Rend. Istit. Mat. Univ. Trieste
Volume 48 (2016), 303–313
DOI: 10.13137/2464-8728/13161

Nondestructive evaluation of inaccessible surface damages by means of active thermography

Gabriele Inglese and Roberto Olmi

Dedicated to Giovanni Alessandrini for his 60th birthday

Abstract. *We derive and test a formal explicit approximated rule for the reconstruction of a damaged inaccessible portion of the boundary of a thin conductor from thermal data collected on the opposite accessible face.*

Keywords: Inverse problems, heat equation, nondestructive evaluation, thin plate approximation.
MS Classification 2010: 65M32, 80A20.

1. Introduction

Let $\Omega = \{(x, y, z) \in [-L, L]^2 \times [0, a] \quad a < 1 \ll L\}$ represent a uniform thin plate of given thermal conductivity κ. We are modeling the following experimental framework:

The half plane $z > a$ is a forbidden aggressive environment, while $z < 0$ is an accessible laboratory. We are able to heat the specimen Ω from below by means of a controlled flux of density Φ_0 generated by lamps or a laser device and we are able to get temperature maps at $z = 0$ by means of an infrared camera (TMC in Figure 1).

Small corrosion damages due to chemical or mechanical aggression may appear on the upper inaccessible boundary of Ω. Since they are not accessible to direct inspection, they must be identified through operations carried out on the laboratory side. If the defect consists of a loss of matter (LOM), the damaged domain is modeled by

$$\Omega_{\epsilon\theta} = \{(x, y, z) : (x, y) \in [-L, L]^2, \quad 0 \le z \le a - \epsilon\theta(x, y)\}.$$

We assume that the geometry of the damage is described by a continuous function $(x, y) \to \epsilon\theta(x, y)$. Here, $\epsilon \ll a$ is a constant dimensional scale factor while $\theta(x, y) \in [0, 1]$ is dimensionless.

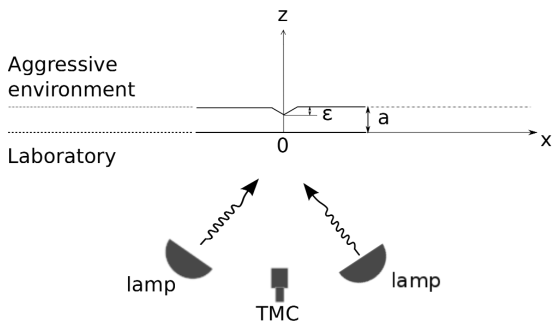


Figure 1: Sketch of the experimental setup

The temperature of the damaged domain solves the following Initial Boundary Value Problem in $D_T = \Omega_{\epsilon\theta} \times (0, T]$:

$$u_t = \alpha \Delta u \,, \tag{1}$$

$$\kappa u_n(x, y, a - \epsilon\theta(x, y), t) + ah(u(x, y, a - \epsilon\theta(x, y), t) - U_0) = 0 \,, \tag{2}$$

$$-\kappa u_z(x, y, 0, t) = \Phi \,, \tag{3}$$

$$u_x(-L, y, z, t) = u_x(L, y, z, t) = 0 \,, \qquad y \in [-L, L], z \in [0, a], t \in (0, T] \,, \tag{4}$$

$$u_y(x, -L, z, t) = u_y(x, L, z, t) = 0 \,, \qquad x \in [-L, L], z \in [0, a], t \in (0, T] \tag{5}$$

and

$$u(x, y, z, 0) = U_0 \,, \tag{6}$$

for all $(x, y, z) \in \Omega_{\epsilon\theta}$ (U_0 is a positive constant). Here, α is the thermal diffusivity, ah is the heat transfer coefficient between the specimen and the upper half-space (see for example [5] and [14]). The positive constant U_0 is both the initial temperature of the specimen and the temperature of the outern environment. The heat flux density Φ is taken constant in space and time for simplicity. In what follows we will refer to (4) and (5) as to "adiabatic conditions on the vertical sides".

Direct model. If $\epsilon\theta$ is given and it is sufficiently smooth, the IBVP (1)-(6) is well posed and it has a unique classical solution u^ϵ [12]. This notation stresses the dependence of the solution on the damage. Hence, the solution u^0 (corresponding to $\epsilon = 0$) is called the *background temperature* of the undamaged specimen.

Inverse Problem. If $\epsilon\theta$ is not known, our goal is to identify it from the knowledge of the thermal contrast $G(x, y, t) = u^\epsilon(x, y, 0, t) - u^0(x, y, 0, t)$ measured from the laboratory side $z = 0$.

This method is called Active Thermography. Thermography is "Active" when an external heat source (in our case, the heat flux Φ) stimulates the specimen for inspection.

Bibliographic remark. See [10] for a complete reference book about thermography. Amongst hundreds of research articles about thermal imaging, we mention [3] because, in our knowledge, it is one of the oldest and [4] because of the close relationship with the present paper. Since the mathematics of stationary thermography is the same used in a class of electrostatic models in nondestructive evaluation, we cite also [1, 7, 8] and references therein.

The idea of loss of matter used in (1)-(6) is very intuitive because LOM is something real and, possibly, measurable in practice.

However, thermal effects of damages on the inaccessible surface can be modeled also by means of perturbed boundary conditions. In this case, the boundary is left unaltered so that the domain (and consequently the mesh in numerical solution with finite elements!) is not dependent on the unknown $\epsilon\theta$.

Here, we assume that $\frac{\epsilon}{a}$ is small enough to use the idea of Domain Derivative ([4, 13]). The domain derivative of u^ϵ can be obtained by formal differentiation as $u' = \frac{du^\epsilon}{d\epsilon}(\epsilon = 0)$ or derived by means of straightforward calculations as done in [4]. The LOM model (1)-(6) in $\Omega_{\epsilon\theta}$ is turned into an Initial Boundary Value Problem in the undamaged domain Ω for the scaled domain derivative $w = \epsilon u'$. It is remarkable that the unknown damage $\epsilon\theta$ appears now in the top boundary condition.

Furthermore, in subsection 2.1, we rescale z and transform it in the new variable $\zeta = \frac{z}{a}$. Since the temperature of the specimen reaches a stationary regime for $t \to \infty$, after a time interval T_α (inversely proportional to the diffusivity α) we focus our attention on the following stationary BVP on the parallelepiped $[-L, L]^2 \times [0, 1]$ (see section 3):

$$a^2(w_{xx} + w_{yy}) + w_{\zeta,\zeta} = 0\,, \tag{7}$$

$$\kappa w_\zeta(x, y, 1) + a^2 h w(x, y, 1) = -a^2 \epsilon\theta(x, y) h \frac{\Phi}{k}\,, \tag{8}$$

$$\kappa w_\zeta(x, y, 0) = 0\,, \tag{9}$$

with adiabatic conditions on the vertical sides.

We expand w and θ in powers of a^2 and plug them into the BVP above: In this way we obtain a perturbative hierarchy of relations amongst their coefficients. This procedure is called Thin Plate Approximation and improves what was done in [9] where perturbations of the heat transfer coefficient h were identified.

In section 3, we derive the TPA formally in any order and implement the following approximated inversion formula for the identification of the damage:

$$\epsilon\theta(x,y) \approx \frac{\kappa^2}{h\Phi}(G_{xx}+G_{yy}) - \frac{\kappa}{\Phi}G(x,y). \tag{10}$$

We succesfully tested this formula using synthetic data. Any difficulties arising from numerical differentiation of an approximately given function like the thermal contrast G are handled by using local weighted regeression [6]. A seminal paper about regularized numerical differentiation is [2].

2. Domain derivative

Domain derivative, introduced in [13], is a techinque for studying PDEs on geometrically perturbed domains. In our case the domain derivative of u^ϵ is the Gateaux derivative of u^ϵ in the direction θ taken for $\epsilon = 0$. This derivative is a function u' that satisfies the heat equation in Ω with boundary conditions

$$\kappa u'_z(x,y,a,t) + ahu'(x,y,a,t) = \theta(x,y)\left(ahu^0_z(x,y,a,t) + \kappa\frac{u^0_t(x,y,a,t)}{\alpha}\right)$$

(derived in [4] in agreement with Theorem 3.2 in [13]),

$$\kappa u'_z(x,y,0,t) = 0$$

and "adiabatic conditions on the vertical sides".

Since we assume Φ constant (in t and (x,y)), the background solution is constant in space variables and, for increasing t, it approaches a stationary value that, after a suitable time interval T_α, is very close to the linear function $u^0_{stat}(z) = U_0 + \frac{\Phi}{h} + \frac{\Phi}{\kappa}(a-z)$ (stationary background temperature).

2.1. Final form of the BVP: domain derivative and scaling

Since we have to recover $\epsilon\theta$ from the thermal contrast $G(x,y,t) = u^\epsilon(x,y,0,t) - u^0(x,y,0,t) \approx \epsilon u'(x,0,t)$, it is convenient to introduce the scaled function $w = \epsilon u'$. Moreover, Thin Plate Approximation (see for example [9]) requires the expansion of w in powers of a^2. For this reason, we scale the variable

$z \to \zeta = \frac{z}{a}$ so that the domain becomes independent of a. Since $w_\zeta = \frac{w_z}{a}$ we have finally

$$\frac{a^2}{\alpha} w_t = a^2(w_{xx} + w_{yy}) + w_{\zeta\zeta} \,,$$

$$\kappa w_\zeta(x, y, 1, t) + a^2 h w(x, y, 1, t) = \epsilon\theta(x, y) \left(a^2 h u_z^0 + a\kappa \frac{u_t^0(x, y, a, t)}{\alpha} \right) ,$$

$$\kappa w_\zeta(x, y, 0, t) = 0 \,,$$

with adiabatic conditions on the vertical sides. Moreover, we have

$$w(x, y, 0, t) \approx G(x, y, t).$$

3. Stationary model, Thin Plate Approximation of the domain derivative

Here, we focus our attention to the stationary heat equation. In what follows, we remove the time variable but, as a rule,we keep the same function names. The stationary heat equation describes well the behavior of the temperature in our model for $t > T_\alpha$. Hence, we introduce a new function w that does not depend on t and solve the elliptic BVP in $[-L, L]^2 \times [0, 1]$

$$a^2(w_{xx} + w_{yy}) + w_{\zeta\zeta} = 0 \,, \tag{11}$$

$$\kappa w_\zeta(x, y, 1) + a^2 h w(x, y, 1) = \epsilon\theta(x, y) a^2 h u_z^0 \,,$$

$$\kappa w_y(x, y, 0) = 0 \,,$$

with adiabatic conditions on the vertical sides. Moreover, we have

$$w(x, y, 0) \approx G(x, y). \tag{12}$$

Remark. The remainder $R_2(h, \epsilon) = max_{x,y}|u^\epsilon(x, y, 0) - u^0 - w(x, y, 0)|$ measures the precision of (12). In Figure 2 we plot $R_2(h, \epsilon)$ for $\epsilon \in \{.003, .005, .007\}$ and $ah \in [20, 200]$ in the framework of the 2D example described in section 4. Observe that the domain derivative is very close to thermal contrast not only for small ϵ (as obviously expected), but also for large values of the heat transfer coefficient ah. We believe that the stabilizing role of increasing ah is related to the instability expected when h goes to zero (it is well known that for $h = 0$ the IBVP (1)-(6) has no stationary solution).

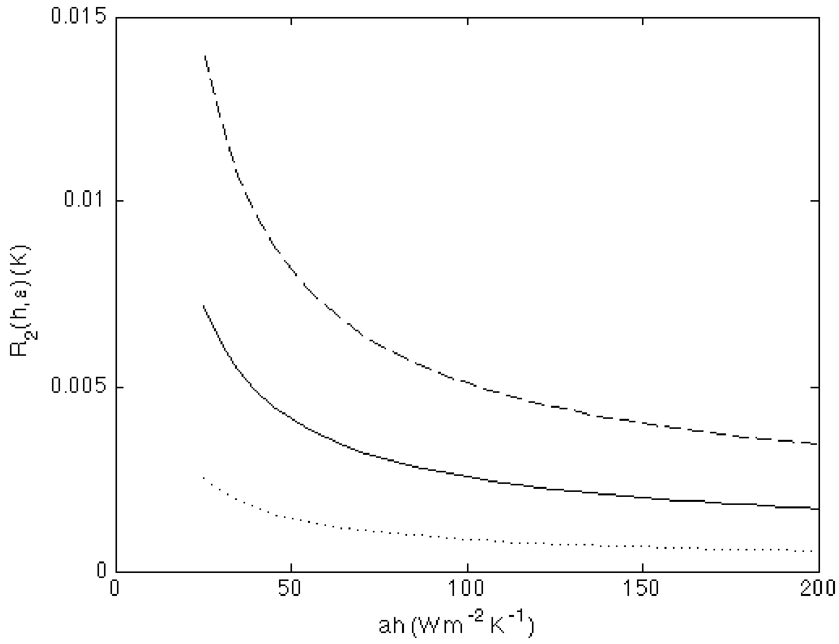


Figure 2: $R_2(h, \epsilon)$ measures how much the scaled domain derivative for $\zeta = 0$ is a good approximation of the thermal contrast.

3.1. Thin Plate Approximation

Plugging the formal expansions

$$w = w_0 + a^2 w_1 + O(a^4)\,, \tag{13}$$

$$\theta = \theta_0 + a^2 \theta_1 + O(a^4) \tag{14}$$

in the BVP, we obtain a hierarchy of relations amongst coefficients which allows us to derive an approximate formula for the unknown $\epsilon\theta$.

Zeroth order relations give $w_{0\zeta}(x, y, 1) = w_{0\zeta}(x, y, 0) = w_{0\zeta\zeta}(x, y, \zeta) = 0$ so that w_0 is actually independent on ζ. Hence, we set $w_0(x, y, \zeta) \equiv w_0(x, y) \approx u^\epsilon(x, y, 0) - u^0(x, y, 0)$ as suggested by Figure 2.

First order relations are

$$\begin{gathered} w_{0xx} + w_{0yy} + w_{1\zeta\zeta} = 0\,, \\ \kappa w_{1\zeta}(x, y, 1) + h w_0(x, y, 1) = -\epsilon\theta^0(x, y)\frac{\Phi}{\kappa}\,, \\ \kappa w_{1\zeta}(x, y, 0) = 0\,, \end{gathered}$$

so that (from the fundamental theorem of calculus)

$$-h w_0(x, y) - \epsilon\theta^0(x, y)\frac{\Phi}{\kappa} = -\kappa(w_{0xx}(x, y) + w_{0yy}(x, y))\,.$$

Hence, we have the following approximation of the boundary damage

$$\epsilon\theta_0(x, y) = \frac{\kappa^2}{h\Phi_0}(w_{0xx}(x, y) + w_{0yy}(x, y)) - \frac{\kappa}{\Phi}w_0(x, y).$$

3.2. The complete hierachic scheme in 2D

We can iterate the perturbative step just described. For simplicity we limit ourselves to the 2D model in the variables (x, ζ). We have

$$w_{0xx} + w_{1\zeta\zeta} = 0,$$

so that it is easy to see that

$$w_1(x, \zeta) = -w_{0xx}(x)\frac{\zeta^2}{2}.$$

Since for all $n \geq 1$ we have

$$w_{nxx} + w_{n+1\zeta\zeta} = 0,$$

we obtain $w_n(x, \zeta) = \frac{d^{2n} w_0(x)}{dx^{2n}}(-1)^n \frac{\zeta^{2n}}{(2n)!}$.

Hence, the coefficients of the expansion of θ are derived plugging expansions (13), (14) in the BVP (11). We have

$$\begin{aligned} \epsilon\theta_n(x) = (-1)^{(n+1)}\frac{\kappa^2}{h\Phi}\frac{d^{2n}w_0}{dx^{2n}}(x)\frac{1}{(2n-1)!} \\ + (-1)^n\frac{\kappa}{\Phi_0}\frac{d^{2(n-1)}w_0}{dx^{2(n-1)}}(x)\frac{1}{2(n-1)!}\,. \end{aligned} \tag{15}$$

Since $z = a\zeta$, the formal expansion in (13) becomes

$$w(x, z) = \sum_{n=0}^{\infty}\frac{d^{2n}w_0(x)}{dx^{2n}}(-1)^n\frac{z^{2n}}{(2n)!}.$$

For x fixed in $[-L, L]$, this is a power series in z that converges uniformly in $[-a, a]$ if, for a positive real number δ, $S = \sum_{n=0}^{\infty} \frac{d^{2n} w_0}{dx^{2n}} \frac{(a+\delta)^{2n}}{(2n)!} < \infty$. Although the Neumann condition $w_z(x, 0) = 0$ allows us to prove the analiticity of $w(x, 0)$, we do not know anything about the convergence of S. In agreement with [11] we must be content of convergence in a smaller interval. It is not a big drawback as long as we keep the formal character of our result.

4. Recovering surface damages using formal TPA. A numerical example.

In our numerical experiment, we fix the following geometrical and physical parameters. The values $L = -.5\ m$, $a = .05\ m$, $\epsilon = .005\ m$, $\theta = e^{-90x^2}$ define the domain $\Omega_{\epsilon\theta}$ in R^2. As for the conducting material we have $\kappa = 100\ \frac{W}{m\ K}$ and $\alpha = 10^{-4}\ \frac{m^2}{s}$ while the heat exchange coefficient is $ah = 100\ \frac{W}{m^2\ K}$. The controlled heat flux is $\Phi = 1000\ \frac{W}{m^2}$.

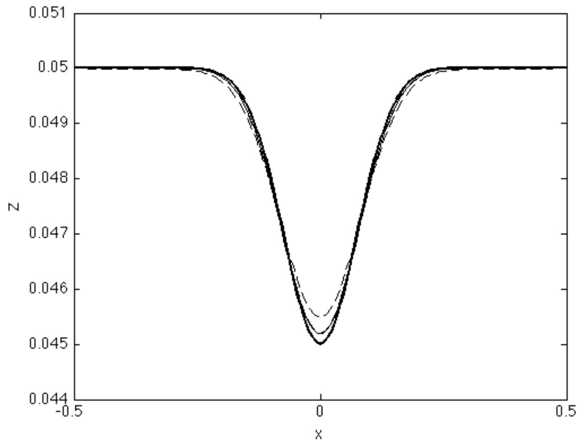
Here we limit ourselves to the second order formal approximation and show some numerical result. The formula comes directly from (15):

$$\epsilon\theta \approx \frac{\kappa^2}{h\Phi}\frac{d^2G}{dx^2} - \frac{\kappa}{\Phi}G(x) + a^2(-\frac{\kappa^2}{3!h\Phi}\frac{d^4G}{dx^4} + \frac{\kappa}{2!\Phi}\frac{d^2G}{dx^2}) + a^4(\frac{\kappa^2}{5!h\Phi}\frac{d^6G}{dx^6} - \frac{\kappa}{4!\Phi}\frac{d^4G}{dx^4}). \quad (16)$$

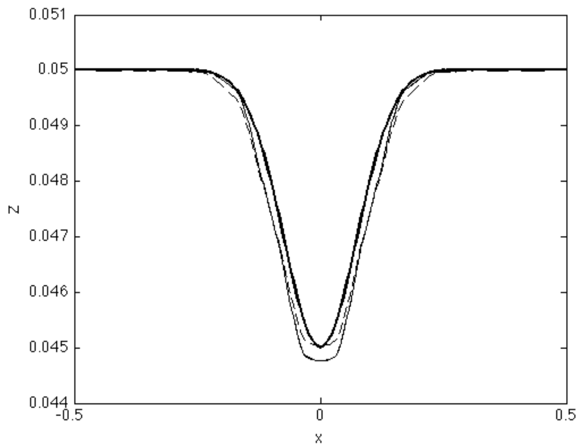
We produce syntetic data of thermal contrast by solving numerically the IBVP (1)-(6). If $t > T_\alpha$, we assume that the thermal contrast is the stationary difference $G(x) \approx u^\epsilon(x, 0, t) - U_0 - (\frac{a}{\kappa} + \frac{1}{h})\Phi$.

Formula (16) gives a good approximation of $\epsilon\theta$: In Figure 3a we show what we obtained by means of (16) when $w(x, 0) = u^\epsilon(x, 0) - u^0(x, 0)$. Convergence at orders > 2 seems to be slow in the neigborhood of $x = 0$ (maximum of the damage size). In Figure 3b it is $w(x, 0) = u^\epsilon(x, 0) - u^0(x, 0) + R_2(h, \epsilon)$. Although the contrast is now affected by noise, TPA still indentify the damage.

We remark that temperature maps at $z = 0$ allow us to localize the inaccessible defect. On the other hand, our goal is to evaluate the health of the specimen. For this reason, we could consider acceptable also the 3D estimate of zeroth order that gives a precise evaluation of the scale parameter ϵ (Figure 4).

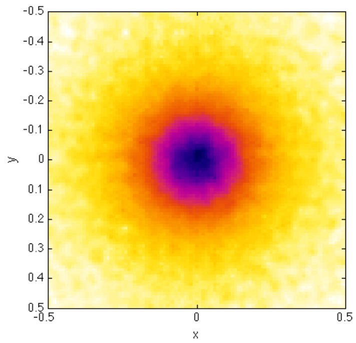


(a)

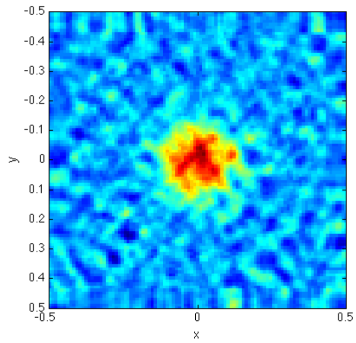


(b)

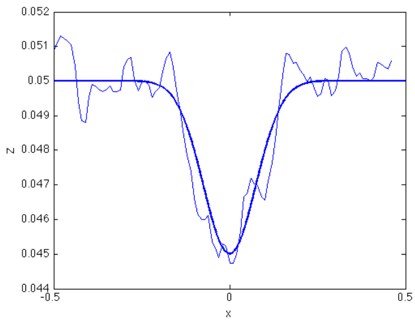
Figure 3: (a) When the thermal contrast is equal to the Domain Derivative ($R_2(h, \epsilon) = 0$), the unknown defect (bold line) is well approximated by the zeroth order TPA (dashed). The reconstruction is improved using the first order TPA (full thin line). The correction due to the second order term (the pointed line overlaps the first order line) seems to be neglectable. (b)Here the TPA is constructed from the thermal contrast (that is $w_0(x) = w(x, 0) + R_2(h, \epsilon)$). It is equivalent to using noisy data. TPA gives anyhow a quite good approximation of the defect. When further noise added, some regularization is required.



(a)



(b)



(c)

Figure 4: (a) Temperature map on the accessibile side: there is a damage spread around the origin of axes. It seems a regular gaussian hole. This image gives an idea of the diameter but we have no information about its depth ϵ. (b) Level sets of the damage as reconstructed in (10). The damaged area is clearly revealed. (c) Section $y = 0$ of the damage (full line) compared to the reconstruction mapped in Figure 4b. The depth is fully identified by (10).

5. Conclusions

We derive here an explicit formal inversion rule for recovering an unknown surface damage from uncomplete thermal data. Our formula is based on the Thin Plate Approximation of the Direct Model. Numerical results are encouraging, but much work is still required: in particular, regularization of numerical differentiation and Cauchy problem for Laplace's equation are expected in the perspective of using real data.

References

[1] G. Alessandrini, E. Beretta, E. Rosset and S. Vessella, *Inverse boundary value problems with unknown boundaries: optimal stability*, C. R. Acad. Sci. Paris 328 (2000), 607–611.

[2] G. Alessandrini, *On differentiation of approximately given functions*, Appl. Anal. **11** (1980), 45–59.

[3] H. T. Banks, F. Kojima and W. P. Winfree, *Boundary estimation problems arising in thermal tomography*, Inverse Problems **6** (1990), 897–921 .

[4] P. Bison, M. Ceseri, D. Fasino and G. Inglese, *Domain derivative approach to Active InfraRed Thermography*, Inverse Probl. Sci. Eng. **18** (2010), 873–889.

[5] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1959.

[6] W.S. Cleveland, *Robust Locally Weighted Regression and Smoothing Scatterplots*, J. Amer. Statist. Assoc. **74** (1979), 829–836.

[7] M. Di Cristo, E. Sincich and S. Vessella, *Size estimates of unknown boundaries with Robin type condition*, to appear on Proc. Roy. Soc. Edinburgh Sect. A (2016).

[8] P. Kaup, F. Santosa and M.Vogelius, *Method for imaging corrosion damage in thin plates from electrostatic data*, Inverse Problems **12** (1996), 279–293.

[9] G.Inglese, *An inverse problem in corrosion detection*, Inverse Problems **13** (1997), 977–994.

[10] X. P. V. Maldague, *Theory and Practice of Infrared Technology for Nondestructive Testing*, Wiley and Sons, New York, 2001.

[11] C. Pucci, *Discussione del problema di Cauchy per le equazioni di tipo ellittico*, Ann. Mat. Pura Appl. **46** (1958) 131–154.

[12] S. Salsa, *Partial Differential Equations in Action*, Springer, Milano, 2008.

[13] J. Simon, *Differentiation with respect to the domain in boundary value problems*, Numer. Funct. Anal. Optim. **2** (1980), 649–687 .

[14] M. Vollmer, *Newton's law of cooling revisited*, European J. Physics **30** (2009), 1063–1084.

Authors' addresses:

Gabriele Inglese
CNR-IAC
via Madonna del Piano, 10
51100, Sesto Fiorentino, Italy
E-mail: `gabriele@fi.iac.cnr.it`

Roberto Olmi
CNR-IFAC
via Madonna del Piano, 10
51100, Sesto Fiorentino, Italy
E-mail: `R.Olmi@ifac.cnr.it`

Received April 26, 2016
Revised May 31, 2016
Accepted June 1, 2016