that if*

$$d_H(\partial D_1, \partial D_2) \leq d_0, \tag{13}$$

then $\partial\Omega_D$ is Lipschitz with constants r_0 and L_0. Moreover, for every $P \in \partial\Omega_D \cap \partial D_1$, up to a rigid transformation of coordinates which maps P into the origin and $e_n = -\nu$, where ν is the outer unit normal to D_1 at P, we have

$$D_i \cap B_{r_0}(P) = \{x \in B_{r_0}(0) |\ x_n > \varphi_i(x')\}, \quad i = 1, 2, \tag{14}$$

$$\varphi_1(0) = 0, \quad \nabla\varphi_1(0) = 0, \tag{15}$$

$$\|\varphi_i\|_{C^{0,1}(B'_{r_0}(0))} \leq L_0 r_0, \quad i = 1, 2. \tag{16}$$

An analogous representation holds for every $P \in \partial\Omega_D \cap \partial D_2$.

Proof of Lemma 2.7. Let

$$d_1 = \frac{d_0}{c_0}, \tag{17}$$

where c_0 is the constant introduced in Lemma 2.6, and let

$$d_2 = \min\{d_1, h_0\rho_0\}, \tag{18}$$

where h_0, $0 < h_0 < 1$, only depending on M_0, has been introduced in Lemma 2.8. We shall distinguish two cases.

Case i) Let $d_\mu \leq d_1$.

Then, by Lemma 2.6 we have $d_H(\partial D_1, \partial D_2) \leq d_0$. Therefore, by Theorem 2.9, $\partial\Omega_D$ is Lipschitz with constants r_0, L_0, where $\frac{r_0}{\rho_0}$ only depends on M_0, and L_0 only depends on M_0 and α. We may apply Lemma 2.8 to $\mathbf{R}^n \setminus \Omega_D$ obtaining that there exists $\widetilde{h}_0$, $0 < \widetilde{h}_0 < 1$, only depending on α and M_0, such that $(\mathbf{R}^n \setminus \Omega_D)_{hr_0}$ is connected for every $h \leq \widetilde{h}_0$.

Let $P \in \partial D_1 \cap \partial\Omega_D$ be such that

$$d_\mu(D_1, D_2) = \operatorname{dist}(P, D_2). \tag{19}$$

Under the coordinate system introduced in Theorem 2.9, let us consider the point $Q = P - \frac{\widetilde{h}_0 r_0}{2} e_n$. We have that

$$\operatorname{dist}(Q, \Omega_D) \geq \frac{\widetilde{h}_0 r_0}{2\sqrt{1 + L_0^2}}. \tag{20}$$

Let us denote $h_1 = \frac{\widetilde{h}_0}{2\sqrt{1+L_0^2}}$. Since $h_1 < \widetilde{h}_0$, the set $\overline{(\mathbf{R}^n \setminus \Omega_D)_{h_1 r_0}}$ is connected and contains Q. Therefore, there exists a path $\gamma \subset \overline{(\mathbf{R}^n \setminus \Omega_D)_{h_1 r_0}}$ joining any point $P_0 \in S_{2\rho_0}$ with Q. Therefore, in the above coordinate system, the set $V(\gamma)$ satisfies

$$V(\gamma) \subset \mathbf{R}^n \setminus \Omega_D, \tag{21}$$

provided

$$d = \frac{\widetilde{h}_0 r_0}{2}, \quad R = \frac{d}{\sqrt{1+L_0^2}}. \tag{22}$$

Case ii) Let $d_\mu \geq d_1$.

Then, trivially, $d_\mu \geq d_2$. Let $\widetilde{P} \in \partial D_1 \cap \partial\Omega_D$ be such that

$$d_\mu(D_1, D_2) = \operatorname{dist}(\widetilde{P}, D_2). \tag{23}$$

Since $d_2 \leq h_0\rho_0$, by Lemma 2.8, $(\mathbf{R}^n \setminus D_2)_{d_2}$ is connected. Therefore, given any point $P_0 \in S_{2\rho_0}$, there exists a path γ, $\gamma : [0,1] \to (\mathbf{R}^n \setminus D_2)_{d_2}$ such that $\gamma(0) \in S_{2\rho_0}$ and $\gamma(1) = \widetilde{P}$. Let $\bar{t} = \inf_{t\in[0,1]} \left\{t |\; dist(\gamma(t), \partial D_1) > \frac{d_2}{2}\right\}$. By definition, $dist(\gamma(\bar{t}), \partial D_1) = \frac{d_2}{2}$, so that there exists $P \in \partial D_1$ satisfying $|P - \gamma(\bar{t})| = \frac{d_2}{2}$. We have that

$$\operatorname{dist}(P, D_2) \geq \operatorname{dist}(\gamma(\bar{t}), D_2) - |\gamma(\bar{t}) - P| \geq d_2 - \frac{d_2}{2} = \frac{d_2}{2}. \tag{24}$$

Let $\overline{\gamma} = \gamma|_{[0,\bar{t}]}$ and let us choose a cartesian coordinate system with origin O at P, and $e_n = -\nu$, where ν is the outer unit normal to D_1 at P. We have that

$$V(\overline{\gamma}) \subset \mathbf{R}^n \setminus \Omega_D, \tag{25}$$

assuming

$$d = \frac{d_2}{2}, \quad R = \frac{d}{\sqrt{1+M_0^2}}. \tag{26}$$

Let

$$\overline{d} = \min\left\{\frac{\widetilde{h}_0 r_0}{2}, \frac{d_0}{2c_0}, \frac{h_0\rho_0}{2}\right\}, \tag{27}$$

and let us notice that $\frac{\overline{d}}{\rho_0}$ only depends on M_0, α. Observing that $L_0 \leq M_0$, formula (11) follows with $\overline{d}$ given in (27). Since there exists a positive constant C only depending on M_0, M_1 such that $diam(\Omega) \leq C\rho_0$, we have that

$$d_\mu \leq \left(\frac{\operatorname{diam}(\Omega)}{\frac{d_2}{2}}\right)\frac{d_2}{2} \leq \widetilde{c}_1\frac{d_2}{2}, \tag{28}$$

with $\widetilde{c}_1$ only depending on M_0, α and M_1. Letting $c_1 = \min\left\{1, \frac{1}{\widetilde{c}_1}\right\}$, inequality (10) follows. $\square$

From now on we will denote by $P = O \in \partial D_1 \cap \partial\Omega$ the point such that

$$d_\mu(D_1, D_2) = \mathrm{dist}(P, D_2). \tag{29}$$

Step 2: Alessandrini's identity.
Let $u_i \in H^1(\Omega)$, $i = 1, 2$, be solutions to (1) when $D = D_1, D_2$ respectively, the following identity holds.

$$\int_\Omega (1 + (k-1)\chi_{D_1})\nabla u_1 \cdot \nabla u_2 - \int_\Omega (1 + (k-1)\chi_{D_2})\nabla u_1 \cdot \nabla u_2 = \int_{\partial\Omega} u_1[\Lambda_{D_1} - \Lambda_{D_2}]u_2. \tag{30}$$

This identity can be obtained by using repeatedly Green's formula. In the context of inverse problems, the prototype of this identity can be traced back to Alessandrini, who first used in [1].

Let $\Gamma_D(x, y)$ be the fundamental solution for the operator $\mathrm{div}((1 + (k - 1)\chi_D)\nabla\cdot)$, thus

$$\mathrm{div}((1 + (k-1)\chi_D)\nabla\Gamma_D(\cdot, y)) = -\delta(\cdot - y), \tag{31}$$

where $y \in \mathbf{R}^n$, δ denotes the Dirac distribution . We shall denote by Γ_{D_1}, Γ_{D_2} such fundamental solutions when $D = D_1$, D_2 respectively. Replacing u_1, u_2 with $\Gamma_{D_1}, \Gamma_{D_2}$ in (30), we get

$$\begin{aligned}\int_\Omega (1 + (k-1)\chi_{D_1})\nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, w) \\ - \int_\Omega (1 + (k-1)\chi_{D_2})\nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, w) \\ = \int_{\partial\Omega} \Gamma_{D_1}(\cdot, y)[\Lambda_{D_1} - \Lambda_{D_2}](\Gamma_{D_2}(\cdot, w))d\sigma,\end{aligned} \tag{32}$$

for any singularities y and w taken in the complement $\mathcal{C}\overline{\Omega}$ of $\overline{\Omega}$. Let us define, for $y, w \in \mathcal{G} \cup \mathcal{C}\Omega$

$$S_{D_1}(y, w) = (k-1)\int_{D_1} \nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, w), \tag{33}$$

$$S_{D_2}(y, w) = (k-1)\int_{D_2} \nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, w), \tag{34}$$

$$f(y, w) = S_{D_1}(y, w) - S_{D_2}(y, w). \tag{35}$$

Thus (32) can be rewritten as

$$f(y,w)=\int_{\partial\Omega}\Gamma_{D_1}(\cdot,y)[\Lambda_{D_1}-\Lambda_{D_2}](\Gamma_{D_2}(\cdot,w))d\sigma \qquad \forall\, y,w\in\mathcal{C}\overline{\Omega}. \tag{36}$$

For $y,w\in\mathcal{C}\overline{\Omega}$, since (2), $f(y,w)$ is small. The idea to get stability is to evaluate how this smallness propagates as y and w move toward the inclusion To perform such analysis, a crucial step is the study of the behavior of the fundamental solution.

Step 3: fundamental solutions.
For $x=(x',x_n)$, where $x'\in\mathbf{R}^{n-1}$ and $x_n\in\mathbf{R}$, we set $x^\star=(x',-x_n)$. We shall denote with χ^+ the characteristic function of the half-space $\{x_n>0\}$ and with Γ_+ the fundamental solution of the operator $\mathrm{div}((1+(k-1)\chi^+)\nabla\cdot)$. If Γ is the standard fundamental solution of the Laplace operator, we have that

$$\Gamma_+(x,y)=\begin{cases}\frac{1}{k}\Gamma(x,y)+\frac{k-1}{k(k+1)}\Gamma(x,y^\star) & \text{for } x_n>0, y_n>0,\\ \frac{2}{k+1}\Gamma(x,y) & \text{for } x_ny_n<0,,\\ \Gamma(x,y)-\frac{k-1}{k+1}\Gamma(x,y^\star) & \text{for } x_n<0, y_n<0.\end{cases} \tag{37}$$

The following Proposition holds.

PROPOSITION 2.10. *Let $D\subset\mathbf{R}^n$ be an open set whose boundary is of class $C^{1,\alpha}$, with constants r_0, M_0.*

(i) There exists a constant $c_1>0$ depending on k, n, α and M_0 only, such that

$$|\nabla_x\Gamma_D(x,y)|\le c_1|x-y|^{1-n}, \tag{38}$$

for every $x,y\in\mathbf{R}^n$,

(ii) There exist constants c_2, $c_3>0$ depending on k, n, α and M_0 only, such that

$$\big|\Gamma_D(x,y)-\Gamma_+(x,y)\big|\le\frac{c_2}{\overline{r}^\alpha}|x-y|^{2-n+\alpha}, \tag{39}$$

$$\big|\nabla_x\Gamma_D(x,y)-\nabla_x\Gamma_+(x,y)\big|\le\frac{c_3}{\overline{r}^{\alpha^2}}|x-y|^{1-n+\alpha^2}, \tag{40}$$

for every $x\in D\cap B_r(P)$, and for every $y=h\nu(P)$, with $0<r<\overline{r}_0$, $0<h<\overline{r}_0$, where $\overline{r}_0=\big(\min\big\{\frac{1}{2}(8M_0)^{-1/\alpha},\frac{1}{2}\big\}\big)\frac{r_0}{2}$.

Proof. The proof of *i)* is based on the $C^{1,\alpha}$ regularity of Γ_D proved in [17], see also [24], and the pointwise bounds of Γ_D with Γ contained in [25].

To prove *ii)* we first flatten the boundary ∂D around the point P through a $C^{1,\alpha}$ diffeomorphism Φ from $\mathbf{R}^n$ into itself. Defining $\tilde{\Gamma}_D(\xi,\eta) = \Gamma_D(x,y)$ where $\xi = \Phi(x)$, $\eta = \Phi(y)$, it is not difficult to check that $\tilde{\Gamma}_D$ solves

$$\operatorname{div}_\xi((1+(k-1)\chi^+)B(\xi)\nabla_\xi\widetilde{\Gamma}_D(\xi,\eta)) = -\delta(\xi-\eta),$$

where B is a C^α matrix such that $B(0) = I$. Considering

$$\tilde{R}(x,y) = \tilde{\Gamma}_D(x,y) - \Gamma_+(x,y),$$

by the properties of Γ_+, $\tilde{R}$ satisfies

$$\operatorname{div}_x((1+(k-1)\chi^+)\nabla_x\widetilde{R}(x,y)) = \operatorname{div}_x((1+(k-1)\chi^+)(I-B)\nabla_x\widetilde{\Gamma}_D(x,y)).$$

Using the fundamental solution Γ_+ of the above operator and estimating the integral that represents the solution $\tilde{R}$, it is possible to show that

$$|\tilde{R}(x,y)| \le c|x-y|^{\alpha+2-n}.$$

Estimate (39) follows going back to the original coordinates and estimate (40) follows by using the interpolation inequality

$$\|\nabla\widetilde{R}(\cdot,y)\|_{L^\infty(Q)} \le c\|\widetilde{R}(\cdot,y)\|_{L^\infty(Q)}^{1-\delta}|\nabla\widetilde{R}(\cdot,y)|_{\alpha,Q}^{\delta},$$

where $\delta = \frac{1}{1+\alpha}$ and

$$|\nabla\widetilde{R}|_{\alpha,Q} = \sup_{x,x'\in Q, x\neq x'} \frac{|\nabla\widetilde{R}(x,y) - \nabla\widetilde{R}(x',y)|}{|x-x'|^\alpha}.$$

We refer to [4, Proposition 3.4] for details. $\square$

Step 4: quantitative estimates.
The next two Propositions provide quantitative estimates on f and S_{D_1} when we move y towards O, along $\nu(O)$.

Proposition 2.11. *Let Ω be an open set in $\mathbf{R}^n$ satisfying the above properties. Let D_1, D_2 be two inclusions in Ω verifying the above properties and let $y = h\nu(O)$, with O defined in (29). If, given $\varepsilon > 0$, we have*

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2},H^{-1/2})} \le \varepsilon,$$

then for every h, $0 < h < \bar{c}\,r_0$, where $0 < \bar{c} < 1$, depends on M_0,

$$|f(y,y)| \le C\frac{\varepsilon^{Bh^F}}{h^A}, \tag{41}$$

where $0 < A < 1$ and $C, B, F > 0$ are constants that depend only on the a priori data.

Proof. To get this upper bound, the procedure is to fix one of the two singularities, say w, in $\mathcal{C}\overline{\Omega}$. It is not difficult to check that $f(y,w)$ is harmonic with respect to y in $\mathcal{C}\overline{\Omega}_D$ and, therefore, we can apply iteratively the three spheres inequality to evaluate the propagation of the $\varepsilon-$smallness as we drag y toward Ω_D. Finally employing this procedure for w, we get the bound. We refer the reader to [4, Proposition 3.5] for details. □

PROPOSITION 2.12. *Let Ω be an open set in $\mathbf{R}^n$ satisfying the above properties. Let D_1, D_2 be two inclusions in Ω verifying the above properties and $y = h\nu(O)$. Then for every h, $0 < h < \overline{r}_0/2$,*

$$|S_{D_1}(y,y)| \geq c_1 h^{2-n} - c_2 d_\mu^{2-2n} + c_3, \tag{42}$$

where c_1, c_2 and c_3 are positive constants only depending on the a priori data. Here $\overline{r}_0$ is the number introduced in Proposition 2.10.

Proof. Choosing $y = h\nu(O)$, where $\nu(O)$ is the exterior outer normal to Ω_D in O, with O defined as in (29), with h sufficiently small, to get the lower bound (42), the crucial ingredient is the following inequality

$$\nabla_x \Gamma_{D_1}(x,y) \cdot \nabla_x \Gamma_{D_1}(x,y) \geq c|x-y|^{2-2n},$$

with $x \in D_1$ sufficiently close to y. This estimate can be derived from [2, Lemma 3.1] once one has at disposal the asymptotic behavior (40) (see [4, Proposition 3.6] for details). □

Step 5: proof of Theorem 2.4.
Let $O \in \partial D_1$ satisfying (29), that is

$$d_\mu(D_1,D_2) = \mathrm{dist}(O,D_2) = d_\mu.$$

Then, for $y = h\nu(O)$, with $0 < h < h_1$, where $h_1 = \min\{d_\mu, \overline{c}\, r_0, \overline{r}_0/2\}$, using (38), we have

$$|S_{D_2}(y,y)| \leq c\int_{D_2} \frac{1}{(d_\mu-h)^{n-1}}\frac{1}{(d_\mu-h)^{n-1}}dx = c\frac{1}{(d_\mu-h)^{2n-2}}|D_2|. \tag{43}$$

Using Proposition 2.11, we have

$$\begin{aligned} |S_{D_1}(y,y)| - |S_{D_2}(y,y)| &\leq |S_{D_1}(y,y) - S_{D_2}(y,y)| \\ &= |f(y,y)| \leq c\frac{\varepsilon^{Bh^F}}{h^A}. \end{aligned}$$

On the other hand, by Proposition 2.12 and (43)

$$|S_{D_1}(y,y)| - |S_{D_2}(y,y)| \geq c_1 h^{2-n} - c_2(d_\mu - h)^{2-2n}.$$

Thus we have

$$c_3 h^{2-n} - c_4 (d_\mu - h)^{2-2n} \leq \frac{\varepsilon^{Bh^F}}{h^A}.$$

That is

$$\begin{aligned} c_4 (d_\mu - h)^{2-2n} &\geq c_3 h^{2-n} - \frac{\varepsilon^{Bh^F}}{h^A} = h^{2-n}(c_3 - \varepsilon^{Bh^F} h^{\widetilde{A}}) \\ &\geq c_5 h^{2-n}\big(1 - \varepsilon^{Bh^F} h^{\widetilde{A}}\big), \end{aligned} \tag{44}$$

where $\widetilde{A} = n - 2 - A$, $\widetilde{A} > 0$. Let $h = h(\varepsilon)$ where $h(\varepsilon) = \min\{|\ln \varepsilon|^{-\frac{1}{2F}}, d_\mu\}$, for $0 < \varepsilon \leq \varepsilon_1$, with $\varepsilon_1 \in (0,1)$ such that $\exp(-B|\ln \varepsilon_1|^{1/2}) = 1/2$. If $d_\mu \leq |\ln \varepsilon|^{-\frac{1}{2F}}$, since, by Lemma 2.6, the Hausdorff distance is dominated by d_μ, estimate (3) follows trivially. In the other case we have

$$\varepsilon^{Bh(\varepsilon)^F} h(\varepsilon)^{\widetilde{A}} \leq \varepsilon^{B|\ln \varepsilon|^{-1/2}} \leq \exp\big(-B|\ln \varepsilon|^{1/2}\big).$$

Then, for any ε, $0 < \varepsilon < \varepsilon_1$,

$$\big(d_\mu - h(\varepsilon)\big)^{2-2n} \geq c_6 h(\varepsilon)^{2-n},$$

that is, solving for d_μ, and recalling that, in this case, $h(\varepsilon) = |\ln \varepsilon|^{-\frac{1}{2F}}$

$$d_\mu \leq c_7 |\ln \varepsilon|^{-\delta \frac{n-2}{2n-2}} \tag{45}$$

where $\delta = 1/(2F)$. When $\varepsilon \geq \varepsilon_1$, then

$$d_\mu \leq \operatorname{diam} \Omega$$

and, in particular when $\varepsilon_1 \leq \varepsilon < 1$

$$d_\mu \leq \operatorname{diam} \Omega \frac{|\ln \varepsilon|^{-\frac{1}{2F}}}{|\ln \varepsilon_1|^{-\frac{1}{2F}}}.$$

Finally, using Lemma 2.6, the theorem follows.

3. Elastic Bodies

Let us consider now the determination of an inclusion D in an elastic body Ω by measuring the displacements and traction on the boundary $\partial\Omega$. More precisely, let Ω be a bounded domain in $\mathbf{R}^3$ and let D be an open set contained in Ω. We deal with the dimension $n = 3$ as it is more relevant for applications. Everything works in any dimension. Assume that both the body Ω and the inclusion D are made by different homogeneous, isotropic, elastic materials, with Lamé moduli μ, λ and μ^D, λ^D, respectively, satisfying the strong convexity conditions $\mu > 0$, $2\mu + 3\lambda > 0$, $\mu^D > 0$, $2\mu^D + 3\lambda^D > 0$. For a given $f \in H^{\frac{1}{2}}(\partial\Omega)$, consider the weak solution $u \in H^1(\Omega)$ to the Dirichlet problem

$$\begin{cases} \operatorname{div}((\mathbf{C}+(\mathbf{C}^D-\mathbf{C})\chi_D)\nabla u)=0, & \text{in } \Omega, \quad (46)\\ u=f, & \text{on } \partial\Omega, \quad (47)\end{cases}$$

where $\mathbf{C}$, $\mathbf{C}^D$ are the elastic tensors of the body and of the inclusion, respectively, and χ_D is the characteristic function of D. We denote by $\Lambda_D : H^{\frac{1}{2}} \to H^{-\frac{1}{2}}$ the Dirichlet-to-Neumann map associated to the problem (46)–(47), that is the operator which maps the Dirichlet data $u|_{\partial\Omega}$ onto the corresponding Neumann data $(\mathbf{C}\nabla u)\nu|_{\partial\Omega}$, where ν is the outer unit normal to Ω. The inverse problem is to determine D when Λ_D is given. In the recent paper [5] it is shown the modulus of continuity of the continuous dependance of the inclusion D from the map Λ_D under mild a priori assumptions on the regularity and the topology. In this section we review the main steps of the proof that is inspired by the argument shown in Section 2. Let us mention here that one of the main difference between the scalar conductivity equation and the vector Lamé is the study of the asymptotic of the fundamental solution. In fact in the scalar case it was possible to prove that $(\Gamma^{D_1} - \Gamma^{D_2})(y, y)$ blows up as $y = w$ tends non-tangentially to $P \in \partial D_1 \setminus \overline{D_2}$, and to evaluate quantitatively the blowup rate. In the present case the situation is more complicated for a number of reasons. First of all the fundamental solutions of the elastic operator are matrix valued (not scalar) functions and, therefore, it is crucial to understand which of the entries of $\Gamma^{D_1} - \Gamma^{D_2}$ has the desired blowup behavior. Second, we are assuming that either $\mu^D \neq \mu$ or $\lambda^D \neq \lambda$ with *no order condition* between such parameters. Hence, we cannot expect, in general, that the difference matrix $\Gamma^{D_1} - \Gamma^{D_2}$ may satisfy any positivity condition. For these reasons we have chosen to examine each diagonal entry of $\Gamma^{D_1} - \Gamma^{D_2}$ separately. Similarly to the scalar case, we can show that, as y, w tend to $P \in \partial D_1 \setminus \overline{D_2}$, $(\Gamma^{D_1} - \Gamma^{D_2})(y, w)$ has, in a suitable reference frame, the same asymptotic behavior of $(\Gamma^+ - \Gamma)(y, w)$. Here Γ is the standard Kelvin fundamental solution with Lamé moduli μ, λ and Γ^+ is the fundamental solution Γ^D when D is replaced by the upper half plane $\{x_3 > 0\}$.

We can take advantage of the fact that Γ^+ is explicitly known, in fact its expression, although complicated, was calculated by Rongved [26] in 1955. With the aid of Rongved's formulas it is possible to estimate the blowup rate of $(\Gamma^+ - \Gamma)_{ii}(y, w)$, $i = 1, 2, 3$, as $y, w \to 0$ vertically along the line $\{x_1 = x_2 = 0\}$ for suitable choices of y, w. The peculiar fact is that we are obliged to pick up very specific choices of y, w, with $w \neq y$. In fact we have found explicit examples of moduli $(\lambda, \mu) \neq (\lambda^D, \mu^D)$ for which $(\Gamma^+ - \Gamma)_{ii}(y, y) = 0$.

Let us consider a elastic body $\Omega \subset \mathbf{R}^3$ and an inclusion D satisfying the assumptions of the previous sections. Moreover we assume the following conditions.

Assumptions on the domain

The body Ω is assumed to be made of linearly elastic, isotropic and homogeneous material, with elastic tensor $\mathbf{C}$ of components

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}), \tag{48}$$

where δ_{ij} is the Kronecker's delta. The *constant* Lamé moduli λ, μ satisfy the strong convexity conditions

$$\mu \geq \alpha_0, \quad 2\mu + 3\lambda \geq \gamma_0, \tag{49}$$

where $\alpha_0 > 0$, $\gamma_0 > 0$ are given constants. We shall also assume upper bounds on the Lamé moduli

$$\mu \leq \overline{\mu}, \quad \lambda \leq \overline{\lambda}, \tag{50}$$

where also $\overline{\mu} > 0$, $\overline{\lambda} \in \mathbf{R}$ are known quantities. In some points of our analysis, we will express the constitutive equation (48) in terms of μ and of Poisson's ratio ν, instead of the Lamé moduli μ, λ. Recalling that

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \tag{51}$$

by (49), (50) we have

$$-1 < \nu_0 \leq \nu \leq \nu_1 < \frac{1}{2}, \tag{52}$$

where ν_0, ν_1 only depend on α_0, γ_0, $\overline{\mu}$, $\overline{\lambda}$. Let us notice that (48) trivially implies that

$$C_{ijkl} = C_{klij} = C_{lkij}, \quad i, j, k, l = 1, 2, 3. \tag{53}$$

We recall that the first equality in (53) is usually named as the major symmetry of the tensor $\mathbf{C}$, whereas the second equality is called the minor symmetry.

Also we note that (49) is equivalent to

$$\mathbf{C}A \cdot A \geq \xi_0 |A|^2 \tag{54}$$

for every 3×3 symmetric matrix A, where $\xi_0 = \min\{2\alpha_0, \gamma_0\}$.

Assumptions on the inclusion

The inclusion D is made of isotropic homogeneous material having elasticity tensor $\mathbf{C}^D$, with *constant* Lamé moduli λ^D, μ^D satisfying the conditions (49), (50) and such that

$$(\lambda - \lambda^D)^2 + (\mu - \mu^D)^2 \geq \eta_0^2 > 0, \tag{55}$$

for a given constant $\eta_0 > 0$.

In what follows we shall refer to the constants M_0, α, M_1, α_0, γ_0, $\overline{\mu}$, $\overline{\lambda}$, η_0 as to the *a-priori data*.

Observe that, in view of (51) and of the a-priori bounds on the Lamé moduli, from (55) it also follows

$$(\nu - \nu^D)^2 + (\mu - \mu^D)^2 \geq C\eta_0^2 > 0, \tag{56}$$

where C only depends on α_0, γ_0, $\overline{\mu}$, $\overline{\lambda}$.

Finally, note that the jump condition (55) does *not* imply any kind of monotonicity relation between $\mathbf{C}$ and $\mathbf{C}^D$.

Before state the stability theorem, we remind that the Dirichlet–to–Neumann map associated to problem (46)–(47) is defined similarly as in Remark 2.3. The stability theorem reads as follows.

THEOREM 3.1. *Let $\Omega \subset \mathbf{R}^3$ and let D_1, D_2 be as above Let $\mathbf{C}$ and $\mathbf{C}^D$ be the constant elastic tensors of the material of Ω and of the inclusions D_i, $i = 1, 2$, respectively, where $\mathbf{C}$ and $\mathbf{C}^D$ satisfy* (48)–(50) *and* (55). *If, for some ε, $0 < \varepsilon < 1$,*

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))} \leq \frac{\varepsilon}{r_0}, \tag{57}$$

then

$$d_H(\partial D_1, \partial D_2) \leq r_0\omega(\varepsilon), \tag{58}$$

where ω is an increasing function on $[0, +\infty)$ satisfying

$$\omega(t) \leq C|\log t|^{-\eta}, \text{ for every } 0 < t < 1, \tag{59}$$

where $C > 0$ and η, $0 < \eta \leq 1$, are constants only depending on the a-priori data.

We will go through the proof of the theorem dividing it in to the same steps of the conductivity problems and underlying the main differences.

Step 1: modified distance.
This part does not change with respect to the impedance tomography case.

Step 2: Alessandrini's identity.
Also in this framework, using Green's formula and the symmetry properties of $\mathbf{C}$, $\mathbf{C}^D$, it is not difficult to get

$$\int_\Omega (\mathbf{C} + (\mathbf{C}^D - \mathbf{C})\chi_{D_1})\nabla u_1 \cdot \nabla u_2 - \int_\Omega (\mathbf{C} + (\mathbf{C}^D - \mathbf{C})\chi_{D_2})\nabla u_1 \cdot \nabla u_2 = \\ = \int_{\partial\Omega} u_1 \cdot (\Lambda_{D_1} - \Lambda_{D_2}) u_2. \tag{60}$$

Arguing similarly as in the previous case, we want to use (60) replacing solutions u_1, u_2 with fundamental solutions with singularities outside Ω. For this purpose let us define them precisely. Given $y \in \mathbf{R}^3$ and a *concentrated force* $l \in \mathbf{R}^3$ applied at y, $|l| = 1$, let us consider the *normalized fundamental solution* $u^D \in L^1_{loc}(\mathbf{R}^3, \mathbf{R}^3)$ defined by

$$\begin{cases} \mathrm{div}_x \big((\mathbf{C} + (\mathbf{C}^D - \mathbf{C})\chi_D)\nabla_x u^D(x, y; l)\big) = -l\delta(x - y), \quad \text{in } \mathbf{R}^3 \setminus \{y\}, \\ \\ \lim_{|x|\to\infty} u^D(x, y; l) = 0, \end{cases} \tag{61}$$

where $\delta(\cdot - y)$ is the Dirac distribution supported at y, that is

$$\int_{\mathbf{R}^3} (\mathbf{C} + (\mathbf{C}^D - \mathbf{C})\chi_D)\nabla_x u^D(x, y; l) \cdot \nabla_x \varphi(x) = l \cdot \varphi(y),$$
$$\text{for every } \varphi \in C^\infty_c(\mathbf{R}^3, \mathbf{R}^3). \tag{62}$$

It is well-known that

$$u^D(x, y; l) = \Gamma^D(x, y)l, \tag{63}$$

where $\Gamma^D = \Gamma^D(\cdot, y) \in L^1_{loc}(\mathbf{R}^3, \mathcal{L}(\mathbf{R}^3, \mathbf{R}^3))$ is the *normalized fundamental matrix* for the operator $\mathrm{div}_x((\mathbf{C} + (\mathbf{C}^D - \mathbf{C})\chi_D)\nabla_x(\cdot))$. The existence of Γ^D is ensured by the following Proposition.

Proposition 3.2. *Under the above assumptions, there exists a unique fundamental matrix* $\Gamma^D(\cdot, y) \in C^0(\mathbf{R}^3 \setminus \{y\})$. *Moreover, we have*

$$\Gamma^D(x, y) = (\Gamma^D(y, x))^T, \quad \textit{for every } x \in \mathbf{R}^3,\ x \neq y, \tag{64}$$

$$|\Gamma^D(x, y)| \leq C|x - y|^{-1}, \quad \textit{for every } x \in \mathbf{R}^3,\ x \neq y, \tag{65}$$

$$|\nabla_x \Gamma^D(x, y)| \leq C|x - y|^{-2}, \quad \textit{for every } x \in \mathbf{R}^3,\ x \neq y, \tag{66}$$

where the constant $C > 0$ *only depends on* M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$.

Proof. Using a result contained in [23] combined with the results presented in [20] it is possible to get the thesis. See [5, Proposition 5.1] for details. □

Let us choose y, $w \in \mathbf{R}^3$, $y \neq w$, and l, $m \in \mathbf{R}^3$ such that $|l| = |m| = 1$. We define the functions

$$S_{D_1}(y, w; l, m) = \int_{D_1} (\mathbf{C}^D - \mathbf{C})\nabla_x(\Gamma^{D_1}(x, y)l) \cdot \nabla_x(\Gamma^{D_2}(x, w)m), \tag{67}$$

$$S_{D_2}(y, w; l, m) = \int_{D_2} (\mathbf{C}^D - \mathbf{C})\nabla_x(\Gamma^{D_1}(x, y)l) \cdot \nabla_x(\Gamma^{D_2}(x, w)m), \tag{68}$$

$$f(y, w; l, m) = S_{D_1}(y, w; l, m) - S_{D_2}(y, w; l, m). \tag{69}$$

Again the leading argument to get stability is to evaluate the function f as we move the singularities y, w quantifying the propagation of the boundary information we have from the measurements. A key ingredient in this analysis is the behavior of fundamental solutions.

Step 3: fundamental solutions.
Let $O \in \partial D$ and $\nu = \nu(O)$ the outer unit normal to D at O. Let us choose a coordinate system with origin O and axis $e_3 = -\nu$, and let $\Gamma^+(x,y) = \Gamma^{\mathbf{R}^3_+}(x,y)$ the normalized fundamental matrix associated to $D = \mathbf{R}^3_+$. We recall that its explicit expression was found by Rongved [26].

Recalling the notation $u^D(x,y) = \Gamma^D(x,y)l$ (see (63)) and defining similarly $u^+(x,y) = \Gamma^+(x,y)l$, for any $l \in \mathbf{R}^3$, $|l| = 1$, the asymptotic approximation of u^D in terms of u^+ reads as follows.

THEOREM 3.3. *Let $y = (0,0,-h)$, $0 < h < \frac{r_0 M_0}{8\sqrt{1+M_0^2}}$. Under the above assumptions and notation, we have*

$$|u^D(x,y) - u^+(x,y)| \le \frac{C}{r_0}\left(\frac{|x-y|}{r_0}\right)^{-1+\alpha}, \qquad \text{for every } x \in Q_{\frac{r_0}{8\sqrt{1+M_0^2}}, \frac{r_0 M_0}{8\sqrt{1+M_0^2}}} \cap D, \tag{70}$$

$$|\nabla_x u^D(x,y) - \nabla_x u^+(x,y)| \le \frac{C}{r_0^2}\left(\frac{|x-y|}{r_0}\right)^{-2+\frac{\alpha^2}{3\alpha+2}}, \qquad \text{for every } x \in Q^+_{\frac{r_0}{12\sqrt{1+M_0^2}}, \frac{r_0 M_0}{12\sqrt{1+M_0^2}}} \cap D, \tag{71}$$

where $C > 0$ only depends on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$.

Proof. The thesis can be obtained defining the function

$$R(x,y) = u^d(x,y) - u^+(x,y)$$

and flattening the boundary ∂D. See [5, Theorem 8.1] for details. □

Step 4: quantitative estimates.
As in the impedance tomography case, in this step we show how the boundary information and the asymptotic behavior of the fundamental solution can be used to estimate the auxiliary function f.

THEOREM 3.4 (Upper bound on the function f). *Under the notation of Lemma 2.7, let*

$$y_h = P - he_3, \tag{72}$$

$$w_h = P - \lambda_w h e_3, \quad 0 < \lambda_w < 1, \tag{73}$$

with

$$0 < h \leq \overline{d}\left(1 - \frac{\sin\widetilde{\vartheta}_0}{4}\right), \tag{74}$$

where $\widetilde{\vartheta}_0 = \arctan\frac{1}{L_0}$ *and* $\nu = -e_3$ *is the outer unit normal to* D_1 *at* P*. Then, for every* l, $m \in \mathbf{R}^3$, $|l| = |m| = 1$, *we have*

$$|f(y_h, w_h; l, m)| \leq \frac{C}{\lambda_w h}\epsilon^{C_1\left(\frac{h}{\rho_0}\right)^{C_2}}, \tag{75}$$

where the constant $C > 0$ *only depends on* M_0, α, M_1, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$*;*

$$C_1 = \gamma\delta^{2+2\frac{|\log A|}{|\log\chi|}},\ C_2 = 2\frac{|\log\delta|}{|\log\chi|},\ A = \frac{\lambda_w}{\frac{\overline{d}}{\rho_0}(1 - \vartheta^*\frac{\sin\widetilde{\vartheta}_0}{8})},\ \chi = \frac{1 - \frac{\sin\widetilde{\vartheta}_0}{8}}{1 + \frac{\sin\widetilde{\vartheta}_0}{8}}, \tag{76}$$

where δ, $0 < \delta < 1$, ϑ^*, $0 < \vartheta^* \leq 1$, *only depend on* α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$*;* $\gamma > 0$ *only depends on* M_0, α, M_1, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$.

Proof. Similarly to the impedance tomography case, the proof is based on the use of the three spheres inequality for solution to the Lamé system. We refer to [5, Theorem 6.4] for details. □

THEOREM 3.5 (Lower bound on the function f). *Under the notation of Lemma 2.7, let*

$$y_h = P - he_3. \tag{77}$$

For every $i = 1, 2, 3$, *there exists* $\lambda_w \in \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}$ *and there exists* $\overline{h} \in \left(0, \frac{1}{2}\right)$ *only depending on* M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, η_0, *such that*

$$|f(y_h, w_h; e_i, e_i)| \geq \frac{C}{h}, \quad \textit{for every } h,\ 0 < h < \overline{h}\rho, \tag{78}$$

where

$$w_h = P - \lambda_w h e_3, \tag{79}$$

$$\rho = \min\left\{dist(P, D_2),\ \frac{r_0}{12\sqrt{1 + M_0^2}} \cdot \min\{1, M_0\}\right\}, \tag{80}$$

and $C > 0$ *only depends on* M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, η_0.

Proof. To obtain such a bound we refer to Theorem 6.5 of [5]. Let us only mention that besides the use of the asymptotic os Γ_D (Theorem 3.3) other ingredients are needed. In particular we point out the identity

$$\int_{\mathbf{R}^3_+}(\mathbf{C}^D - \mathbf{C})\nabla_x(\Gamma^+(x, y_0)l)\cdot\nabla_x(\Gamma(x, w_0)m) = (\Gamma(y_0, w_0) - \Gamma^+(y_0, w_0))m\cdot l,$$

$$\text{for every } y_0, w_0 \in \mathbf{R}^3,\ y_0 \neq w_0,$$

(See [5, Lemma 9.2]) that is a special case of [10, Prposition 3.2] and the bound

$$\left|(\Gamma^+(y_0,w_0)-\Gamma(y_0,w_0))e_i\cdot e_i\right|\geq \mathcal{C},$$

where $y_0=(0,0,-1)$, $w_0=(0,0,-\lambda_w)$, with $\lambda_w\in\left\{\frac{2}{3},\frac{3}{4},\frac{4}{5}\right\}$ for $i=1,2,3$ (see [5, Proposition 9.3]). $\square$

Step 5: proof of Theorem 3.1.
From the combination of the upper bound (75), with $l=m=e_i$ for $i\in\{1,2,3\}$, and from the lower bound (78), we have

$$C\leq \varepsilon^{C_1\left(\frac{h}{r_0}\right)^{C_2}},\quad \text{for every } h,\ 0<h\leq \overline{h}\rho, \tag{81}$$

where ρ is given in (80), the constants $C_1>0$, $C_2>0$ are defined in (76) and depend only on M_0, α, M_1, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, and the constants $C\in(0,1)$, $\overline{h}\in\left(0,\frac{1}{2}\right)$ only depend on M_0, α, α_0, γ_0, $\overline{\lambda}$, $\overline{\mu}$, η_0.

Passing to the logarithm and recalling that $\varepsilon\in(0,1)$, we have

$$h\leq Cr_0\left(\frac{1}{|\log\varepsilon|}\right)^{\frac{1}{C_2}},\quad \text{for every } h,\ 0<h\leq\overline{h}\rho, \tag{82}$$

In particular, choosing $h=\overline{h}\rho$, we have

$$\rho\leq Cr_0\left(\frac{1}{|\log\varepsilon|}\right)^{\frac{1}{C_2}}. \tag{83}$$

If $\rho=dist(P,D_2)$, by Lemmas 2.6 and 2.7, the thesis follows. If, otherwise, $\rho=\frac{r_0}{12\sqrt{1+M_0^2}}\min\{1,M_0\}$, the thesis follows by noticing that $d_H(\partial D_1,\partial D_2)\leq diam(\Omega)\leq Cr_0$, with $C>0$ only depending on M_0, M_1.

4. Thermal Conductors

In this section we go through the problem of determining an inclusion, whose shape can vary with the time, within a thermal conductor. Let T be a given positive number. Let Ω be a bounded domain of $\mathbf{R}^n$ with a sufficiently smooth boundary and let Q be a domain contained in $\Omega\times(0,T)$. Assume that for every $\tau\in(0,T)$ the intersection of Q with the hyperplane $t=\tau$ is a nonempty set and denote by k, $k\neq 1$ a positive constant. Let u be the weak solution to the following parabolic initial-boundary value problem

$$\begin{cases} \partial_t u-\operatorname{div}((1+(k-1)\chi_Q)\nabla u)=0 & \text{in } \Omega\times(0,T),\\ u(\cdot,0)=0 & \text{in } \overline{\Omega},\\ u=g & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{84}$$

where g is a prescribed function on $\partial\Omega \times (0,T)$. The inverse problem we are addressing to is to determine the region Q when infinitely many boundary measurements $\left\{ g, \frac{\partial u}{\partial \nu}_{|\partial\Omega\times(0,T)} \right\}$ are available. A uniqueness result was proved in 1997 by Elayyan and Isakov [18]. We want to discuss the stability issue proved in [15]. We will show that also in this case the stability estimates are of logarithmic type. The argument to get such a rate of continuity follows the line of the impedance tomography case, using singular solutions of Green's type. Let us emphasize here that one of the main difference with respect to the previous cases is the lack of an explicit representation of the fundamental solution when the interface is flat. To overcome this difficulty we will use some formulas proved by [22] involving the Fourier transform of the fundamental solution that will lead to an estimate from below (see Proposition 4.5).

Another difficulty that characterizes the parabolic case consists in employing a precise evaluation of the smallness propagation based on the two-sphere and one-cylinder inequality for solution of parabolic equation [19], [27] (see Theorem 4.7 below).

Let us first premise a definition.

Definition 4.1. *Let Q be a domain in $\mathbf{R}^{n+1}$. We shall say that Q (or equivalently ∂Q) is of class $\mathcal{K}$ with constants r_0, M_0 if for all $X_0 \in \partial Q$ there exists a rigid transformation of space coordinates under which we have $X_0 = (0,0)$ such that*

$$Q \cap \left(B_{r_0}(0) \times (-r_0^2, r_0^2)\right) = \{X \in B_{r_0}(0) \times (-r_0^2, r_0^2) \ : \ x_n > \varphi(x', t)\},$$

where φ is endowed with second derivatives with respect to x_i, $i = 1, \cdots, n$, with the t-derivative and with second derivatives with respect to x_i and t and it satisfies the following conditions $\varphi(0,0) = |\nabla_{x'}\varphi(0,0)| = 0$ and

$$r_0^2 \|D^2_{x'}\varphi\|_{L^\infty(B'_{r_0}\times(-r_0^2,r_0^2))} + r_0^2\|\partial_t\varphi\|_{L^\infty(B'_{r_0}\times(-r_0^2,r_0^2))} + r_0^3\|\nabla_{x'}\partial_t\varphi\|_{L^\infty(B'_{r_0}\times(-r_0^2,r_0^2))} \le M_0 r_0.$$

Assumptions on the domain

Let r_0, M_0, M_1 be given positive numbers. We assume that Ω is a bounded domain in $\mathbf{R}^n$ satisfying

$$|\Omega| \le M_1 r_0^n, \tag{85a}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. We also assume that

$$\partial\Omega \text{ is of class } \mathcal{C}^{1,1} \text{ with constants } r_0, M_0. \tag{85b}$$

Assumptions on the inclusion
Denoting by $Q = \bigcup_{t\in\mathbf{R}} D(t) \times \{t\}$, we assume the following conditions

$$\partial Q \text{ is of class } \mathcal{K} \text{ with constans } r_0, M_0, \tag{86a}$$

$$\operatorname{dist}(D(t), \partial\Omega) \geq r_0 \quad \forall\, t \in [0,T], \tag{86b}$$

$$\Omega \setminus \overline{D(t)} \text{ is connected } \forall\, t \in [0,T]. \tag{86c}$$

Before stating the stability result, let us define the Dirichlet–to–Neumann map in this framework. We denote by $H = H^{3/2,3/4}_{,0}(\partial\Omega \times (0,T))$, its dual $H' = H^{-3/2,-3/4}(\partial\Omega \times (0,T))$, and

$$W(\Omega \times (0,T)) = \left\{ v \in L^2((0,T), H^1(\Omega)) \,:\, \partial_t v \in L^2((0,T), H^{-1}(\Omega)) \right\}.$$

For any $g \in H$, let $u \in W(\Omega \times (0,T))$ be the weak solution of the initial–boundary value problem

$$\begin{cases} \partial_t u - \operatorname{div}((1 + (k-1)\chi_Q)\nabla u) = 0, & \text{in } \Omega \times (0,T), \\ u(x,0) = 0, & x \in \overline{\Omega}, \\ u(x,t) = g(x,t), & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{87}$$

where χ_Q is the characteristic function of the set Q. Then for any $g \in H$, we set

$$\Lambda_Q g = \frac{\partial u}{\partial \nu}|_{\partial\Omega\times(0,T)}, \quad u \text{ solution to (87)}.$$

We can also consider Λ_Q as a linear and bounded operator between H and H', by setting

$$\langle \Lambda_Q g, \phi \rangle_{H',H} = \langle \frac{\partial u}{\partial \nu}|_{\partial\Omega\times(0,T)}, \phi \rangle_{H',H} = \int_{\partial\Omega\times(0,T)} \frac{\partial u}{\partial \nu}\phi, \quad \text{for any } g, \phi \in H,$$

where u solves (87) and $\langle\cdot,\cdot\rangle_{H',H}$ is the duality pairing between H' and H.

THEOREM 4.2. *Let $\Omega \subset \mathbf{R}^n$ satisfying* (85). *Let $k > 0$, $k \neq 1$ be given. Let $\{D_1(t)\}_{t\in\mathbf{R}}$, $\{D_2(t)\}_{t\in\mathbf{R}}$ be two families of domains satisfying* (86). *Assume that $D_1(0) = D_2(0)$ and, for $\varepsilon > 0$,*

$$\|\Lambda_{Q_1} - \Lambda_{Q_2}\|_{\mathcal{L}(H,H')} \leq \varepsilon, \tag{88}$$

where $Q_i = D_i((-\infty,+\infty))$, $i = 1,2$. Then

$$d_{\mathcal{H}}(D_1(t), D_2(t)) \leq \omega_t(\varepsilon), \qquad t \in (0,T], \tag{89}$$

where $\omega_t(s)$ is such that

$$\omega_t(s) \leq C|\log s|^{-\eta}, \qquad 0 < s < 1, \tag{90}$$

with $C = C(t) > 0$ and $0 < \eta \leq 1$ depend on the a priori data only.

Remark 4.3. Let us observe that for the case of more general thermal conductivities with local Dirichlet–to–Neumann map has been studied in [16].

Step 1: modified distance.
This part can be obtained through minor modifications form the impedance tomography case (see [15, Proposition 3.2, 3.3] for further details).

Step 2: Alessandrini's identity.
For the sake of brevity we name $a_j = 1 + (k-1)\chi_{Q_j}$, $j = 1, 2$. We fix $g \in H$. We shall denote by u_j, $j = 1, 2$ the solution of (84) when $Q = Q_j$. For $\psi \in H^{1,1}(\Omega \times (0,T))$ such that

$$\psi(\cdot, T) = 0 \quad \text{in } \Omega, \tag{91}$$

using the weak formulation of (84) we have

$$\int_{\partial\Omega\times(0,T)} a_j \frac{\partial u_j}{\partial \nu}\psi dS + \int_\Omega u_j(x,0)\psi(x,0)dx \\ - \int_{\Omega\times(0,T)} (a_j \nabla u_j \cdot \nabla\psi - u_j \partial_t \psi)\, dxdt = 0 \qquad \text{for } j = 1, 2.$$

Subtracting the two equations we obtain

$$\int_{\Omega\times(0,T)} (a_1 \nabla(u_1 - u_2)\cdot\nabla\psi - (u_1 - u_2)\partial_t\psi)\, dxdt \\ + \int_{\Omega\times(0,T)} (a_1 - a_2)\nabla u_2 \cdot \nabla\psi =< (\Lambda_{Q_1} - \Lambda_{Q_2})g, \psi >_{H',H}, \tag{92}$$

(we notice here that in these identities it is possible to have $u_i(\cdot, 0) \neq 0$ for $i = 1, 2$). Taking ψ such that it satisfies (91) and

$$\partial_t\psi + \operatorname{div}(a_1\nabla\psi) = 0 \qquad \text{in } \Omega\times(0,T), \tag{93}$$

by (92) we have (recalling that on $\partial\Omega\times(0,T)$ $u_1 = u_2 = g$)

$$\int_{\Omega\times(0,T)} (a_1 - a_2)\nabla u_2 \cdot \nabla\psi =< (\Lambda_{Q_1} - \Lambda_{Q_2})g, \psi >_{H',H}, \qquad \forall\, g \in H$$

or, equivalently,

$$\int_0^T \int_\Omega (\chi_{Q_1} - \chi_{Q_2})\nabla u_2 \cdot \nabla\psi dxdt = \frac{1}{k-1} < (\Lambda_{Q_1} - \Lambda_{Q_2})u_2, \psi >_{H',H} . \tag{94}$$

Let us denote by $\Gamma_2(x,t;y,s)$ and $\Gamma_1^*(x,t;y,s)$ the fundamental solutions of the operator $\partial_t - \mathrm{div}((1+(k-1)\chi_{Q_2})\nabla)$ and $\partial_t + \mathrm{div}((1+(k-1)\chi_{Q_1})\nabla)$ respectively, that is

$$\partial_t\Gamma_2(x,t;y,s) - \mathrm{div}((1+(k-1)\chi_{Q_2})\nabla_x\Gamma_2(x,t;y,s)) = -\delta(x-y,t-s),$$

$$\partial_t\Gamma_1^*(x,t;y,s) + \mathrm{div}((1+(k-1)\chi_{Q_1})\nabla_x\Gamma_1^*(x,t;y,s)) = -\delta(x-y,t-s),$$

where δ denotes the Dirac distribution. Choosing in (94) $u_2(x,t) = \Gamma_2(x,t;y,s)$ and $\psi(x,t) = \Gamma_1^*(x,t;\xi,\tau)$, with (y,s) and $(\xi,\tau) \notin \Omega\times(0,T)$, $0 \le s < \tau \le T$, we obtain

$$\int_0^T\int_\Omega(\chi_{Q_1}-\chi_{Q_2})\nabla_x\Gamma_2(x,t;y,s)\cdot\nabla_x\Gamma_1^*(x,t;\xi,\tau)dxdt = \frac{1}{k-1} < (\Lambda_{Q_1}-\Lambda_{Q_2})\Gamma_2(\cdot,\cdot;y,s),\Gamma_1^*(\cdot,\cdot;\xi,\tau) >_{H',H}\,. \tag{95}$$

For $t\in[0,T]$ we shall define $\mathcal{G}(t)$ as the connected component of $\Omega\setminus(\overline{D_1(t)}\cup\overline{D_2(t)})$ that contains $\partial\Omega$, $\tilde{\mathcal{G}}(t) = (\mathbf{R}^n\setminus\Omega)\cup\mathcal{G}(t)$ and $\tilde{\mathcal{G}}((0,T)) := \bigcup_{t\in(0,T)}\tilde{\mathcal{G}}(t)\times\{t\}$. For (y,s), $(\xi,\tau)\in\tilde{\mathcal{G}}((0,T))$ with $0\le s<\tau\le T$, we set

$$S_1(y,s;\xi,\tau) = \int_{Q_1}\nabla_x\Gamma_2(x,t;y,s)\cdot\nabla_x\Gamma_1^*(x,t;\xi,\tau)dxdt,$$

$$S_2(y,s;\xi,\tau) = \int_{Q_2}\nabla_x\Gamma_2(x,t;y,s)\cdot\nabla_x\Gamma_1^*(x,t;\xi,\tau)dxdt$$

$$\mathcal{U}(y,s;\xi,\tau) := S_1(y,s;\xi,\tau) - S_2(y,s;\xi,\tau).$$

By (95) we have

$$\mathcal{U}(y,s;\xi,\tau) = \frac{1}{k-1} < (\Lambda_{Q_1}-\Lambda_{Q_2})\Gamma_2(\cdot,\cdot;y,s),\Gamma_1^*(\cdot,\cdot;\xi,\tau) >_{H',H}, \tag{96}$$

for all $y,\xi\notin\Omega$, $0\le s<\tau\le T$.

Step 3: fundamental solutions.
We denote by $\Gamma_0(x-y,t-s)$ the standard fundamental solution of $\partial_t - \Delta$ which is

$$\Gamma_0(x-y,t-s) = \frac{1}{[4\pi(t-s)]^{n/2}}\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}, \qquad t>s$$

and by denote by $\Gamma(x,t;y,s)$ the fundamental solution of the operator $\partial_t - \mathrm{div}((1+(k-1)\chi_Q)\nabla_x)$ (see [9]). We recall that Γ satisfies the following properties

$$\Gamma(x,t;y,s) = \Gamma(y,s;x,t) \qquad \forall\,(x,t),(y,s)\in Q,\ (x,t)\ne(y,s) \tag{97}$$

and

$$0 < \Gamma(x,t;y,s) \leq \frac{C}{[4\pi(t-s)]^{n/2}} \mathrm{e}^{-\frac{|x-y|^2}{C(t-s)}} \chi_{[s,+\infty)}(t), \tag{98}$$

where $C \geq 1$ depends on k and M_0 only. Furthermore we have also the following estimate for the gradient of Γ.

PROPOSITION 4.4. *Let $\Gamma(x,t;y,s)$ be the fundamental solution of the operator $\partial_t - \operatorname{div}\left((1+(k-1)\chi_Q)\nabla_x\right)$. There exists $C \geq 1$, depending on k and E only such that*

$$|\nabla_x \Gamma(x,t;y,s)| \leq \frac{C}{(t-s)^{\frac{n+1}{2}}} \mathrm{e}^{-\frac{|x-y|^2}{C(t-s)}}, \tag{99}$$

for almost every $x, y \in \mathbf{R}^n$ and $t, s \in \mathbf{R}$, $t > s$.

Proof. See [15, Proposition 3.6]. □

In the sequel we need the fundamental solution of the operator $\mathcal{L}_+ = \partial_t - \operatorname{div}((1+(k-1)\chi_+)\nabla)$ where $\chi_+ = \chi_{\{(x,t)\in\mathbf{R}^{n+1}\,:\,x_n>0\}}$. We shall denote by Γ_+ such a fundamental solution and by Γ_+^* the fundamental solution of the adjoint operator of $\mathcal{L}_+$. Observe that $\Gamma_+(x,t;y,s) = \Gamma_+(x,t-s;y,0)$ and $\Gamma_+^*(x,t;y,s) = -\Gamma_+(x,s-t;y,0)$. For a given function $f(x',x_n)$, $\mathcal{F}_{\zeta'}(f(\cdot,x_n))$ will be the Fourier transform of f with respect to the variable x'. Thus

$$\mathcal{F}_{\zeta'}(f(\cdot,x_n)) = \int_{\mathbf{R}^{n-1}} f(x',x_n)\mathrm{e}^{-ix'\cdot\zeta'}dx',$$

for every $\zeta' \in \mathbf{R}^{n-1}$.

In [22] it has been proved some formulas for $\mathcal{F}_{\zeta'}\left(\Gamma_+(.,x_n,t;y)\right)$. The technique to prove such formulas is rather classical and lengthy. For this reason we display only the ones that we need corresponding to the case in which $x_n > 0$, $y_n < 0$.

Case $k > 1$. Denote by

$$E(\zeta',x_n,t;\rho) = \exp\left[-t(k-(k-1)\rho)|\zeta'|^2 - \sqrt{\frac{k-1}{k}}x_n|\zeta'|\sqrt{\rho}\right], \tag{100}$$

$$F(\zeta',y_n;\rho) = \operatorname{Im}\left(A_1(\rho)\mathrm{e}^{iy_n\sqrt{k-1}\sqrt{1-\rho}|\xi'|}\right), \tag{101}$$

where, for complex number $z = a+ib$, $\operatorname{Im}(z)$ denotes the imaginary part b of z, and

$$A_1(\rho) = \frac{\sqrt{k-1}}{\pi}\frac{1}{i\sqrt{k-1}\sqrt{1-\rho}+\sqrt{k}\sqrt{\rho}}. \tag{102}$$

Then

$$\mathcal{F}_{\zeta'}(\Gamma_+(\cdot,x_n,t;y,0)) = \int_0^1 |\zeta'|\mathrm{e}^{-iy'\cdot\zeta'}E(\zeta',x_n,t;\rho)F(\zeta',y_n;\rho)d\rho, \tag{103}$$

for every $x_n > 0$, $y_n < 0$.

Case $0 < k < 1$. Denote by

$$G(\zeta', y_n, t; \rho) = \exp\left[-t(1-(1-k)\rho)|\zeta'|^2 + \sqrt{1-k}\, y_n |\zeta'|\sqrt{\rho}\right],$$

$$H(\zeta', x_n; \rho) = \operatorname{Im}\left(A_2(\rho) \mathrm{e}^{-ix_n\sqrt{\frac{1-k}{k}}\sqrt{1-\rho}|\zeta'|}\right),$$

where

$$A_2(\rho) = \frac{\sqrt{1-k}}{\pi} \frac{1}{\sqrt{k}\sqrt{\rho} - i\sqrt{1-k}\sqrt{1-\rho}}.$$

Then

$$\mathcal{F}_{\zeta'}(\Gamma_+(\cdot, x_n, t; y, 0)) = \int_0^1 |\zeta'| \mathrm{e}^{-iy'\cdot\zeta'} G(\zeta', y_n, t; \rho) H(\zeta', x_n; \rho) d\rho,$$

for every $x_n > 0$, $y_n < 0$.

PROPOSITION 4.5. *For every $\lambda_0 \in (0,1]$ there exist $\lambda_1, \lambda_2, \lambda_3 \in (0, \lambda_0]$ such that for every $h > 0$ the following inequality holds true*

$$I^{(h)} := \left| \int_0^{\lambda_2 h^2} dt \int_{\mathbf{R}^n_+} \nabla_x \Gamma_+^*(x, t; -\lambda_1 h e_n, \lambda_2 h^2) \cdot \nabla_x \Gamma_0(x, t; -\lambda_3 h e_n, 0) dx \right| \geq \frac{1}{Ch^n}, \quad (104)$$

where C, $C \geq 1$, depends on $\lambda_1, \lambda_2, \lambda_3$ and k only.

Proof. See [15, Proposition 3.7]. □

Step 4: quantitative estimates.
For $\bar{t} \in (0, T]$ fixed, we can assume, without loosing generality, that there exists $O \in \partial D_1(\bar{t}) \cap \Omega_D(\bar{t})$ such that

$$d_\mu(\bar{t}) = \operatorname{dist}(O, D_2(\bar{t})). \quad (105)$$

Denote by

$$\rho = \min\{d_\mu(\bar{t}), \rho_0\}.$$

Furthermore, denote by $\nu(O, \bar{t})$ the exterior unit normal to $\partial D_1(\bar{t})$ in O. Choosing parameters $\lambda_1, \lambda_2, \lambda_3 \in (0,1]$ satisfying inequality (104) and $\delta \in (0,1]$, we set

$$t_1 = \bar{t} - \lambda_2 h^2, \qquad \bar{y} = \lambda_1 h \nu(0, \bar{t}), \qquad y_1 = \lambda_3 h \nu(0, \bar{t}), \quad (106)$$

where

$$0 < h \leq \delta \min\{\rho, \sqrt{\overline{t}}\}. \tag{107}$$

By using (86a) it is simple to check that there exists C_1, $C_1 \geq 1$, depending on M_0 only such that if

$$0 < \delta \leq \frac{\lambda_3}{C_1} \tag{108}$$

then, for every $t \in [t_1, \overline{t}]$, we have

$$\operatorname{dist}(\overline{y}, D_1(t)) \geq \frac{1}{2} \min\{\lambda_1, \lambda_2, \lambda_3\} h, \tag{109}$$

$$\operatorname{dist}(y_1, D_1(t)) \geq \frac{1}{2} \min\{\lambda_1, \lambda_2, \lambda_3\} h. \tag{110}$$

On the other side, using the inequality [27, Proposition 4.1.6]

$$\left|\operatorname{dist}(O, D_2(t)) - \operatorname{dist}(O, D_2(\overline{t}))\right| \leq \frac{C_0}{\rho_0}|t - \overline{t}|, \tag{111}$$

where C_0 depends on M_0 and M_1 only, for $t \in [t_1, \overline{t}]$ and by using the triangle inequality we have that there exists C_2, $C_2 \geq 1$, depending on M_0 and M_1 only such that if

$$0 < \delta \leq \frac{1}{C_2} \tag{112}$$

then

$$\operatorname{dist}(z, D_2(t)) \geq \frac{1}{2}\rho, \qquad \text{with } z = \overline{y}, y_1. \tag{113}$$

Proposition 4.6. *Let $\{D_1(t)\}_{t \in \mathbf{R}}$, $\{D_2(t)\}_{t \in \mathbf{R}}$ be two families of domains satisfying* (86) *and let $\lambda_1, \lambda_2, \lambda_3 \in (0,1)$ be such that the inequality* (104) *is satisfied. Then there exist C, $C \geq 1$, and $\tilde{C}$, $\tilde{C} \geq 1$, C depending on k only and $\tilde{C}$ depending on $k, M_0, M_1, \lambda_1, \lambda_2$ and λ_3 only such that*

$$|\mathcal{U}(y_1, t_1; \overline{y}, \overline{t})| \geq \frac{1}{Ch^n}, \tag{114}$$

for $0 < h \leq \frac{1}{\tilde{C}} \min\{\rho, \sqrt{\overline{t}}\}$, where $y_1, t_1, \overline{y}, \overline{t}$, and ρ are defined in (106).

Proof. See [15]. □

Theorem 4.7 (Two-spheres and one-cylinder inequality). *Let λ, Λ and M positive numbers with $\lambda \in (0,1]$. Let P be the parabolic operator*

$$P = \partial_t - \partial_i \left(a^{ij} \partial_j\right), \tag{115}$$

where $\{a^{ij}(x,t)\}_{i,j=1}^{n}$ *is a symmetric* $n\times n$ *matrix. For* $\xi\in\mathbf{R}^n$ *and* $(x,t),(y,s)\in\mathbf{R}^{n+1}$ *assume that*

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \tag{116a}$$

and

$$\left(\sum_{i,j=1}^{n}\left(a^{ij}(x,t)-a^{ij}(y,s)\right)^2\right)^{1/2} \le \frac{\Lambda}{R}\left(|x-y|^2+|t-s|\right)^{1/2}. \tag{116b}$$

Let u *be a function in* $H^{2,1}\left(B_R\times(0,R^2)\right)$ *satisfying the inequality*

$$|Pu| \le \Lambda\left(\frac{|\nabla u|}{R}+\frac{|u|}{R^2}\right)^{1/2} \qquad \text{in } B_R\times[0,R^2). \tag{117}$$

Then there exist constants $\eta_1\in(0,1)$ *and* $C\in[1,+\infty)$*, depending on* λ*,* Λ *and* n *only such that for every* r_1,r_2*,* $0<r_1\le r_2\le\eta_1 R$ *we have*

$$\|u(\cdot,0)\|_{L^2(B_{r_2})} \le \frac{CR}{r_2}\|u\|_{L^2(B_R\times(0,R^2))}^{1-\theta_1}\|u(\cdot,0)\|_{L^2(B_{r_1})}^{\theta_1}, \tag{118}$$

where $\theta_1=\frac{1}{C\log\frac{R}{r_1}}$.

Proof. See [27]. □

Step 5: proof of Theorem 4.2.
For the proof of the theorem we refer to [15, Theorem 2.7] as it is rather technical and lengthy.

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