Rend. Istit. Mat. Univ. Trieste
Volume 48 (2016), 407–431
DOI: 10.13137/2464-8728/13204

# Stable determination at the boundary of the optical properties of a medium: the static case

Romina Gaburro

*This manuscript is dedicated to Giovanni Alessandrini on the occasion of his $60^{th}$ birthday, to honour his outstanding contribution to the field of inverse problems and mathematical analysis and to thank him for having been such an inspirational guide to the author*

Abstract. *The problem of the stable determination of the coefficients of second order elliptic partial differential equations arising in inverse problems is considered. Results of uniqueness and stability at the boundary were obtained in [3] and extended in [8, 9] for the conductivity equation. The common features of these papers are the employment of the singular solutions and the monotonicity assumption introduced in [3]. We revisit the techniques adopted in these papers to stably determine the absorption coefficient in anisotropic media by means of Optical Tomography (OT) in the so-called static case. This also shows that the monotonicity assumption is realistic at least in the context of OT.*

Keywords: Stability at the boundary, EIT, OT.
MS Classification 2010: 35J25, 35R30.

## 1. Introduction

We start by considering the well known inverse conductivity problem. In absence of internal sources, the electrostatic potential $u$ in a conducting body, described by a domain $\Omega \subset \mathbb{R}^n$, is governed by the elliptic equation

$$\mathrm{div}(\sigma\nabla u) = 0 \qquad \text{in} \quad \Omega, \tag{1}$$

where the symmetric, positive definite matrix $\sigma = \sigma(x)$, $x \in \Omega$, represents the (possibly anisotropic) electric conductivity. The inverse conductivity problem consists of finding $\sigma$ when the so called Dirichlet-to-Neumann (D-N) map

$$\Lambda_\sigma : u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega) \longrightarrow \sigma\nabla u \cdot \nu|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$$

is given for any $u \in H^1(\Omega)$ solution to (1.1). Here, $\nu$ denotes the unit outer normal to $\partial\Omega$. If measurements can be taken only on one portion $\Gamma$ of $\partial\Omega$, then the relevant map is called the local D-N map.

This problem arises in electrical resistivity tomography (ERT) (or more generally electrical impedance tomography EIT), a method used for subsurface geophysical imaging, industrial process monitoring and as an experimental medical imaging technique. Different materials display different electrical properties, so that a map of the conductivity $\sigma(x)$, $x \in \Omega$ can be used to investigate internal properties of $\Omega$. The first mathematical formulation of the inverse conductivity problem is due to A. P. Calderón [19], where he addressed the problem of whether it is possible to determine the (isotropic) conductivity by the D-N map.
The case when measurements can be taken all over the boundary has been studied extensively in the past and fundamental papers like [3, 37, 38, 54] show that the isotropic case can be considered solved. On the other hand the anisotropic case is still open and different lines of research have been pursued. One direction has been to find the conductivity up to a diffeomorphism which keeps the boundary fixed (see [39, 40, 41, 46, 53]). The original work of [41] assumed that the metric was real-analytic with topological assumptions subsequently relaxed in [39, 40] in the context of local data. We also refer to the work [22] which introduced methods for studying the anisotropic Calderón problem on manifolds which are not real-analytic, but where the metric has a certain form. This result is based on the concept of limiting Carleman weights, earlier introduced in [36] for the Euclidean case and partial data. We refer to [20] and [35] for related works on the stability and reconstruction respectively of anisotropic conductivities. We also mention that the results obtained in [22] have been improved in [23]. Another direction has been the one to assume that the anisotropic conductivity is *a priori* known to depend on a restricted number of spatially–dependent parameters (see [3, 8, 9, 24, 25, 42]).
Alessandrini [3] considered the case when $\sigma(x)$ is anisotropic and it is *a priori* known to have the structure $\sigma(x) = \sigma(a(x))$, where $t \to \sigma(t)$ is a given matrix-valued function and $a = a(x)$ is an unknown scalar function. In [3] results of uniqueness and stability at the boundary are proven by using the method of singular solutions under the additional assumption of monotonicity

$$D_t\sigma(t) \geq Const.I > 0.$$

These results have been extended in [8] and[9] to the case when $\sigma$ has the more general structure

$$\sigma(x) = \sigma(x, a(x)), \tag{2}$$

where $a(x)$ is an unknown scalar function and $\sigma(x,t)$ is given and satisfies the monotonicity assumption

$$D_t\sigma(x,t) \geq Const.I > 0, \tag{3}$$

in the case of full and local data respectively. The singular solutions introduced in [3] have been extended in [51] for the more general operator of type

$$Lu = -\mathrm{div}(\sigma \nabla u + Pu) + Q \cdot \nabla u + qu, \tag{4}$$

where the leading order coefficients matrix $\sigma = \sigma(x)$ is merely Hölder continuous and some positivity condition is imposed on the lower order terms. We recall that singular solutions have also been used by Isakov [31] to determine discontinuities in the conductivity for the isotropic case. However, only Green's function type singularities were needed for this purpose.
In the present paper the author considers the inverse problem of determining the optical properties of a medium and shows that the structure (2) introduced in [8, 9] is appropriate in optical tomography (OT). This is the problem of determining the spacially dependent optical properties (the absorption and the scattering coefficients $\mu_a$, $\mu_s$ respectively) when light in a narrow-wavelength band in the near infrared is employed to transilluminate tissue (see [11, 13, 14]). We also refer to [28, 29, 30] for related topis in OT. The resulting measurements of intensity on the tissue boundary are then used to reconstruct a map of the optical properties within the tissue. In the so-called OT static case the integral equation (*Radiative Transfer Equation*) typically used to model this problem can be reduced (under certain conditions) to an elliptic partial differential equation of type

$$\mathrm{div}(\sigma \nabla u) - qu = 0 \qquad \text{in} \quad \Omega, \tag{5}$$

with

$$\sigma = \sigma(x, \mu_a(x)), \qquad q = \mu_a(x), \tag{6}$$

where $\mu_a(x)$ is a function (the absorption coefficient) to be determined and $\sigma(x,t)$ is given and satisfies the monotonicity assumption

$$D_t \sigma(x,t) \leq Const.I < 0. \tag{7}$$

Notice that although $D_t\sigma(x,t)$ is a negative definite matrix in (7), whereas the case of a positive definite $D_t\sigma(x,t)$ was considered in [8, 9], the arguments used in the current paper and in [8, 9] continue to work if $D_t\sigma(x,t)$ satisfies either (3) or (7). In other words a monotonicity assumption of either type (3) or (7) seems to be a realistic hypothesis that is satisfied for example in the OT problem considered in this manuscript. The result presented here also shows that the machinery of the stability proofs at the boundary via singular solutions introduced in [3] works also in the more general case (5), where the equation has an extra zero order term. The OT formulation given by (5), (6) is achieved in the static case if it is assumed that the scattering coefficient $\mu_s$ has been determined by employing a different imaging modality (like MRI) prior to the application of OT and the structural information we are interested in

is the determination of $\mu_a$. The main focus of the present paper is indeed on the stable determination at the boundary of $\mu_a$ and its derivatives by pursuing the same line of investigation of [8, 9]. This is done by considering anisotropic diffusion tensors $\sigma(x,t)$ arising in OT that are real matrix-valued functions. The time-harmonic case where $\sigma(x,t)$ is a complex matrix-valued function will be investigated in future work. The case in which $\mu_a$ is known and the scattering coefficient $\mu_s$ is to be determined can be treated in a similar manner to the one considered in this work. In medical applications, while the scattering coefficient $\mu_s$ varies from tissue to tissue, it is the absorption coefficient $\mu_a$ that carries the more interesting physiological information as it is related to the global concentrations of certain metabolites in their oxygenated and deoxygenated states. Moreover, many tissues including parts of the brain, muscle and breast tissue have a fibrous structure on a microscopic scale which results in anisotropic physical properties on a larger scale.
We shall also emphasize that the stable determination of $\mu_a$ (or equivalently of $\mu_s$) and its derivatives at the boundary are useful tools to infer uniqueness and stability of $\mu_a$ (or $\mu_s$) in the interior, which represents the preliminary goal to achieve an image of the interior of $\Omega$ (the body under investigation). On the other hands, it is well known that the inverse boundary value problem of determining $\sigma$ in (1) from the knowledge of the D-N map is severely ill-posed. Indeed, regarding the stability of the inverse conductivity problem, Alessandrini [2] proved that, assuming $n \geq 3$ and *a-priori* bounds on $\sigma$ of the form

$$||\sigma||_{H^s(\Omega)} \leq E, \quad \text{for some } s > \frac{n}{2} + 2, \tag{8}$$

$\sigma$ depends continuously on $\Lambda_\sigma$ with a modulus of continuity of logarithmic type. For subsequent results of this type we also refer to [3, 4] and to [15, 16, 43] for the two-dimensional case. The common logarithmic type of stability cannot be avoided ([5, 44]). However, the ill-posed nature of this problem can be modified to be conditionally well-posed by restricting the conductivity to certain function subspaces. Well-posedness is here expressed by Lipschitz stability. A first result of this kind was established by Alessandrini and Vessella [10], where the authors proved global stability of $\sigma$ in terms of the local D-N map, for the case when $\sigma$ is isotropic and piecewise constant on a given finite partition of $\Omega$. This fundamental result was extended later on to different types of inverse problems. In the context of the inverse conductivity problem to which we refer in this work, we wish to recall the results of [7, 17] for the cases of real piecewise linear and complex piecewise constant isotropic conductivity respectively and to [25] for the case of a conformal class of piecewise anisotropic conductivities. All of these results are obtained in terms of local data. We also refer to [50] where it was shown that the Lipschitz stability constant appearing in the above mentioned results grows exponentially with the number of domains partitioning $\Omega$ and to [6] for a recent result of global uniqueness for anisotropic

conductivities that are piecewise constant in the context of local data too.
To conclude, we shall point out that the problem of recovering the conductivity $\sigma$ by local measurements has been treated more recently. In this context we wish to recall also [18, 21, 27, 33, 34, 47, 48, 49]. The results obtained in the current paper could be adapted to the case of local data too.
The paper is organized as follows. Section 2 contains the formulation of the problem in OT for the static case (subsection 2.1) and the main results (subsection 2.2, Theorems 2.5, 2.6). Section 3 is devoted to a review of the construction of singular solutions for equations of type (5) having a singularity of arbitrarily high order at a given point. This is done by following the same line of [3] (see also [51] for the more general case (4)). The proofs of Theorems 2.5, 2.6 are given in section 4.

## 2. The main result

### 2.1. Formulation of the problem

Although Maxwell's equations provide a complete model for the light propagation in a scattering medium on a micro scale, on the scale suitable for medical OT an appropriate model is given by the *Radiative Transfer Equation* (or *Boltzmann equation*)[14]. If $\Omega$ is a domain in $\mathbb{R}^n$, with $n \geq 2$ with smooth boundary $\partial\Omega$ and radiation is considered in the body $\Omega$, then it is well known that if the input field is modulated with a fixed harmonic frequency $\omega$, the so-called *Diffusion Approximation* leads to the elliptic equation (see [11]) for the energy current density $u$

$$\mathrm{div}\,(K\nabla u) - (\mu_a - \mathrm{i}k)u = 0, \qquad \text{in } \Omega, \tag{9}$$

where $k = \frac{\omega}{c}$ is the wave number and $K$ is the complex matrix valued function

$$K = \frac{1}{n}\Big((\mu_a - \mathrm{i}k)I + (I-B)\mu_s\Big)^{-1},$$

where $B_{ij}(x) = B_{ji}(x)$ is a real matrix valued function and $I - B$ is positive definite ([11, 29, 30]). The spacially varying coefficients $\mu_a$ and $\mu_s$ are called the *absorption* and the *scattering coefficients* of the medium $\Omega$ and represent the optical properties of $\Omega$. Here we consider the simpler static case $k = 0$ for which $K$ reduces to the real matrix valued function

$$K = \frac{1}{n}\Big(\mu_a I + (I-B)\mu_s\Big)^{-1}. \tag{10}$$

Although it is common practise in OT to use the Robin-to-Robin map to describe the boundary measurements (see [11]), the D-N map will be employed in this manuscript instead. The rigorous definition of this map for an equation

of type (9) will be given in subsection 2.1.1. For now, we just recall that prescribing its inverse, called the Neumann-to-Dirichlet (N-D) map, is equivalent to prescribe in OT the more commonly used Robin-to-Robin map. It can also be shown that prescribing the N-D map is insufficient to recover both coefficients $\mu_a$ and $\mu_s$ uniquely [13] unless *a priori* smoothness assumptions are employed [26]. In this paper we consider the problem of determining $\mu_a$ and its derivatives when $\mu_s$ and $B$ are assumed known. More precisely, we show that $\mu_a$ and its derivatives at the boundary depend upon $\Lambda_{K,\mu_a}$ with a modulus of continuity of Lipschitz and Hölder type respectively. These are the main results of this paper and are contained in Theorems 2.5, 2.6.
We rigorously formulate the problem by introducing the following notation, definitions and assumptions.
For $n \geq 3$, a point $x \in \mathbb{R}^n$ will be denoted by $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Moreover, given a point $x \in \mathbb{R}^n$, we will denote with $B_r(x), B'_r(x')$ the open balls in $\mathbb{R}^n, \mathbb{R}^{n-1}$ respectively centred at $x$ and $x'$ with radius $r$ and by $Q_r(x)$ the cylinder

$$Q_r(x) = B'_r(x') \times (x_n - r, x_n + r).$$

We will also denote $B_r = B_r(0)$, $B'_r = B'_r(0)$ and $Q_r = Q_r(0)$.

Definition 2.1. *Let $\Omega$ be a domain in $\mathbb{R}^n$. We say that $\partial\Omega$ is of Lipschitz class with constants $L$, $r > 0$ if for any $P \in \partial\Omega$ there exists a rigid transformation of $\mathbb{R}^n$ under which we have $P = 0$ and*

$$\Omega \cap Q_r = \{x \in Q_r \,:\, x_n > \varphi(x')\},$$

*where $\varphi$ is a Lipschitz function on $B'_{r_0}$ satisfying*

$$\varphi(0) = 0; \qquad \|\varphi\|_{C^{0,1}(B'_r)} \leq Lr.$$

**Assumption** (on the known parameters $\mu_s$ and $B$)**:** we assume that $\mu_s, B \in W^{1,\infty}(\Omega)$ and that for some positive constants $\lambda, E$

$$\lambda^{-1} \leq \mu_s(x) \leq \lambda, \qquad for\ every \quad x \in \Omega, \tag{11}$$

and

$$||\mu_s||_{W^{1,\infty}(\Omega)} \leq E, \tag{12}$$
$$||B||_{W^{1,\infty}(\Omega)} \leq E. \tag{13}$$

We introduce the following class of matrix valued functions $\sigma(x,t)$ on $\Omega \times [\lambda^{-1}, \lambda]$.

Definition 2.2. *Given $p > n$ , we say that $\sigma(\cdot,\cdot) \in \mathcal{H}'_p$ if there are positive constants $\lambda$, $\mathcal{E}$, $\mathcal{F} > 0$, such that, denoting by* $\mathrm{Sym}_n$ *the class of $n \times n$ real symmetric matrices, the following conditions are satisfied*

$$\sigma \in W^{1,p}(\Omega \times [\lambda^{-1}, \lambda], \mathrm{Sym}_n), \tag{14}$$

$$D_t\sigma \in W^{1,p}(\Omega \times [\lambda^{-1}, \lambda], \mathrm{Sym}_n), \tag{15}$$

$$\begin{aligned}\text{ess sup}_{t\in[\lambda^{-1},\lambda]} \Big( & \| \sigma(\cdot,t) \|_{L^p(\Omega)} + \| D_x\sigma(\cdot,t) \|_{L^p(\Omega)} \\ & + \| D_t\sigma(\cdot,t) \|_{L^p(\Omega)} + \| D_tD_x\sigma(\cdot,t) \|_{L^p(\Omega)} \Big) \le \mathcal{E},\end{aligned} \tag{16}$$

$$\lambda^{-1}|\xi|^2 \le \sigma(x,t)\xi\cdot\xi \le \lambda|\xi|^2, \qquad \textit{for almost every } x \in \Omega, \quad \textit{for every } t \in [\lambda^{-1}, \lambda],\ \xi \in \mathbb{R}^n, \tag{17}$$

$$D_t\sigma(x,t)\,\xi\cdot\xi \le -\mathcal{F}|\xi|^2, \qquad \textit{for almost every } x \in \Omega\,, \quad \textit{for every } t \in [\lambda^{-1}, \lambda]\,,\ \xi \in \mathbb{R}^n. \tag{18}$$

Remark 2.3. We observe that properties (14) - (17) were satisfied by the one-parameter family of conductivities $\sigma(x,t)$ belonging to the class $\mathcal{H}$ introduced in [8, 9]. Property (18), which is a property of monotonicity of $D_t\sigma(x,t)$ with respect to the variable t, replaces the monotonicity assumption (3) in $\mathcal{H}$. (18) states that $D_t\sigma(x,t)$ is a negative definite matrix for almost every $x \in \Omega$, where the monotonicity assumption (3) of $\mathcal{H}$ in [8, 9] required $D_t\sigma(x,t)$ to be positive definite instead. In this work we will show that the results obtained in [8, 9] can be similarly obtained when (3) is replaced by (18) and equation (1) is replaced by the more general one in (5).

Let us rigorously define the D-N map for (5).

#### 2.1.1. The Dirichlet-to-Neumann map.

If $n \ge 3$ and $\Omega$ is a domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ (with constants $L, r$) as in Definition 2.1, we assume that $\sigma \in L^\infty(\Omega, Sym_n)$, $q \in L^\infty(\Omega)$ satisfy the ellipticity condition

$$\lambda^{-1}|\xi|^2 \le \sigma(x)\xi\cdot\xi \le \lambda|\xi|^2, \qquad \textit{for almost every } x \in \Omega, \quad \textit{for every } \xi \in \mathbb{R}^n. \tag{19}$$

and

$$\lambda^{-1} \le q(x) \le \lambda, \quad \text{for almost every } x \in \Omega \tag{20}$$

respectively. We denote by $\langle\cdot,\cdot\rangle$ the $L^2(\partial\Omega)$-pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and its dual $H^{-\frac{1}{2}}(\partial\Omega)$.

DEFINITION 2.4. *The Dirichlet-to-Neumann (D-N) map associated with $\sigma$, $q$ is the operator*

$$\Lambda_{\sigma,\, q} : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega) \tag{21}$$

*defined by*

$$\langle \Lambda_{\sigma,\, q}\, f,\, g\rangle \;=\; \int_{\Omega} \Big(\sigma(x)\nabla u(x)\cdot\nabla\varphi(x) + q(x)u(x)\varphi(x)\Big)\, dx, \tag{22}$$

*for any $f,\, g \in H^{\frac{1}{2}}(\partial\Omega)$, where $u \in H^1(\Omega)$ is the weak solution to*

$$\begin{cases} \operatorname{div}(\sigma(x)\nabla u(x)) - q(x)u(x) = 0, & \text{in} \quad \Omega, \\ u = f, & \text{on} \quad \partial\Omega \end{cases}$$

*and $\varphi \in H^1(\Omega)$ is any function such that $\varphi|_{\partial\Omega} = g$ in the trace sense.*

Note that, by (22), it is easily verified that $\Lambda_{\sigma,\, q}$ is selfadjoint and that given $\sigma_i \in L^\infty(\Omega\,, Sym_n)$, for $i = 1, 2$, $q_i \in L^\infty(\Omega)$, satisfying (19) and (20) respectively, the well known Alessandrini's identity (see [32, (5.0.4), p.129])

$$\begin{aligned} \langle \Lambda_{\sigma_1,\, q_1} - \Lambda_{\sigma_2,\, q_2} f_1, f_2\rangle \quad &= \quad \int_{\Omega} (\sigma_1(x) - \sigma_2(x))\, \nabla u_1(x)\cdot\nabla u_2(x)\, dx \\ &\quad + \int_{\Omega} (q_1(x) - q_2(x))\, u_1(x) u_2(x)\, dx, \end{aligned} \tag{23}$$

holds true for any $f_i \in H^{\frac{1}{2}}(\partial\Omega)$, where $u_i \in H^1(\Omega)$ is the unique weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma_i(x)\nabla u_i(x)) - q_i(x)u_i(x) = 0, & \text{in} \quad \Omega, \\ u_i = f_i, & \text{on} \quad \partial\Omega, \end{cases}$$

for $i = 1, 2$.
In the sequel we will denote the D-N map $\Lambda_{K,\mu_a}$ corresponding to (9) (for $k = 0$) by

$$\Lambda_{\mu_a}$$

to simplify our notation. We will also denote by $\|\cdot\|_*$ the norm on the Banach space of bounded linear operators between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$.

## 2.2. The main result

The following theorems are the main results of this paper.

Theorem 2.5 (Lipschitz stability of boundary values). *Let $n \geq 3$, $p > n$ and $\Omega$ be a bounded domain with Lipschitz boundary with constants $L$, $r$ as in Definition 2.1. Let $\mu_{s_i}$ satisfy* (11), (12), *$i = 1, 2$ and $B$ satisfy* (13). *If $\mu_{a_i}$ satisfies*

$$\lambda^{-1} \leq \mu_{a_i}(x) \leq \lambda, \qquad \text{for every} \quad x \in \Omega, \tag{24}$$

$$\| \mu_{a_i} \|_{W^{1,p}(\Omega)} \leq E, \tag{25}$$

*for $i = 1, 2$, then we have*

$$\| \mu_{a_1}(x) - \mu_{a_2}(x) \|_{L^\infty(\partial\Omega)} \leq C \| \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}} \|_* \, . \tag{26}$$

*Here $C > 0$ is a constant depending on $n$, $p$, $L$, $r$, $diam(\Omega)$, $\lambda$, $\mathcal{E}$, $\mathcal{F}$ and $E$.*

Theorem 2.6 (Hölder stability of derivatives at the boundary). *Let $n \geq 3$, $p$, $\Omega$, $\mu_{a_i}$, $\mu_{s_i}$, $i = 1, 2$ and $B$ be as in Theorem 2.5. Given $y \in \partial\Omega$ and a neighborhood $U$ of $y$ in $\overline{\Omega}$, assume that for some positive integer $k$ and some $\alpha$, $0 < \alpha < 1$ we have*

$$\| \mu_{s_i} \|_{C^{k,\alpha}(\bar{U})}, \, \| B \|_{C^{k,\alpha}(\bar{U})} \leq E_k, \tag{27}$$

*for $i = 1, 2$ and*

$$\| \mu_{a_1} - \mu_{a_2} \|_{C^{k,\alpha}(\bar{U})} \leq E_k. \tag{28}$$

*Then, for every neighborhood $W$ of $y$ in $\overline{\Omega}$ such that $\overline{W} \subset U$,*

$$\| D^k(\mu_{a_1} - \mu_{a_2}) \|_{L^\infty(\partial\Omega \cap \overline{W})} \leq C \| \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}} \|_*^{\delta_k \alpha}, \tag{29}$$

*where*

$$\delta_k = \prod_{j=0}^{k} \frac{\alpha}{\alpha + j}. \tag{30}$$

*Here $C > 0$ is a constant which depends only on $n$, $p$, $L$, $r$, $diam(\Omega)$, $dist(W \cap \partial\Omega, \Omega \setminus U)$, $\lambda$, $E$, $\mathcal{F}$, $\mathcal{E}$ $\alpha$, $k$, and $E_k$.*

## 3. Singular solutions

This section is devoted to a review of the construction of singular solutions of an elliptic equation in divergence form with a lower extra term of order zero. This type of solutions were introduced by Alessandrini in [3] for an equation of type (1) and have been extended to solutions of a more general equation of type (4). The decision to expose in this manuscript the key-points necessary for the constructions of such solutions in the OT context is driven by the willingness of keeping the manuscript as self-contained as possible. It is also hoped that the details highlighted here will be of use for the more physically relevant time-harmonic case in OT, where the matrix valued function $K$ is complex and the zero order term in (9) is complex too. Here we consider an operator of type

$$L = \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial}{\partial x_j}\right) - q, \qquad \text{in} \quad B_R, \tag{31}$$

where the leading order coefficients $\sigma_{ij}(x)$, $i, j = 1, \dots, n$ and the zero order coefficient $q(x)$ satisfy

$$\lambda^{-1}\,|\xi|^2 \le \sigma_{ij}(x)\xi_i\xi_j \le \lambda\,|\xi|^2, \quad \text{for every } x,\ \xi, \quad x \in B_R,\ \xi \in \mathbb{R}^n, \tag{32}$$

$$\|\,\sigma_{ij}\,\|_{W^{1,\,p}(B_R)} \le E, \qquad i,\ j = 1, \dots, n, \tag{33}$$

for some $p > n$ and

$$\lambda^{-1} \le q(x) \le \lambda, \quad \text{for any } x, \quad x \in B_R. \tag{34}$$

Theorem 3.1 (Singular solutions for $L = \mathrm{div}(\sigma\nabla\cdot) - q$). *Let $L$ satisfy* (31)-(34). *For any spherical harmonic $S_m$ of degree $m = 0,\ 1,\ 2, \dots$, there exists $u \in W^{2,\,p}_{loc}(B_R \setminus \{0\})$ such that*

$$Lu = 0, \ \ in\ B_R \setminus \{0\} \tag{35}$$

*and furthermore*

$$u(x) \;=\; \log|Jx| S_0\left(\frac{Jx}{|Jx|}\right) + w(x), \qquad when\ n = 2\ and\ \ m = 0, \tag{36}$$

$$u(x) \;=\; |Jx|^{2-n-m}\, S_m\left(\frac{Jx}{|Jx|}\right) + w(x) \quad otherwise, \tag{37}$$

*where $J$ is a positive definite symmetric matrix such that $J = \sqrt{(\sigma_{ij}(0))^{-1}}$ and $w$ satisfies*

$$|\,w(x)| + |\,x\,|\,|Dw(x)| \le C\,|\,x\,|^{\,2-n-m+\alpha}, \quad in \quad B_r \setminus \{0\}, \tag{38}$$

$$\left(\int_{r<|x|<2r} |D^2 w|^p\right)^{\frac{1}{p}} \le C\, r^{-n-m+\alpha+\frac{n}{p}}, \quad for\,every \quad r,\ 0 < r < R/2. \tag{39}$$

*Here $\alpha$ is any number such that $0 < \alpha < 1 - \frac{n}{p}$, and $C$ is a constant depending only on $\alpha$, $n$, $p$, $r$, $\lambda$, and $\mathcal{E}$.*

Next we consider three technical lemmas. The proofs of these results for the case where $L = \mathrm{div}(\sigma\nabla\cdot)$ are treated in details in [2] and their extension to the more general case $L = \mathrm{div}(\sigma\nabla\cdot) - q$ is quite straightforward, therefore only the key points of their proofs will be highlighted here. In what follows $A$ denotes a positive constant.

LEMMA 3.2. *Let $p > n$ and $u \in W^{2,p}_{loc}(B_R \backslash \{0\})$ be such that, for some positive $s$,*

$$|u(x)| \leq |x|^{2-s}, \qquad \text{for any} \quad x \in B_R \setminus \{0\}, \tag{40}$$

$$\left( \int_{r<|x|<2r} |Lu|^p \right)^{\frac{1}{p}} \leq A r^{\frac{n}{p}-s}, \qquad \text{for any } r, \quad 0 < r < \frac{R}{2}. \tag{41}$$

*Then we have*

$$|Du(x)| \leq C|x|^{1-s}, \quad \text{for any} \quad x \in B_R \setminus \{0\}, \tag{42}$$

$$\left( \int_{r<|x|<2r} |D^2u|^p \right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-s} \quad \text{for any } r, \quad 0 < r < \frac{R}{4}, \tag{43}$$

*where $C$ is a positive constant depending only on $A$, $n$, $p$, $\lambda$ and $E$.*

*Proof of Lemma 3.2.* The proof is a consequence of the $L^p$ interior Schauder estimate

$$||D^2u||_{L^p(B_{\rho_1\rho_2})} \leq \frac{C}{(1-\rho_1^2)\rho_2^2} \left( \rho_2^2 ||Lu||_{L^p(B_{\rho_2})} + ||u||_{L^p(B_{\rho_2})} \right), \tag{44}$$

where $C = C(n, p, \lambda, E)$ is a positive constant, $0 < \rho_1 < 1$ and $B_{\rho_2}$, $B_{\rho_1\rho_2}$ are two concentric balls such that $u \in W^{2,p}(B_{\rho_2})$ (see [45, Lemma 5.6.1]). We refer to [2, Proof of Lemma 2.1] for a detailed proof of this lemma. □

LEMMA 3.3. *Let $f \in L^p_{loc}(B_R \setminus \{0\})$ satisfy*

$$\left( \int_{r<|x|<2r} |f|^p \right)^{\frac{1}{p}} \leq A r^{\frac{n}{p}-s}, \qquad \text{for any } r, \quad 0 < r < \frac{R}{2}, \tag{45}$$

*with $2 < s < n < p$. Then there exists $u \in W^{2,p}_{loc}(B_R \setminus \{0\})$ satisfying*

$$Lu = f, \qquad \text{in} \quad B_R \setminus \{0\} \tag{46}$$

*and*

$$|u(x)| \leq C|x|^{2-s}, \quad \text{for any} \quad x \in B_R \setminus \{0\}, \tag{47}$$

*where $C$ is a positive constant depending only on $A$, $s$, $n$, $p$, $R$, $\lambda$ and $E$.*

*Proof of Lemma 3.3.* The proof is based on the construction of a fundamental solution $\Gamma$ of the equation $Lu = 0$ so that

$$|\Gamma(x, y)| \leq C(n, \lambda)|x - y|^{2-n}, \qquad \text{for any } x \neq y \tag{48}$$

(see [52]). See also [1, section 4] for a brief description of this construction and [2, Proof of Lemma 2.2] for a complete proof of this lemma. □

Definition 3.4. *We shall denote solution $u$ of* (46) *by*

$$u = T_L u.$$

The last technical result that we recall involves pointwise estimates of some solution of the Laplace equation and we refer to [2, Proof of Lemma 2.3] for its proof.

Lemma 3.5. *Let $s > n$ be a non-integer real number. Let $f$ be as in lemma 3.3 and satisfying* (45) *with $p > n$. Then there exists $u \in W^{2,p}_{loc}(B_R \setminus \{0\})$ satisfying*

$$\Delta u = f, \text{ in } B_R \setminus \{0\} \tag{49}$$

*and such that* (47) *holds true with $C > 0$ a constant depending only on $A, s, n, p$ and $R$.*

Definition 3.6. *We shall denote solution $u$ of* (49) *by*

$$u = T_S u.$$

We proceed next with the proof of 3.1.

*Proof of Theorem 3.1.* The proof follows the same line of [2, Proof of Theorem 1.1]. We will therefore only rephrase the key points of this proof showing how it can be adapted to the more general case treated here. For simplicity we first assume that $\sigma(0) = I$, where $I$ denotes the $n \times n$ identity matrix and prove that, under this additional assumption, for any spherical harmonic $S_m$ of degree $m = 0, 1, 2, \ldots$, there exists $u \in W^{2,p}_{loc}(B_R \setminus \{0\})$ such that

$$Lu = 0, \text{ in } B_R \setminus \{0\} \tag{50}$$

and

$$u(x) = \log|x| S_0\left(\frac{x}{|x|}\right) + w(x), \quad \text{when } n = 2 \text{ and } m = 0, \tag{51}$$

$$u(x) = |x|^{2-n-m} S_m\left(\frac{x}{|x|}\right) + w(x) \quad \text{otherwise,} \tag{52}$$

where $w$ satisfies (38), (39). For this, we consider in $B_R \setminus \{0\}$ the harmonic

$$H(x) = |x|^{2-n-m} S_m\left(\frac{x}{|x|}\right).$$

As in [2, Proof of Theorem 1.1] the idea is to find $w$ satisfying (38), (39) and such that

$$Lw = -LH, \text{ in } B_R \setminus \{0\}.$$

We have

$$-LH = (\Delta - L)H = (\delta_{ij} - a_{ij})\,\frac{\partial^2 H}{\partial x_i \partial x_j} - \frac{\partial a_{ij}}{\partial x_i}\,\frac{\partial H}{\partial x_j} - qH. \tag{53}$$

From [2, Proof of Theorem 1.1] we have

$$\left(\int_{r<|x|<2r} |\delta_{ij} - a_{ij}|^p \left|\frac{\partial^2 H}{\partial x_i \partial x_j}\right|^p\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-n-m+\beta}, \tag{54}$$

$$\left(\int_{r<|x|<2r} \left|\frac{\partial a_{ij}}{\partial x_i}\right|^p \left|\frac{\partial H}{\partial x_j}\right|^p\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-n-m+\beta}, \tag{55}$$

where $\beta = 1 - \frac{n}{p}$. Here the extra lower order term $-qH$ can be estimated as follows

$$\begin{aligned}
\left(\int_{r<|x|<2r} |qH|^p\right)^{\frac{1}{p}} &\leq C(\lambda, R)\left(\int_{r<|x|<2r} |x|^{(2-n-m)p}\right)^{\frac{1}{p}} \\
&\leq C(\lambda, R)\left(\int_r^{2r} \rho^{(2-n-m)p+n-1}\right)^{\frac{1}{p}} \\
&\leq C r^{\frac{n}{p}-n-m+\beta}
\end{aligned} \tag{56}$$

and by combining (54)-(56) together we obtain

$$\left(\int_{r<|x|<2r} |LH|^p\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-n-m+\beta}. \tag{57}$$

Let $\alpha$ be an irrational number such that $0 < \alpha < \beta$ and define

$$K = \left[\frac{m}{\alpha}\right].$$

If $w_0 = T_S(-LH)$, then we have

$$|w_0(x)| \leq C|x|^{2-n-m+\beta}, \qquad \text{for any } x,\ x \in B_R \setminus \{0\}.$$

We define

$$w_j = \begin{cases} w_0, & j = 0 \\ T_S f, \quad f = (\Delta - L)w_{j-1}, & j = 1, \dots, K-1. \end{cases} \tag{58}$$

Lemma 3.7. *For any $j = 0, \dots, K-1$ we have*

$$|w_j(x)| \le C|x|^{2-n-m+(j+1)\alpha}, \tag{59}$$

$$\left(\int_{r<|x|<2r} |(\Delta - L)w_j|^p\right)^{\frac{1}{p}} \le Cr^{\frac{n}{p}-n-m+(j+2)\alpha}. \tag{60}$$

*Proof of Lemma 3.7.* . We prove (59), (60) by induction on $j$. For $j = 0$ we have

$$|w_0(x)| \le C|x|^{2-n-m+\beta} \le C|x|^{2-n-m+\alpha}$$

and

$$\begin{aligned}\left(\int_{r<|x|<2r} |(\Delta-L)w_j|^p\right)^{\frac{1}{p}} &\le Cr^{\frac{n}{p}-n-m+2\alpha} + C\left(\int_{r<|x|<2r} |(cw_0|^p\right)^{\frac{1}{p}} \\ &\le Cr^{\frac{n}{p}-n-m+2\alpha} + C\left(\int_{r<|x|<2r} |x|^{(2-n-m+\alpha)p}\right)^{\frac{1}{p}} \\ &\le Cr^{\frac{n}{p}-n-m+2\alpha} + Cr^{\frac{n}{p}-n-m+\alpha} \\ &\le Cr^{\frac{n}{p}-n-m+\alpha}.\end{aligned}$$

Suppose now that (59), (60) are true for $j$, i.e.

$$|w_j(x)| \le C|x|^{2-n-m+(j+1)\alpha},$$

$$\left(\int_{r<|x|<2r} |(\Delta - L)w_j|^p\right)^{\frac{1}{p}} \le Cr^{\frac{n}{p}-n-m+(j+2)\alpha},$$

then if we define $s = n + m - (j+2)\alpha$, we have that $s > n$ and if we take

$$w_{j+1} = T_S f, \qquad \text{with} \quad f = (\Delta - L)w_j,$$

then

$$|w_{j+1}(x)| \le C|x|^{2-n-m+(j+2)\alpha} \tag{61}$$

and

$$\begin{aligned}\left(\int_{r<|x|<2r} |(\Delta-L)w_{j+1}|^p\right)^{\frac{1}{p}} &\le Cr^{\frac{n}{p}-n-m+(j+3)\alpha} + C\left(\int_{r<|x|<2r} |cw_{j+1}|^p\right)^{\frac{1}{p}} \\ &\le Cr^{\frac{n}{p}-n-m+(j+3)\alpha} \\ &\quad + C\left(\int_{r<|x|<2r} |x|^{(2-n-m+(j+2)\alpha)\,p}\right)^{\frac{1}{p}} \\ &\le Cr^{\frac{n}{p}-n-m+(j+3)\alpha} + Cr^{\frac{n}{p}-n-m+(j+2)\alpha} \\ &\le Cr^{\frac{n}{p}-n-m+(j+3)\alpha},\end{aligned} \tag{62}$$

which conclude the proof. $\square$

(60) with $j = K-1$ gives

$$\left(\int_{r<|x|<2r} |(\Delta - L)w_{K-1}|^p\right)^{\frac{1}{p}} \leq Cr^{\frac{n}{p}-n-m+(K+1)\alpha}$$

and if we define $s = n+m-(K+1)\alpha$, we have $s < n$. If we define

$$W_K = T_L f, \qquad \text{with} \quad f = (\Delta - L)w_{K-1},$$

we have

$$|W_K(x)| \leq C|x|^{2-n-m+(K+1)\alpha}, \qquad \text{for any} \quad x \in B_R \setminus \{0\}. \tag{63}$$

We define as in [2, Proof of Theorem 1.1] the function $w$

$$w = \sum_{j=0}^{K-1} w_j + W_K. \tag{64}$$

$w \in W^{2,p}_{loc}(B_R \setminus \{0\})$ and satisfies

$$|w(x)| \leq C|x|^{2-n-m+\alpha} \qquad \text{for any} \quad x \in B_R \setminus \{0\},$$

moreover

$$\begin{aligned}
\left(\int_{r<|x|<2r} |Lw|^p\right)^{\frac{1}{p}} &\leq Cr^{\frac{n}{p}-n-m+\alpha} + \left(\int_{r<|x|<2r} |qw|^p\right)^{\frac{1}{p}} \\
&\leq Cr^{\frac{n}{p}-n-m+\alpha} + C\left(\int_{r<|x|<2r} |x|^{(2-n-m+\alpha)p}\right)^{\frac{1}{p}} \\
&\leq Cr^{\frac{n}{p}-n-m+\alpha} + Cr^{\frac{n}{p}+2-n-m+\alpha} \\
&\leq Cr^{\frac{n}{p}-n-m+\alpha}.
\end{aligned} \tag{65}$$

Estimate (65), together with Lemma 3.2, leads to

$$|Dw(x)| \leq C|x|^{1-n-m+\alpha}, \tag{66}$$

$$\left(\int_{r<|x|<2r} \left|D^2 w\right|^p\right)^{\frac{1}{p}} \leq Cr^{\frac{n}{p}-n-m+\alpha}. \tag{67}$$

In the general case in which the extra assumption $\sigma(0) = I$ is not satisfied, we consider the linear change of variable $\xi = Jx$, with $J = \sqrt{(\sigma_{ij}(0))^{-1}}$, so that in the new coordinate system the above mentioned extra assumption is satisfied. In this case (51), (52) must be replaced by (36), (37) respectively, which concludes the proof. $\square$

We shall also need the following lemma.

Lemma 3.8. *Let the hypotheses of Theorem 3.1 be satisfied. For every* $m = 1, 2, \dots$ *there exists a spherical harmonic* $S_m$ *of degree* $m$ *such that the solution* $u$ *given by Theorem 3.1 also satisfies*

$$|Du(x)| > |x|^{1-(n+m)}, \quad \text{for every}\, x, 0 < |x| < r_0, \tag{68}$$

*where* $r_0$ *depends only on* $\lambda$, $E$, $p$, $m$ *and* $R$.

*Proof.* The proof of this lemma can be obtained along the same lines as of [2, Lemma 3.1] and [8, Section 3]. □

## 4. Proof of the main result.

Since the boundary $\partial\Omega$ is Lipschitz, the normal unit vector field might not be defined on $\partial\Omega$. We shall therefore introduce a unitary vector field $\widetilde{\nu}$ locally defined near $\partial\Omega$ such that: (i) $\widetilde{\nu}$ is $C^\infty$ smooth, (ii) $\widetilde{\nu}$ is non-tangential to $\partial\Omega$. At this point we would need to quantify $\partial\Omega$ in terms of its compactness and the constants $L$, $r$ introduced in definition 2.1. We think that this goes beyond the scope of this paper, therefore we choose to refer to [8, Lemmas 3.1-3.3] for a precise introduction of $\widetilde{\nu}$. Here we will simply recall that the point $z_\tau = x^0 + \tau\widetilde{\nu}$, where $x^0 \in \partial\Omega$, satisfies

$$C\,\tau \le d(z_\tau,\, \partial\Omega) \le \tau, \quad \text{for any} \quad \tau, \quad 0 \le \tau \le \tau_0, \tag{69}$$

where $\tau_0$ and $C$ depend on $L$ and $r$ only.

Lemma 4.1. *If* $\mu_s$, $B$ *satisfy conditions* (11), (12) *and* (13) *respectively, then* $K(x,t)$ *given by* (10) *belongs to the class* $\mathcal{H}'_\infty$ *with* $\mathcal{E}$ *being a positive constant depending only on* $n$, $\lambda$ *and* $E$.

*Proof of Lemma 4.1.* Notice that if $\mu_s$ and $B$ satisfy (11), (12) and (13) respectively, then

$$K(x,t) \in L^\infty(\Omega). \tag{70}$$

We also have

$$\begin{aligned} D_t K(x,t) &= -nK^2(x,t) & (71)\\ D_x K(x,t) &= nK(x,t)\big[(D_x B)\mu_s - (I-B)D_x\mu_s\big]K(x,t) & (72)\\ D_t D_x K(x,t) &= -2n^2K^2(x,t)\big[(D_x B)\mu_s - (I-B)D_x\mu_s\big]K(x,t). & (73)\end{aligned}$$

By combining (70) together with (71)-(73) and recalling that $I - B$ is positive definite, we obtain that $K \in \mathcal{H}'_\infty$. □

Note that if $K$ is given by (10), $\mu_s$, $B$ satisfy conditions (11), (12) and (13) respectively and $\mu_a$ satisfies (24), (25), then

$$K(\cdot, \mu_a(\cdot)) \in W^{1,p}(\Omega, \mathrm{Sym}_n), \tag{74}$$

where $p$ is the number introduced in (25). Furthermore

$$||K(\cdot, \mu_a(\cdot))||_{W^{1,p}(\Omega)} \leq C\mathcal{E}(1 + ||\mu_a||_{W^{1,p}(\Omega)}), \tag{75}$$

where $C$ is a positive constant depending only on $\lambda$, $\Omega$, $n$ and $p$ (see for instance [8, Lemma 3.6]).
In the following two proofs of the main result the appearance of positive constants that depend on the various quantities $n$, $p$, $\alpha$, $\beta$, $k$ $L$, $r$, $E$, $\mathcal{E}$, $\mathcal{F}$ and $\Omega$ will be common. These quantities represent our *a-priori* information, therefore, we will denote by $C$ any of these positive constants arising in the proofs in order to keep the notation simple.

*Proof of Theorem 2.5.* Let $x^0 \in \partial\Omega$ be such that

$$(\mu_{a_2} - \mu_{a_1})(x^0) \ = \| \ \mu_{a_1} - \mu_{a_2} \ \|_{L^\infty(\partial\Omega)}$$

and $z_\tau = x^0 + \tau\widetilde{\nu}$, with $0 < \tau \leq \min\left\{\tau_0, \ \frac{r_0}{4}\right\}$, where $\tau_0$ is the number fixed in (69) and $r_0$ is the number appearing in (68). We set $\sigma_i = K(\cdot, \mu_{a_i})$, $q_i = \mu_{a_i}$, for $i = 1, 2$ and $m = 0$ in Theorem 3.1. The corresponding singular solution $u_i \in W^{2,p}(\Omega)$ of

$$\mathrm{div}\,(K(\cdot, \mu_{a_i})\nabla u_i) - \mu_{a_i} u_i = 0 \quad \text{in}\, \Omega$$

have a Green's function type of singularity at $z_\tau$ outside $\Omega$

$$u_i(x) = \left|J_{\mu_{a_i}}(x - z_\tau)\right|^{2-n} + O\left(\left|x - z_\tau\right|^{2-n+\alpha}\right), \tag{76}$$

for $i = 1, 2$. By setting $\rho = r_0$ we have that $B_\rho(z_\tau) \cap \Omega \neq \emptyset$ and, recalling (23), we have

$$\begin{aligned}
&\left|\int_{\Omega\cap B_\rho(z_\tau)} \left(K(x,\ \mu_{a_1}) - K(x,\ \mu_{a_2})\right) \nabla u_1 \cdot \nabla u_2\right| \\
&\quad \leq \int_{\Omega\cap B_\rho(z_\tau)} |\mu_{a_1} - \mu_{a_2}|\,|u_1||u_2| \\
&\qquad + \int_{\Omega\setminus B_\rho(z_\tau)} |K(x,\ \mu_{a_1}) - K(x,\ \mu_{a_2})|\ |\nabla u_1|\,|\nabla u_2| \\
&\qquad + \int_{\Omega\setminus B_\rho(z_\tau)} |\mu_{a_1} - \mu_{a_2}|\,|u_1||u_2| \\
&\qquad + ||\Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}}||_*\ ||u_1||_{H^{\frac{1}{2}}(\partial\Omega)}\ ||u_2||_{H^{\frac{1}{2}}(\partial\Omega)}.
\end{aligned} \tag{77}$$

By combining (76) with (77) and the fact that $K(x,\ \mu_{a_i})$ is Hölder continuous with exponent $\beta = 1 - \frac{n}{p}$, we obtain

$$
\begin{aligned}
&\int_{\Omega\cap B_\rho(z_\tau)} \frac{J^2_{\mu_{a_2}}\left(K(x^0,\ \mu_{a_1}) - K(x^0,\ \mu_{a_2}\right) J^2_{\mu_{a_1}}\ (x-z_\tau)\cdot(x-z_\tau)}{|J_{\mu_{a_1}}(x-z_\tau)|^n\ |J_{\mu_{a_2}}(x-z_\tau)|^n} \\
&\quad \le\ C\Big\{ \int_{\Omega\cap B_\rho(z_\tau)} |x-\ z_\tau|^{2-2n+\alpha} \\
&\qquad + \int_{\Omega\cap B_\rho(z_\tau)} |x-\ z_\tau|^{2-2n}\ |x-\ x^0|^{\beta} \\
&\qquad + \int_{\Omega\cap B_\rho(z_\tau)} |x-\ z_\tau|^{4-2n} \\
&\qquad + \int_{\Omega\setminus B_\rho(z_\tau)} |K(x,\ \mu_{a_2}) - K(x,\ \mu_{a_1})|\ |x-z_\tau|^{2-2n} \\
&\qquad + \int_{\Omega\setminus B_\rho(z_\tau)} |\mu_{a_1}-\mu_{a_2}|\ |x-\ z_\tau|^{4-2n} \Big\} \\
&\quad + \|\ \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}}\ \|_*\ \|\ u_1\ \|_{H^{\frac{1}{2}}(\partial\Omega)}\ \|\ u_2\ \|_{H^{\frac{1}{2}}(\partial\Omega)}\ .
\end{aligned}
$$

Since $\left|J_{\mu_{a_i}} - K(x^0,\mu_{a_i})^{-1}\right| \le C\tau^{\beta}$, for $i=1,2$, we have

$$
\begin{aligned}
&J^2_{\mu_{a_2}}\left(K(x^0,\ \mu_{a_1}) - K(x^0,\ \mu_{a_2})\right) J^2_{\mu_1}\ (x-z_\tau)\cdot(x-z_\tau) \\
&\quad \ge \left(K(x^0,\mu_{a_2})^{-1} - K(x^0,\mu_{a_1})^{-1})(x-z_\tau\right)\cdot(x-z_\tau) \\
&\qquad -\ C\tau^{\beta}(\mu_{a_1}-\mu_{a_2})(x^0)\ |x-z_\tau|^2 \qquad (78)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big(K(x^0,\mu_{a_2})^{-1} - K(x^0,\mu_{a_1})^{-1})(x-z_\tau\big)\cdot(x-z_\tau)\\
&= \int_{\mu_{a_1}(x^0)}^{\mu_{a_2}(x^0)} D_t\left(K(x^0,t)\right)^{-1}(x-z_\tau)\cdot(x-z_\tau)\,dt\\
&= \int_{\mu_{a_1}(x^0)}^{\mu_{a_2}(x^0)} -K^{-1}(x^0,t)\,D_t\,K(x^0,t)\,K^{-1}(x^0,t)(x-z_\tau)\cdot(x-z_\tau)\,dt\\
&= \int_{\mu_{a_2}(x^0)}^{\mu_{a_1}(x^0)} -D_t\,K(x^0,t)\,K^{-1}(x^0,t)(x-z_\tau)\cdot K^{-1}(x^0,t)\,(x-z_\tau)\,dt\\
&\geq \mathcal{F}\int_{\mu_{a_1}(x^0)}^{\mu_{a_2}(x^0)} \left|\,K^{-1}(x^0,t)(x-z_\tau)\right|^2\,dt\\
&\geq \mathcal{F}\lambda^{-2}\left(\mu_{a_2}(x^0)-\mu_{a_1}(x^0)\right)\,|x-z_\tau|^2\,. \qquad (79)
\end{aligned}
$$

By combining (78) together with (79) we obtain

$$
\begin{aligned}
&J^2_{\mu_{a_2}}\left(K(x^0,\,\mu_{a_1})-K(x^0,\,\mu_{a_2})\right)J^2_{\mu_{a_1}}\,(x-z_\tau)\cdot(x-z_\tau)\\
&\quad\geq\left(\mathcal{F}\lambda^{-2}+C\tau^\beta\right)\left(\mu_{a_2}(x^0)-\mu_{a_1}(x^0)\right)\,|x-z_\tau|^2\\
&\quad\geq C\left(\mu_{a_2}(x^0)-\mu_{a_1}(x^0)\right)\,|x-z_\tau|^2. \qquad (80)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\|\,\mu_{a_1}-\mu_{a_2}\,\|_{L^\infty(\partial\Omega)}\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{2-2n}\\
&\leq\;C\Big\{\int_{\Omega\cap B_\rho(z_\tau)}|x-\;z_\tau|^{2-2n+\alpha}\\
&\qquad+\int_{\Omega\cap B_\rho(z_\tau)}|x-\;z_\tau|^{2-2n}\,|x-\;x^0|^\beta\\
&\qquad+\int_{\Omega\cap B_\rho(z_\tau)}|\mu_{a_2}-\mu_{a_1}|\;|x-\;z_\tau|^{4-2n}\\
&\qquad+\int_{\Omega\setminus B_\rho(z_\tau)}|K(x,\,\mu_{a_2})-K(x,\,\mu_{a_1})|\,|x-z_\tau|^{2-2n}\\
&\qquad+\int_{\Omega\setminus B_\rho(z_\tau)}|\mu_{a_2}-\mu_{a_1}|\,|x-\;z_\tau|^{4-2n}\Big\}\\
&\quad+\;\|\,\Lambda_{\mu_{a_1}}-\Lambda_{\mu_{a_2}}\,\|_*\,\|\,u_1\,\|_{H^{\frac{1}{2}}(\partial\Omega)}\,\|\,u_2\,\|_{H^{\frac{1}{2}}(\partial\Omega)}
\end{aligned}
$$

and by estimating the above integrals and the $H^{\frac{1}{2}}(\partial\Omega)$ norm of $u_i$ for $i=1,2$ (see [2, 8]) we obtain

$$\begin{aligned}\| \mu_{a_1} - \mu_{a_2} \|_{L^\infty(\partial\Omega)}\, \tau^{2-n} \quad &\leq C\Big\{ \tau^{2-n+\alpha} + \tau^{2-n+\beta} + \tau^{4-n} + C \\ &\qquad + \| \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2})} \|_* \, \tau^{2-n} \Big\}, \end{aligned} \tag{81}$$

which leads to

$$\| \mu_{a_1} - \mu_{a_2} \|_{L^\infty(\partial\Omega)} \leq C \left\{ \omega(\tau) + \| \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}} \|_* \right\}, \tag{82}$$

where $\omega(\tau) \to 0$ as $\tau \to 0$, which concludes the proof. $\square$

*Proof of Theorem 2.6.* Let $\widetilde{\nu}$ be the unit vector field introduced in this section. We shall prove that

$$\left\| \frac{\partial^j}{\partial \widetilde{\nu}^j} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial\Omega \cap \overline{W})} \leq C \, \| \Lambda_1 - \Lambda_{2)} \|_*^{\delta_j}, \qquad \text{for every } j \leq k, \tag{83}$$

where $\delta_j$ is given by (30). We proceed by induction on $k$ by following the same line of [8, Proof of Theorem 2.2] and therefore only the points where the two proofs differ will be highlighted. From theorem 2.5 we have that (83) holds true for $k=0$. Let us assume that (83) holds true for $j=k-1$ and prove that it is true for $j=k$ too.
Let $m$ be a positive integer and $x^0 \in \partial\Omega \cap \overline{W}$ be such that

$$(-1)^k \, \frac{\partial^k}{\partial \widetilde{\nu}^k} (\mu_{a_2} - \mu_{a_1})(x^0) = \left\| \frac{\partial^k}{\partial \widetilde{\nu}^k} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial\Omega \cap \overline{W})}. \tag{84}$$

Let $z_\tau = x^0 + \tau \widetilde{\nu}$, with $\tau \leq \min\left\{\tau_0, \frac{\rho}{2}\right\}$, where $\tau_0$ is the number fixed in (69) and $\rho = \min\left\{r_0, \frac{h}{4L}\right\}$, where $r_0$ is the number depending on the choice of $m$ which was introduced in (68). With these choices $B_\rho(z_\tau) \cap \bar{\Omega}$ is nonempty and

$$B_\rho(z_\tau) \cap \overline{\Omega} \subset U. \tag{85}$$

For the choice of $\rho$ and (85) we recall [8, Lemmas 3.1-33] as explained at the beginning of this section. Let $u_i$ be the singular solution of Theorem 3.1 corresponding to $\mu_{a_i}$, for $i=1,2$ and $m$. By Lagrange theorem, for every $x \in \overline{U}$ there exists $t(x)$, $0 < t(x) < 1$, such that

$$K(x, \mu_{a_1}) - K(x, \mu_{a_2}) = (\mu_{a_1}(x) - \mu_{a_2}(x)) \, D_t K(x,t)|_{t=c(x)}, \tag{86}$$

where $c(x) = a(x) + t(x)(\mu_{a_2}(x) - \mu_{a_1}(x))$ and

$$|Du_1 - Du_2| \leq C\big(|x - z_\tau|^{1-n-m} |\mu_{a_1}(x^0) - \mu_{a_2}(x^0)| + |x - z_\tau|^{1-n-m+\alpha}\big), \tag{87}$$

which leads to

$$D_t K(x,t)|_{t=c(x)} Du_1 \cdot Du_2 \leq -C|x-z_\tau|^{2-2(n+m)}, \tag{88}$$

for almost every $x \in B_\rho(z_\tau) \cap \Omega$. Noting that every $x \in U$ can be uniquely represented as $x = y - s\widetilde{\nu}$, where $y \in \partial\Omega$, $0 \leq s \leq \tau_0$, with $0 < \tau_0 < h - Lr$, Taylor's formula for $\mu_{a_2} - \mu_{a_1}$ leads to

$$\begin{aligned} k!\,(\mu_{a_2} - \mu_{a_1})(x) \quad \geq \quad & \left|\left|\frac{\partial^k}{\partial\widetilde{\nu}^k}(\mu_{a_1} - \mu_{a_2})\right|\right|_{L^\infty(\partial\Omega\cap\bar{W})} \\ & -C\Bigg\{\sum_{j=0}^{k-1}\left|\left|\frac{\partial^j}{\partial\widetilde{\nu}^j}(\mu_{a_1} - \mu_{a_2})\right|\right| \; s^j \\ & \qquad - s^k\,|x - x^0|^\alpha\Bigg\} \end{aligned} \tag{89}$$

and by combining Alessandrini's identity (23) together with (88) and (89) we obtain

$$\begin{aligned} & ||\Lambda_{\mu_1} - \Lambda_{\mu_2}||_* ||u_1||_{H^{\frac{1}{2}}(\partial\Omega)} ||u_2||_{H^{\frac{1}{2}}(\partial\Omega)} \\ & \geq \left|\left|\frac{\partial^k}{\partial\widetilde{\nu}^k}(\mu_{a_1} - \mu_{a_2})\right|\right|_{L^\infty(\partial\Omega\cap\bar{W})} \int_{\Omega\cap B_\rho(z_\tau)} (d(x,\partial\Omega))^k |x-z_\tau|^{2-2(n+m)} \\ & \quad - \sum_{j=0}^{k-1}\left|\left|\frac{\partial^j}{\partial\widetilde{\nu}^j}(\mu_{a_1} - \mu_{a_2})\right|\right|_{L^\infty(\partial\Omega\cap\bar{W})} \int_{\Omega\cap B_\rho(z_\tau)} (d(x,\partial\Omega))^j |x-z_\tau|^{2-2(n+m)} \\ & \quad - \int_{\Omega\cap B_\rho(z_\tau)} (d(x,\partial\Omega))^k \, |x-x^0|^\alpha |x-z_\tau|^{2-2(n+m)} \\ & \quad - \int_{\Omega\setminus B_\rho(z_\tau)} |K(x,\,\mu_{a_1}) - K(x,\,\mu_{a_2})|\,|x-z_\tau|^{2-2(n+m)} \\ & \quad - \int_{\Omega\cap B_\rho(z_\tau)} |(\mu_{a_1} - \mu_{a_2})(x)||x-z_\tau|^{4-2(n+m)} \\ & \quad - \int_{\Omega\setminus B_\rho(z_\tau)} |(\mu_{a_1} - \mu_{a_2})(x)||x-z_\tau|^{4-2(n+m)}. \end{aligned} \tag{90}$$

Estimating the above integrals and the norms $||u_i||_{H^{\frac{1}{2}}(\partial\Omega)}$, for $i = 1, 2$ as in [8, Proof of Theorem 2.2] leads to

$$\left|\left|\frac{\partial^k}{\partial\widetilde{\nu}^k}(\mu_{a_1}-\mu_{a_2})\right|\right|_{L^\infty(\partial\Omega\cap\bar{W})}\tau^{2-n-2m+k}\leq C\Bigg\{\sum_{j=0}^{k-1}||\Lambda_{\mu_{a_1}}-\Lambda_{\mu_{a_2}}||_*^{\delta_j}\tau^{2-n-2m+j}$$
$$+\tau^{2-n-2m+\alpha+k}+C+\tau^{4-n-2m}$$
$$+||\Lambda_{\mu_{a_1}}-\Lambda_{\mu_{a_2}}||_*^{\delta_j}\tau^{2-n-2m}\Bigg\}, \quad (91)$$

therefore to

$$\left|\left|\frac{\partial^k}{\partial\widetilde{\nu}^k}(\mu_{a_1}-\mu_{a_2})\right|\right|_{L^\infty(\partial\Omega\cap\overline{W})}\leq C\Big\{||\Lambda_{\mu_{a_1}}-\Lambda_{\mu_{a_2}}||_*^{\delta_{k-1}}\tau^{-k}+\tau^{\alpha}\Big\}. \quad (92)$$

(83) is then derived for $j=k$ by optimizing the choice of $\tau$ in (92). We recall for sake of completeness that (29) is obtained by combining (83) together with an iterated use of the following interpolation inequality

$$||Df||_{L^\infty(\partial\Omega\cap\overline{U})}\leq C\left\{\left|\left|\frac{\partial}{\partial\widetilde{\nu}}f\right|\right|_{L^\infty(\partial\Omega}+||f||_{L^\infty(\partial\Omega\cap\overline{U})}^{\frac{\alpha}{1-\alpha}}\,||f||_{C^{1+\alpha}(\overline{U})}^{\frac{\alpha}{1+\alpha}}\right\}, \quad (93)$$

for every $f\in C^{1,\alpha}(\overline{\Omega})$. Such an interpolation inequality can be found for example in [2, Lemma 3.2]. □

## References


[1] S. Agmon, S. Douglis, and Nirenberg L., *Estimates near the boundary for solutions of elliptic partial differential equation satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), no. 4, 623–727.

[2] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. **27** (1988), 153–172.

[3] G. Alessandrini, *Singular solutions of elliptic equations and the determination of conductivity by boundary measurements*, J. Differential Equations **84** (1990), no. 2, 252–272.

[4] G. Alessandrini, *Determining conductivity by boundary measurements, the stability issue*, Appl. Industrial Math. **27** (1991), 317–324.

[5] G. Alessandrini, *Open issues of stability for the inverse conductivity problem*, J. Inverse Ill-Posed Probl. **15** (2007), 1–10.

[6] G. Alessandrini, de Hoop M. V., and R. Gaburro, *Uniqueness for the electrostatic inverse boundary value problem with piecewise constant anisotropic conductivities*, arXiv preprint, arXiv:1604.02948 (2016).

[7] G. Alessandrini, de Hoop M. V., R. Gaburro, and E. Sincich, *Lipschitz stability for the electrostatic inverse boundary value problem with piecewise linear conductivities*, arXiv preprint arXiv:1509.06277 (2015).

[8] G. Alessandrini and R. Gaburro, *Determining conductivity with special anisotropy by boundary measurements*, SIAM J. Math. Anal. **33** (2001), 153–171.

[9] G. Alessandrini and R. Gaburro, *The local Calderón problem and the determination at the boundary of the conductivity*, Comm. Partial Differential Equations **34** (2009), 918–936.

[10] G. Alessandrini and S. Vessella, *Lipschitz stability for the inverse conductivity problem*, Adv. in Appl. Math. **35** (2005), 207–241.

[11] S. R. Arridge, *Optical tomography in medical imaging*, Inverse Problems **15** (1999), no. 2, R41.

[12] S. R. Arridge and J. C. Hebden, *Optical imaging in medicine II: modelling and reconstruction*, Physics in Medicine and Biology **42** (1997), no. 5, 841.

[13] S. R. Arridge and W.R.B Lionheart, *Nonuniqueness in diffusion-based optical tomography*, Optics Letters **23** (1998), 882–884.

[14] S. R Arridge and J. C. Schotland, *Optical tomography: forward and inverse problems*, Inverse Problems **25** (2009), no. 12, 123010.

[15] J. A. Barceló, T. Barceló, and A. Ruiz, *Stability of the inverse conductivity problem in the plane for less regular conductivities*, J. Differential Equations **173** (2001), 231–270.

[16] T. Barceló, D. Faraco, and A. Ruiz, *Stability of Calderón inverse conductivity problem in the plane*, J. Math. Pures Appl. **88** (2007), no. 6, 522–556.

[17] E. Beretta and E. Francini, *Lipschitz stability for the electrical impedance tomography problem: the complex case*, Comm. Partial Differential Equations **36** (2011), 1723–1749.

[18] R. Brown, *Recovering the conductivity at the boundary from the local Dirichlet-to-Neumann map: a pointwise result*, J. Inverse Ill-posed Prob. **9** (2001), no. 6, 567–574.

[19] A. P. Calderón, *On an inverse boundary value problem, seminar on numerical analysis and its applications to continuum physics (Rio de Janeiro, 1980)*, Soc. Brasil. Mat., Rio de Janeiro 1980, 1980, pp. 65–73.

[20] P. Caro and M. Salo, *Stability of the Calderón problem in admissible geometries*, arXiv preprint arXiv:1404.6652 (2014).

[21] M. Di Cristo, *Stable determination of an inhomogeneous inclusion by local boundary measurements*, J. Comput. Appl. Math. **198** (2007), 414–425.

[22] D. Dos Santos Ferreira, C. E. Kenig, M. Salo, and G. Uhlmann, *Limiting Carleman weights and anisotropic inverse problems*, Invent. Math. **178** (2009), no. 1, 119–171.

[23] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas, and M. Salo, *The Calderón problem in transversally anisotropic geometries*, arXiv preprint arXiv:1305.1273 (2013).

[24] R. Gaburro and W. R. B. Lionheart, *Recovering Riemannian metrics in monotone families from boundary data*, Inverse Problems **25** (2009), no. 4, 045004.

[25] R. Gaburro and E. Sincich, *Lipschitz stability for the inverse conductivity problem for a conformal class of anisotropic conductivities*, Inverse Problems **31** (2015), no. 1, 015008.

[26] B. Harrach, *On uniqueness in diffuse optical tomography*, Inverse Problems **25** (2009), no. 5, 055010.
[27] H. Heck and J. N. Wang, *Stability estimates for the inverse boundary value problem by partial Cauchy data*, Inverse Problems **22** (2006), no. 5, 1787.
[28] J. Heino, S. Arridge, J. Sikora, and E. Somersalo, *Anisotropic effects in highly scattering media*, Phys. Rev. E **68** (2003), no. 3, 031908.
[29] J. Heino and E. Somersalo, *Estimation of optical absorption in anisotropic background*, Inverse Problems **18** (2002), no. 3, 559.
[30] N. Hyvönen, *Characterizing inclusions in optical tomography*, Inverse Problems **20** (2004), no. 3, 737.
[31] V. Isakov, *On uniqueness of recovery of a discontinuous conductivity coefficient*, Comm. Pure Appl. Math. **41** (1988), 865–878.
[32] V. Isakov, *Inverse problems for partial differential equations*, no. 127, Springer Sciences & Business Media, New York, 2006.
[33] V. Isakov, *On the uniqueness in the inverse conductivity problem with local data*, Inverse Probl. Imaging **1** (2007), no. 1, 95.
[34] H. Kang and K. Yun, *Boundary determination of conductivities and Riemannian metrics via local Dirichlet-to-Neumman operator*, SIAM J. Math. Anal. **34** (2002), no. 3, 719–735.
[35] C. E. Kenig, M. Salo, and G. Uhlmann, *Reconstructions from boundary measurements on admissible manifolds*, Inverse Probl. Imaging **5** (2011), 859–877.
[36] C. E. Kenig, J. Sjöstrand, and G. Uhlmann, *The Calderón problem with partial data*, Ann. of Math. **165** (2007), no. 2, 567–591.
[37] R. Kohn and M. Vogelius, *Identification of an unknown conductivity by means of measurements at the boundary*, SIAM-AMS Proc. **14** (1984), 113–123.
[38] R. Kohn and M. Vogelius, *Determining conductivity by boundary measurements II. interior results*, Comm. Pure Appl. Math. **38** (1985), 643–667.
[39] M. Lassas, M. Taylor, and G. Uhlmann, *The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary*, Comm. Anal. Geom. **11** (2003), no. 2, 207–221.
[40] M. Lassas and G. Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map*, Ann. Sci. Éc. Norm. Supér. **34** (2001), no. 5, 771–787.
[41] J. M. Lee and G. Uhlmann, *Determining anisotropic real-analytic conductivities by boundary measurements*, Comm. Pure Appl. Math **42** (1989), 1097–1112.
[42] W. R. B. Lionheart, *Conformal uniqueness results in anisotropic electrical impedance imaging*, Inverse Problems **13** (1997), no. 1, 125.
[43] L. Liu, *Stability estimates for the two-dimensional inverse conductivity problem*, Ph.D. thesis, University of Rochester, New York, 1987.
[44] N. Mandache, *Exponential instability in an inverse problem for the Schrödinger equation*, Inverse Problems **17** (2001), no. 5, 1435.
[45] C. B. Morrey, *Multiple integrals in the calculus of variations*, Springer, Berlin, 1966.
[46] A. Nachman, *Global uniqueness for a two dimensional inverse boundary value problem*, Ann. Math. **142** (1995), 71–96.

[47] G. Nakamura and K. Tanuma, *Direct determination of the derivatives of conductivity at the boundary from the localized Dirichlet to Neumann map*, Comm. Korean Math. Soc. **16** (2001), 415–425.

[48] G. Nakamura and K. Tanuma, *Local determination of conductivity at the boundary from the Dirichlet-to-Neumann map*, Inverse Problems **17** (2001), 405–419.

[49] G. Nakamura and K. Tanuma, *Formulas for reconstructing conductivity and its normal derivative at the boundary from the localized Dirichlet to Neumann map*, Recent Development in Theories and Numerics 2003, Birkhäuser Boston, 2003, pp. 192–201.

[50] L. Rondi, *A remark on a paper by Alessandrini and Vessella*, Adv. in Appl. Math. **36** (2006), no. 1, 67–69.

[51] M. Salo, *Inverse problems for nonsmooth first order perturbations of the Laplacian*, Ph.D. thesis, University of Helsinki , Helsinki, 2004.

[52] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Ist. Fourier (Grenoble) **15** (1965), 189–258.

[53] J. Sylvester, *An anisotropic inverse boundary value problem*, Comm. Pure Appl. Math. **43** (1990), 201–232.

[54] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary valued problem*, Ann. of Math. **125** (1987), 153–169.

[55] G. Uhlmann, *Electrical impedance tomography and Calderón's problem*, Inverse Problems **25** (2009), no. 12, 123011.

Author's address:

Romina Gaburro
Department of Mathematics and Statistics
University of Limerick
Ireland
E-mail: romina.gaburro@ul.ie

Received March 31, 2016
Revised July 1, 2016
Accepted July 14, 2016