Rend. Istit. Mat. Univ. Trieste
Volume 48 (2016), 261–283
DOI: 10.13137/2464-8728/13205

# Carleman estimates with two large parameters for an anisotropic system of elasticity

Victor Isakov

*Dedicated to Professor Giovanni Alessandrini*

Abstract. *We consider the system of partial differential equations of transversely isotropic elasticity with residual stress. Completing previous results we derive Carleman estimates for this system containing time derivatives. This permits to obtain exact observability inequalities for this system with the Cauchy data on the whole lateral boundary.*

Keywords: Carleman estimates, inverse problems, elasticity theory.
MS Classification 2010: 35R30, 35B60, 74B10.

## 1. Introduction

We consider a transversely isotropic elasticity system with residual stress [2, 15]. We let $x \in \mathbf{R}^3$ and $(x,t) \in \Omega$ which is a bounded domain in $\mathbf{R}^4$. Let $\mathbf{u}(x,t) = (u_1, u_2, u_3)^\top : \Omega \to \mathbf{R}^3$ be the displacement vector in $\Omega$. We introduce the operator of the transversely isotropic elasticity

$$(\mathbf{A}_T \mathbf{u})_i = \sum_{j,k,l=1}^{3} \partial_j \Big( C_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \Big), \tag{1}$$

where $C_{ijlk}$ are elastic parameters. In general, they enjoy the following symmetry properties

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}. \tag{2}$$

In the transversely ($(x_1, x_2)$-) isotropic case, in addition,

$$\begin{gathered} C_{1111} = C_{2222} = c_{11},\ C_{1122} = c_{12},\ C_{1133} = C_{2233} = c_{13},\ C_{3333} = c_{33}, \\ C_{2323} = C_{3131} = c_{23},\ C_{1212} = \tfrac{1}{2}(c_{11} - c_{12}),\ C_{ijkl} = 0 \text{ otherwise.} \end{gathered} \tag{3}$$

We assume that $c_{jk}$ are functions on $\bar{\Omega}$ and impose a sufficient condition of strict positivity of the elastic tensor:

$$\begin{aligned} &\varepsilon_0 < c_{11},\ \varepsilon_0 < c_{11} - c_{12},\ \varepsilon_0 < c_{12} + c_{11}, \\ &\varepsilon_0 < c_{23},\ \varepsilon_0 < c_{33},\ \varepsilon_0 < c_{13} + c_{23}, \\ &\varepsilon_0 + 2c_{13}^2 < (c_{11} + c_{12})c_{33},\ \varepsilon_0 + c_{13}^2 < c_{11}c_{33} \qquad \text{on } \Omega \end{aligned} \tag{4}$$

for some $\varepsilon_0 > 0$. We also introduce the scalar partial differential operator $R = \sum_{j,k=1}^{3} r_{jk}\partial_j\partial_k$ used to model the residual stress.

To state the main results we introduce pseudo convexity condition for a general scalar partial differential operator of second order $P = \sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k$ in $\Omega$ with the real-valued coefficients $a^{jk} \in C^1(\bar{\Omega})$. The principal symbol of this operator is $P(X;\zeta) = \sum_{j,k=1}^{n} a_{jk}(X)\zeta_j\zeta_k,\ X = (x,t)$. We will assume that the coefficients of $P$ admit the following bound $|a_{jk}|_2(\Omega) \le M$.

Let $K$ be a positive constant. A function $\psi$ is called $K$-pseudo-convex on $\Omega$ with respect to $P$ if $\psi \in C^2(\bar{\Omega})$, $P(X, \nabla\psi(X)) \neq 0,\ X \in \bar{\Omega}$, and

$$\sum_{j,k=1}^{4} \left(\partial_j\partial_k\psi \frac{\partial P}{\partial\zeta_j}\frac{\partial P}{\partial\zeta_k}\right)(X;\xi) + \sum_{j,k=1}^{4} \left(\left(\frac{\partial P}{\partial\zeta_k}\partial_k\frac{\partial P}{\partial\zeta_j} - \partial_k P \frac{\partial^2 P}{\partial\zeta_j\partial\zeta_k}\right)\partial_j\psi\right)(X,\xi) \ge K|\xi|^2$$

for any $\xi \in \mathbf{R}^n$ and any point $X$ of $\bar{\Omega}$ provided

$$P(X;\xi) = 0, \qquad \sum_{j=1}^{4} \frac{\partial P}{\partial\zeta_j}(X,\xi)\partial_j\psi(X) = 0.$$

We use the following convention and notations. Let $\partial = (\partial_1, \dots, \partial_4), D = -i\partial$, $\alpha = (\alpha_1, \dots, \alpha_4)$ is a multi-index with integer components, $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_4^{\alpha_4}$, $D^\alpha$ and $\partial^\alpha$ are defined similarly. $x_4 = t$. $\nabla$ denotes the gradient with respect to spatial variables $x_1, x_2, x_3$. $\nu$ is the outward normal to the boundary of a domain. $\Omega_\varepsilon = \Omega \cap \{\psi(x) > \varepsilon\}$. We recall that

$$\|u\|_{(k)}(\Omega) = \Big(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u|^2\Big)^{\frac{1}{2}}$$

is the norm in the Sobolev space $H_{(k)}(\Omega)$ and $\|\ \|_2 = \|\ \|_{(0)}$ is the $L^2$-norm. Let $C$ be generic constants (different at different places) depending only on $M$, on $K$, on the function $\psi$, on $C^2(\Omega)$-norms of the coefficients $\rho, c_{jk}, r_{jk}$ of the elasticity system, on $\varepsilon_0$, and on the domain $\Omega$. Any additional dependence will be indicated.

We let

$$a_1 = \frac{c_{11} - c_{12}}{c_{11} + c_{12}}, \quad a_2 = 2\frac{c_{23}}{c_{11} + c_{12}}, \quad a_3 = 2\frac{c_{13} + c_{23}}{c_{11} + c_{12}},$$
$$a_4 = \frac{(c_{11} - c_{12})(c_{13} + c_{23})}{(c_{11} + c_{12})c_{23}}, \tag{5}$$

and

$$A = a_1(\partial_1^2 + \partial_2^2) + a_2\partial_3^3,\ div_T\mathbf{u} = \partial_1 u_1 + \partial_2 u_2 + a_3\partial_3 u_3,$$
$$curl_T\mathbf{u} = (\partial_2 u_3 - a_4\partial_3 u_2, a_4\partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)\,. \tag{6}$$

We introduce the following conditions

$$(c_{11}^2 - c_{12}^2)c_{33} - 2(c_{13} + c_{23})^2(c_{11} - c_{12}) - 2c_{23}^2(c_{11} + c_{12}) = 0,$$
$$c_{11} - c_{12} = 2c_{23} \text{ on } \Omega \tag{7}$$

and the weight and scaling functions

$$\varphi = e^{\gamma\psi},\ \sigma = \gamma\tau\varphi. \tag{8}$$

THEOREM 1.1. *Let $\psi \in C^3(\bar\Omega)$ be $K$- pseudo convex with respect to $\rho\partial_t^2 - A - R$, $\rho\partial_t^2 - A - \partial_1^2 - \partial_2^2 - a_3a_4\partial_3^2 - R$ in $\bar\Omega$ and let $|\rho|_2(\Omega) + |c_{jk}|_2(\Omega) + |r_{jk}|_2(\Omega) \leq M$. Let the conditions (7) be satisfied.*

*Then there are constants $C, C_0(\gamma)$ such that*

$$\int_\Omega (\gamma\sigma^2|\mathbf{u}|^2 + \sigma(|div_T\mathbf{u}|^2 + |curl_T\mathbf{u}|^2) + \gamma(|\partial_t\mathbf{u}|^2 + |\nabla\mathbf{u}|^2))e^{2\tau\varphi}$$
$$\leq C\int_\Omega |(\rho\partial_t^2 - \mathbf{A}_T - R)\mathbf{u}|^2 e^{2\tau\varphi} \tag{9}$$

*for all $\mathbf{u} \in H_0^2(\Omega), C < \gamma, C_0 < \tau$.*

This estimate for isotropic elasticity with residual stress was obtained in [11] and for more general transversely isotropic elasticity in [10] without the terms with $\gamma$ on the left side.

Let us consider the following Cauchy problem

$$(\rho\partial_t^2 - \mathbf{A}_T - R)\mathbf{u} = \mathbf{f} \text{ in } \Omega,\ \mathbf{u} = \mathbf{g}_0, \quad \partial_\nu\mathbf{u} = \mathbf{g}_1 \quad \text{on} \quad \Gamma \subset \partial\Omega, \tag{10}$$

where $\Gamma \in C^3$. Let $\Omega_\delta = \Omega \cap \{\psi > \delta\}$. The Carleman estimate of Theorem 1.1 by standard argument ([9], section 3.2) implies the following conditional Hölder stability estimate for (10) in $\Omega(\delta)$ (and hence uniqueness in $\Omega(0)$).

THEOREM 1.2. *Let $\psi \in C^3(\bar\Omega)$ be $K$- pseudo convex with respect to $\rho\partial_t^2 - A - R, \rho\partial_t^2 - A - \partial_1^2 - \partial_2^2 - a_3a_4\partial_3^2 - R$ in $\bar\Omega$ and let $|\rho|_2(\Omega) + |c_{jk}|_2(\Omega) + |r_{jk}|_2(\Omega) \leq M$. Let the condition (7) be satisfied. Assume that $\bar\Omega_0 \subset \Omega \cap \Gamma$.*

*Then there exist* $C = C(\delta), \kappa = \kappa(\delta) \in (0,1)$ *such that for a solution* $\mathbf{u} \in H^2(\Omega)$ *to* (10) *one has*

$$\|\mathbf{u}\|_{(0)}(\Omega_\delta) + \|\nabla_x \mathbf{u}\|_{(0)}(\Omega_\delta) + \|\partial_t \mathbf{u}\|_{(0)}(\Omega_\delta) \le C(F + M_1^{1-\kappa} F^\kappa), \tag{11}$$

*where* $F = \|\mathbf{f}\|_{(0)}(\Omega_0) + \|\mathbf{g}_0\|_{(\frac{3}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{1}{2})}(\Gamma)$, $M_1 = \|\mathbf{u}\|_{(1)}(\Omega)$.

In Theorems 1.3, 1.4 we assume that $\Omega = G \times (-T,T), \partial G \in C^3$ and that $R = 0$.

Due to (4) the system (10) is $t$-hyperbolic and from known results (e.g. [3], III.4, p.123) it follows that the first initial boundary value problem for this system is uniquely solvable in standard energy spaces, moreover the conventional energy integral

$$E(t;\mathbf{u}) = \int_G (|\partial_t \mathbf{u}|^2 + |\nabla \mathbf{u}|^2 + |\mathbf{u}|^2)(,t)$$

is bounded by the initial energy and the right side (more detail in the proof of Theorem 1.3). Repeating the argument in [3] one can obtain the same result when the smallest eigenvalue of the matrix $r_{jk}$ is greater than than $-\frac{\varepsilon_0}{2}$.

Theorem 1.3. *Let* $\psi \in C^3(\bar{\Omega})$ *be* $K$*- pseudo convex with respect to* $\rho\partial_t^2 - A - R, \rho\partial_t^2 - A - \partial_1^2 - \partial_2^2 - a_3 a_4 \partial_3^2 - R$ *in* $\bar{\Omega}$ *and let* $|\rho|_2(\Omega) + |c_{jk}|_2(\Omega) + |r_{jk}|_2(\Omega) \le M$. *Assume that*

$$\psi < 0 \text{ on } \bar{G} \times \{-T,T\},\ 0 < \psi \text{ on } G \times \{0\}. \tag{12}$$

*Then there exist* $C$ *such that for a solution* $\mathbf{u} \in H^2(\Omega)$ *to* (10) *one has*

$$E(t;\mathbf{u}) \le C(\|\mathbf{f}\|_{(0)}(\Omega) + \|\mathbf{g}_0\|_{(\frac{3}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{1}{2})}(\Gamma)). \tag{13}$$

Now we state results about identification of a source from additional boundary data.

Let $\mathbf{u}$ be a solution to

$$\begin{gathered}(\rho\partial_t^2 - \mathbf{A}_T - R)\mathbf{u} = \mathcal{A}\mathbf{f} \text{ in } \Omega,\\ \mathbf{u} = \mathbf{0},\ \partial_t \mathbf{u} = \mathbf{0} \text{ on } G \times \{0\},\ \mathbf{u} = \mathbf{0} \text{ on } \partial G \times (-T,T).\end{gathered} \tag{14}$$

We will assume that $\mathcal{A} \in C(\bar{\Omega})$.

We will consider the boundary stress data as measurements (observations). We introduce the norm of the of the lateral Cauchy data

$$F = \|\partial_t^2 \partial_\nu \mathbf{u}\|_{(\frac{1}{2})}(\Gamma). \tag{15}$$

To guarantee the uniqueness, we impose some non-degeneracy condition on the matrix $\mathcal{A}$. We assume that

$$\det\mathcal{A} > \varepsilon_0 > 0 \text{ on } G \times \{0\}. \tag{16}$$

THEOREM 1.4. *Let $\psi \in C^3(\bar{\Omega})$ be $K$- pseudo convex with respect to $\rho\partial_t^2 - A - R, \rho\partial_t^2 - A - \partial_1^2 - \partial_2^2 - a_3 a_4 \partial_3^2 - R$ in $\bar{\Omega}$. Assume that $\rho, c_{jk}, r_{jk}$ do not depend on $t$ and $|\rho|_2(\Omega) + |c_{jk}|_2(\Omega) + |r_{jk}|_2(\Omega) + |\partial_t^2 \mathcal{A}|_0(\Omega) \leq M$. Assume that the condition (12) is satisfied. Let the matrix function $\mathcal{A}$ satisfy (16).*

*Then there exist $C$ such that*

$$\|\mathbf{f}\|_{(0)}(\Omega) \leq CF. \tag{17}$$

Observe that the classical isotropic elasticity is a particular case of the system under consideration, when $c_{11} = c_{33} = \lambda + 2\mu, c_{12} = c_{13} = \lambda, c_{23} = \mu$. In particular, the conditions (7) are satisfied.

Carleman estimates were introduced by Carleman in 1939 to demonstrate uniqueness in the Cauchy problem for a system of first order in $\mathbf{R}^2$ with non analytic coefficients. Carleman type estimates and uniqueness of the continuation theorems have been obtained for wide classes of scalar partial differential equations [6, 9]. But useful concept of pseudo convexity is not available for systems, and Carleman estimates were derived only in particular cases, like for classical isotropic dynamical Maxwell's and elasticity systems [5] (by using principal diagonalization). Two large parameters were introduced in [8]. They were a main tool in the first proof of uniqueness and stability of all three elastic parameters in dynamical isotropic Lame system from two sets of boundary data [7]. A system of transversely isotropic elasticity with residual stress was recently studied in [10, 11, 12, 14] where there are Carleman estimates, uniqueness and stability of the continuation and of the identification of elastic coefficients.

In this paper for the transversely isotropic system with residual stress we obtain Carleman estimates including time derivative. Most advanced previous results [10] handled only spatial derivatives. Observe that our results are new for the classical isotropic elasticity system. Including temporal derivative enables to obtain exact controllability (Lipschitz) bounds in the lateral Cauchy and inverse problems under minimal regularity assumptions. So far our results need a special condition (7). The main idea is to use principal upper triangular reduction, scalar Carleman estimates with two large parameters, and spatial smoothing (pseudo-differential) operator with parameter. The crucial part is $L^2$ bounds on commutators of this operator and of differential operators with parameters.

We stated our basic results in section 1. In section 2 we obtain auxiliary results where crucial are bounds on commutators of multiplication and of smoothing operator and especially Lemma 2.4 on certain localization of this pseudo-differential operator. In section 3 we prove estimates of Theorem 1.1 and in section 4 apply them to stability estimates in the continuation and inverse problems. We tried to minimize technicalities and refer as much as possible to known results.

It is not easy to find functions $\psi$ which are pseudo convex with respect to a general operator. In an isotropic case explicit and verifiable conditions for $\psi(x,t) = |x-\beta|^2 - \theta^2 t^2$ were found by Isakov in 1980 and their simplifications are given in [9], section 3.4. In general anisotropic case Khaidarov [13] showed that under certain conditions the same $\psi$ is pseudo convex if the speed of the propagation determined by $A$ is monotone in a certain direction.

In the following Lemma for a general hyperbolic operator we give the condition of $K$-pseudo convexity of $\psi(x,t) = |x-\beta|^2 - \theta^2 t^2$.

LEMMA 1.5. *Let*

$$P = \partial_t^2 - \sum_{j,k=1}^{3} a_{jk}\partial_j\partial_k, \ \ a_{jk} = a_{kj},$$

*where $a_{jk} \in C^1$ satisfy the uniform ellipticity condition*

$$\sum_{j,k=1}^{3} a_{jk}(X)\xi_j\xi_k \geq \varepsilon_0|\xi|^2, \ X \in \Omega \ \ \xi \in \mathbf{R}^3, \ \ \varepsilon_0 > 0.$$

*Let*

$$\psi(x,t) = |x-\beta|^2 - \theta^2 t^2, \ \ \beta = (0,0,\beta_3).$$

*Assume that*

$$\sum_{j,l=1}^{3}\left(\sum_{k=1}^{3} a_{3k}\partial_k a_{jl} - 2\sum_{k=1}^{2} a_{lk}\partial_k a_{j3}\right)\xi_j\xi_l \geq \varepsilon_1|\xi|^2, \quad \xi \in \mathbf{R}^3.$$

*for some $\varepsilon_1 > 0$.*

*Then there is large $\beta_3$ such that the function $\psi$ is $K$-pseudo convex with respect to $P$ in $\bar{\Omega}$.*

A proof is given in [11].

## 2. Auxiliary results.

For a linear partial differential operator $\mathbf{A}$ ( with matrix coefficients) we introduce $\mathbf{A}_\varphi$ by the equality $(\mathbf{A}_\varphi v)e^{-\tau\varphi} = \mathbf{A}(ve^{-\tau\varphi})$. From the Leibniz formula it follows that $\mathbf{A}_\varphi$ is the linear partial differential operator with the same principal part as $\mathbf{A}$. We observe that

$$(\partial_j)_\varphi = \partial_j - \sigma\partial_j\psi \tag{18}$$

and

$$(\mathbf{A}_1\mathbf{A}_2)_\varphi = (\mathbf{A}_1)_\varphi(\mathbf{A}_2)_\varphi. \tag{19}$$

Indeed, according to the definition,

$$((\mathbf{A}_1\mathbf{A}_2)_\varphi v)e^{-\tau\varphi} = \mathbf{A}_1(\mathbf{A}_2(ve^{-\tau\varphi}))$$
$$= \mathbf{A}_1(((\mathbf{A}_2)_\varphi v)e^{-\tau\varphi}) = ((\mathbf{A}_1)_\varphi(\mathbf{A}_2)_\varphi v)e^{-\tau\varphi}.$$

In particular,

$$P_\varphi(D) = P(D + i\tau\nabla\varphi) = P(D) + \tau P_1(D) + \tau^2 P(\nabla\varphi) \tag{20}$$

where $P_1$ is a first order differential operator with coefficients depending on $\gamma$. We will use the notation $<\xi>=(|\xi|^2+1)^{\frac{1}{2}}$ and the pseudo-differential operator $\Lambda^s_\tau f = \mathcal{F}^{-1}(<\xi>+\tau)^s\mathcal{F}f$, where $\mathcal{F}$ is the Fourier transform in $\mathbf{R}^3$ and $\xi\in\mathbf{R}^3$. Let $\Omega^*$ be a bounded domain in $\mathbf{R}^4$ with a smooth boundary such that $\overline{\Omega}\subset\Omega^*$. We can extend all coefficients of the operators $\mathbf{A}_T, R$ and functions $\rho,\psi$ onto $\mathbf{R}^4$ preserving the regularity in such a way that they have support in $\Omega^*$ and their $C^2$-norms are bounded by $C$.

In next Lemmas we fix $x^0$ with $(x^0,t^0)\in\overline{\Omega}$ and introduce $\sigma(t)=\sigma(x^0,t)$

LEMMA 2.1. *There exists a constant $C(\gamma)$ such that*

$$\|\Lambda^{-1}_{\sigma(t)}\partial_t u - \partial_t\Lambda^{-1}_{\sigma(t)}u\|_{(0)}(\mathbf{R}^4) \le C(\gamma)\|\Lambda^{-1}_{\sigma(t)}u\|_{(0)}(\mathbf{R}^4), \tag{21}$$

$$\|\sigma^{\frac{1}{2}}(\Lambda^{-1}_{\sigma(t)}div_{T,\varphi}\mathbf{u} - div_{T,\varphi}\Lambda^{-1}_{\sigma(t)}(\mathbf{u}))\|_{(0)}(\mathbf{R}^4) \le C(\gamma)\tau^{-\frac{1}{2}}\|\mathbf{u}\|_{(0)}(\Omega),$$

$$\|\sigma^{\frac{1}{2}}(\Lambda^{-1}_{\sigma(t)}curl_{T,\varphi}\mathbf{u} - curl_{T,\varphi}\Lambda^{-1}_{\sigma(t)}(\mathbf{u}))\|_{(0)}(\mathbf{R}^4) \le C(\gamma)\tau^{-\frac{1}{2}}\|\mathbf{u}\|_{(0)}(\Omega) \tag{22}$$

*and*

$$\|\Lambda^{-1}_{\sigma(t)}(P_\varphi u) - (P_\varphi\Lambda^{-1}_{\sigma(t)}u)\|_{(0)}(\mathbf{R}^4) \le C(\gamma)(\|u\|_{(0)}(\Omega) + \|\Lambda^{-1}_{\sigma(t)}\partial_t u\|_{(0)}(\mathbf{R}^4)) \tag{23}$$

*for all $u,\mathbf{u}\in H^2_0(\Omega)$.*

*Proof.* We first prove (21). Observe that $\sigma=\tau\gamma\varphi, \partial_t\sigma=\gamma\sigma\partial_t\psi, \partial_t^2\sigma = \gamma\sigma(\partial_t^2\psi+\gamma(\partial_t\psi)^2)$ , that

$$\partial_t\Lambda^{-1}_{\sigma(t)}u = \mathcal{F}^{-1}\left(\frac{-\partial_t\sigma(t)}{(<\xi>+\sigma(t))^2}\mathcal{F}u + \frac{1}{<\xi>+\sigma(t)}\mathcal{F}\partial_t u\right) \tag{24}$$

and

$$\partial_t^2\Lambda^{-1}_{\sigma(t)}u = \mathcal{F}^{-1}\left(\left(\frac{-\partial_t^2\sigma(t)}{(<\xi>+\sigma(t))^2} + \frac{(\partial_t\sigma(t))^2}{(<\xi>+\sigma(t))^3}\right)\mathcal{F}u \right.$$
$$\left. +2\frac{-\partial_t\sigma}{(<\xi>+\sigma(t))^2}\mathcal{F}\partial_t u + \frac{1}{<\xi>+\sigma(t)}\mathcal{F}\partial_t^2 u\right). \tag{25}$$

Formula (24) implies that

$$\Lambda^{-1}_{\sigma(t)}\partial_t u - \partial_t \Lambda^{-1}_{\sigma(t)} u = \mathcal{F}^{-1}\left(\frac{\partial_t \sigma(t)}{(<\xi> + \sigma(t))^2}\mathcal{F}u\right),$$

so

$$\begin{aligned}
&\|\Lambda^{-1}_{\sigma(t)}\partial_t u - \partial_t \Lambda^{-1}_{\sigma(t)} u\|^2_{(0)}(\mathbf{R}^4)\\
&\leq \int_{\mathbf{R}}\int_{\mathbf{R}^3}\left|\mathcal{F}^{-1}(\frac{\partial_t \sigma(t)}{(<\xi> + \sigma(t))^2}\mathcal{F}u(x,t)\right|^2 dxdt\\
&\leq \int_{\mathbf{R}}\int_{\mathbf{R}^3}\left|\frac{\partial_t \sigma(t)}{(<\xi> + \sigma(t))^2}\right|^2 |\mathcal{F}u(\xi,t)|^2 d\xi dt\\
&\leq C(\gamma)\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left|\frac{1}{(<\xi> + \sigma(t))}\mathcal{F}u(\xi,t)\right|^2 d\xi dt
\end{aligned}$$

and again using the Parseval equality we yield (21).

Due to (18), $div_{T,\varphi}\mathbf{u}$ is the sum of terms $a(\partial_j - \sigma\partial_j\psi)u_k$, where $j,k = 1,2,3$, $|a|_2(\mathbf{R}^4) \leq C$, and $a = 0$ outside $\Omega^*$. Hence it suffices to show that

$$\|\Lambda^{-1}_{\sigma(t)}(b\partial^\beta u) - b\partial^\beta \Lambda^{-1}_{\sigma(t)} u\|_{(0)}(\mathbf{R}^4) \leq C(\gamma)\tau^{-1}\|u\|_{(0)}(\Omega), \tag{26}$$

for all $\beta$ with $|\beta| = 1, \beta_4 = 0$, and that

$$\|\Lambda^{-1}_{\sigma(t)}(bu) - b\Lambda^{-1}_{\sigma(t)} u\|_{(0)}(\mathbf{R}^4) \leq C(\gamma)\tau^{-2}\|u\|_{(0)}(\Omega), \tag{27}$$

when $b \in C^1(\mathbf{R}^4)$, $|b|_1(\mathbf{R}^4) < C(\gamma)$, and $b = 0$ outside $\Omega^*$.

To prove (26) we introduce $u_1 = \Lambda^{-1}_{\sigma(t)}\partial^\beta u$. Using also that $\Lambda_\sigma = \Lambda_0 + \sigma$, we have

$$\Lambda^{-1}_{\sigma(t)} b\partial^\beta u - b\partial^\beta\Lambda^{-1}_{\sigma(t)} u = \Lambda^{-1}_{\sigma(t)}(b\Lambda_{\sigma(t)} - \Lambda_{\sigma(t)} b)u_1 = \Lambda^{-1}_{\sigma(t)}(b\Lambda_0 - \Lambda_0 b)u_1.$$

As above, from the Parseval identity, $\|u_1\|_{(0)}(\mathbf{R}^4) \leq C\|u\|_{(0)}(\Omega)$. By known (e.g. Coifman and Meyer ([1])) estimates of commutators of pseudo-differential operators and of multiplication operators

$$\|(b\Lambda_0 - \Lambda_0 b)u_1(,t)\|^2_{(0)}(\mathbf{R}^3) \leq C(\gamma)\|u_1(,t)\|^2_{(0)}(\mathbf{R}^3).$$

Using that $\|\Lambda^{-1}_{\sigma(t)} v\|^2_{(0)}(\mathbf{R}^3) \leq C(\gamma)\tau^{-2}\|v\|^2_{(0)}(\Omega)$ and integrating with respect to $t$ we complete the proof of (26).

Proofs of (27) and for *curl* are similar.

Due to (18), (19), $P_\varphi u$ is the sum of terms

$$a(\partial_j - \sigma\partial_j\psi)(\partial_k - \sigma\partial_k\psi)u$$

where $j,k = 1,2,3,4$, $|a|_2(\mathbf{R}^4) \leq C$, $a = 1$ when $j = k = 4$, $a = 0$ when $j = 1,2,3, k = 4$, and $a = 0$ outside $\Omega^*$ otherwise. Elementary calculations show that this expression equals to

$$a\partial_j\partial_k u - a\sigma(\partial_k\psi\partial_j u + \partial_j\psi\partial_k u) + a\sigma((\sigma-1)\partial_j\psi\partial_k\psi - \partial_j\partial_k\psi)u$$

Hence it suffices to show that

$$\|\Lambda^{-1}_{\sigma(t)} a\partial^\alpha u - a\partial^\alpha\Lambda^{-1}_{\sigma(t)} u\|_{(0)}(\mathbf{R}^4) \leq C(\gamma)(\|u\|_{(0)}(\Omega) + \|\Lambda^{-1}_{\sigma(t)}\partial_t u\|_{(1)}(\mathbf{R}^4)), \quad (28)$$

for all $\alpha$ with $|\alpha| \leq 2$, that

$$\tau\|\Lambda^{-1}_{\sigma(t)} b\partial^\beta u - b\partial^\beta\Lambda^{-1}_{\sigma(t)} u\|_{(0)}(\mathbf{R}^4) \leq C(\gamma)\|u\|_{(0)}(\Omega), \quad \text{for all } |\beta| \leq 1 \qquad (29)$$

for all $\beta$ with $|\beta| \leq 1$, and that

$$\tau^2\|\Lambda^{-1}_{\sigma(t)}(bu) - b\Lambda^{-1}_{\sigma(t)} u\|_{(0)}(\mathbf{R}^4) \leq C(\gamma)\|u\|_{(0)}(\Omega), \qquad (30)$$

when $b \in C^1(\overline{\Omega^*})$, $|b|_1(\bar{\Omega}) < C$, and $b = 0$ outside $\Omega^*$.

To show (28) let first $\alpha_4 = 2$.

As above, (25) implies that

$$\|\Lambda^{-1}_{\sigma(t)}\partial_t^2 u - \partial_t^2\Lambda^{-1}_{\sigma(t)} u\|_{(0)}(\mathbf{R}^4) \leq C(\gamma)(\|u\|_{(0)}(\Omega) + \|\Lambda^{-1}_{\sigma(t)}\partial_t u\|_{(0)}(\mathbf{R}^4)).$$

To complete a proof of (28) we now consider $\alpha_4 = 0$. Let $\alpha_j > 0$ and $\beta_j = 1$ while other components of $\beta$ be zero. We introduce $u_1 = \Lambda^{-1}_{\sigma(t)}\partial^{\alpha-\beta}u$. Using also that $\Lambda_\sigma = \Lambda_0 + \sigma$, we have

$$\begin{aligned}
\Lambda^{-1}_{\sigma(t)} a\partial^\alpha u - a\partial^\alpha\Lambda^{-1}_{\sigma(t)} u &= \Lambda^{-1}_{\sigma(t)}(a\Lambda_{\sigma(t)} - \Lambda_{\sigma(t)} a)\partial_j u_1 \\
&= \Lambda^{-1}_{\sigma(t)}(a\Lambda_0 - \Lambda_0 a)\partial_j u_1 = \Lambda^{-1}_{\sigma(t)}(a\partial_j\Lambda_0 - \partial_j(\Lambda_0 a) + \Lambda_0\partial_j a)u_1 \\
&= \Lambda^{-1}_{\sigma(t)}(\partial_j(a\Lambda_0 - \Lambda_0 a) + (\Lambda_0\partial_j a - \partial_j a\Lambda_0))u_1.
\end{aligned}$$

As above, from the Parseval identity, $\|u_1\|_{(0)}(\mathbf{R}^4) \leq C\|u\|_{(0)}(\Omega)$. By known (e.g. Coifman and Meyer [1]) estimates of commutators of pseudo-differential operators and of multiplication operators

$$\|(a\Lambda_0 - \Lambda_0 a)u_1(,t)\|_{(0)}(\mathbf{R}^3) \leq C(\gamma)\|u_1(,t)\|_{(0)}(\mathbf{R}^3).$$

A similar estimate is valid when we replace $a$ by $\partial_j a$. Using, as above, that the norm of the operator $\Lambda^{-1}_{\sigma(t)}\partial_j$ from $L^2(\mathbf{R}^3)$ into itself is bounded by $C(\gamma)$ and integrating with respect to $t$ we complete the proof of (28).

Next we demonstrate (29). Let first $\beta = (0,0,0,1)$. Using (24) we have

$$\|\partial_t\Lambda^{-1}_{\sigma(t)} u - \Lambda^{-1}_{\sigma(t)}\partial_t u\|_{(0)}(\mathbf{R}^3) \leq C(\gamma)\|u\|_{(0)}(\mathbf{R}^3)$$

so it suffices to bound $\Lambda^{-1}_{\sigma(t)} b\partial_t u - b\Lambda^{-1}_{\sigma(t)}\partial_t u$. To do this, let $u_2 = \Lambda^{-1}_{\sigma(t)}\partial_t u$, then we need to bound

$$\Lambda^{-1}_{\sigma(t)} b\Lambda_{\sigma(t)}u_2 - bu_2 = \Lambda^{-1}_{\sigma(t)}(b\Lambda_{\sigma(t)}u_2 - \Lambda_{\sigma(t)}(bu_2)) = \Lambda^{-1}_{\sigma(t)}(b\Lambda_0 u_2 - \Lambda_0(bu_2))$$

because $\Lambda_\sigma = \Lambda_0 + \sigma$. As above, from known bounds of commutators and the definition of $u_2$ it follows that

$$\tau\|\Lambda^{-1}_{\sigma(t)}(b\Lambda_0 u_2 - \Lambda_0(bu_2))\|_{(0)}\|(\mathbf{R}^4) \leq C(\gamma)\|\Lambda^{-1}_{\sigma(t)}\partial_t u\|_{(0)}(\mathbf{R}^4)$$

Proofs of (29) for general $\beta$ and of (30) are similar. □

LEMMA 2.2. *Let $K(x,y;t)$ be the Schwartz kernel of the pseudo-differential operator $\Lambda^{-1}_{\sigma(t)}$ with $\tau > 1$.*

*Then*

$$|\partial^\alpha_x K(x,y;t)| \leq C(\gamma)\tau^{-2}|x-y|^{-8}$$

*provided $|\alpha| \leq 2$.*

A proof is similar to [7], Lemma 3.4.

*Proof.* The Schwartz kernel $K(x,y;t)$ is the oscillatory integral

$$\int_{\mathbf{R}^3} e^{i(x-y)\cdot\xi}(<\xi> + \sigma(t))^{-1}d\xi$$
$$= -|x-y|^{-2}\int_{\mathbf{R}^3}(\Delta_\xi e^{i(x-y)\cdot\xi})(<\xi> + \sigma(t))^{-1}d\xi$$
$$= -|x-y|^{-2}\int_{\mathbf{R}^3} e^{i(x-y)\cdot\xi}\Delta_\xi(<\xi> + \sigma(t))^{-1}d\xi$$
$$= \cdots = (-1)^l|x-y|^{-2l}\int_{\mathbf{R}^3} e^{i(x-y)\cdot\xi}\Delta^l_\xi(<\xi> + \sigma(t))^{-1}d\xi$$

where we did integrate by parts. Observing that

$$|\Delta^l_\xi(<\xi> + \sigma(t))^{-1}| \leq C(l)(<\xi> + \sigma(t))^{-2} <\xi>^{-2l+1},\ l = 1,2,...,$$

and letting $l = 4$ we complete the proof. □

We denote by $S'$ the orthogonal projection of a set $S$ in $\mathbf{R}^4$ onto $\mathbf{R}^3$ and let $Cyl(x^0;\delta) = (B'(x^0;\delta)\times\mathbf{R})\cap\Omega^*$.

LEMMA 2.3. *We have*

$$\int_{\mathbf{R}^4\setminus Cyl(x^0;3\delta)}\left(\tau^3|\Lambda^{-1}_{\sigma(t)}v|^2 + \tau\sum_{|\alpha|=1}|\partial^\alpha\Lambda^{-1}_{\sigma(t)}v|^2\right)$$
$$\leq C(\gamma,\delta)\tau^{-1}\int_{\mathbf{R}^4}\sigma\left(|v|^2 + |\partial_t\Lambda^{-1}_{\sigma(t)}v|^2\right) \quad (31)$$

*for all $v \in H^1_0(Cyl(x^0;\delta)), x^0 \in \bar\Omega'$.*

*Proof.* We can assume that $x^0 = 0$ and drop $x^0$.

We first consider the case when $\alpha_4 = 0$. Since $suppv \subset Cyl(\delta)$,

$$|\partial^\alpha \Lambda^{-1}_{\sigma(t)} v(x,t)| \le \int_{B(\delta)} |v(y,t)|\, |\partial^\alpha K(x,y;t)|\, dy$$
$$\le C(\gamma,\delta)\tau^{-2} \int_{B(\delta)} |x-y|^{-8} |v(y,t)|\, dy$$

by Lemma 2.2, provided $x \in \mathbf{R}^3 \setminus B(3\delta)$. When $y \in B(2\delta)$,

$$|x-y| \ge \frac{1}{2}|x-y| + \frac{1}{8}|x-y| \ge \frac{\delta}{2} + \frac{1}{8}|x| - \frac{1}{8}|y| \ge \frac{\delta}{4} + \frac{1}{8}|x| \ge \frac{1+|x|}{C(\delta)}. \quad (32)$$

Hence by using the Schwarz inequality

$$|\partial^\alpha \Lambda^{-1}_{\sigma(t)} v(x,t)| \le C(\gamma,\delta)\tau^{-2}(1+|x|)^{-8} \Big( \int_{B(\delta)} |v(,t)|^2 \Big)^{\frac{1}{2}} \quad \text{for all } |\alpha| \le 1,$$

provided $x \in \mathbf{R}^3 \setminus B(3\delta)$. Using this estimate we conclude that the last integral on the left side of (31) is less than $C(\gamma,\delta)\int_{Cyl(\delta)} |v|^2$. Similarly we bound the first integral.

Now we will handle the most delicate case of $\alpha = (0,0,0,1)$, i.e. $\partial^\alpha = \partial_t$. Let $w = \partial_t v$. Due to (21), it suffices to show that

$$\tau \int_{\mathbf{R}^3 \setminus B(3\delta)} |\Lambda^{-1}_{\sigma(t)} w(,t)|^2 \le C(\gamma) \int_{\mathbf{R}^3} |\Lambda^{-1}_{\sigma(t)} w|^2(,t). \quad (33)$$

To do so we will make use of the integral operator $\Lambda^*_{\sigma(t)} w = \mathcal{F}^{-1}(|\xi|^2 + \sigma(t))^{-1}\mathcal{F}w$ which is obviously a fundamental solution of the differential operator $-\Delta + \sigma(t)$ in $\mathbf{R}^3$. So for $W = \Lambda^*_{\sigma(t)} w$,

$$(-\Delta + \sigma(t))W = w \text{ in } \mathbf{R}^3.$$

We have

$$(|\xi|^2 + \sigma(t))^{-1} \le C(\gamma)(< \xi > + \sigma(t))^{-1}$$

and hence

$$\int_{\mathbf{R}^3} |\Lambda^*_{\sigma(t)} w|^2 \le C(\gamma) \int_{\mathbf{R}^3} |\Lambda^{-1}_{\sigma(t)} w|^2. \quad (34)$$

Let a cut-off function $\chi_\delta = 1$ on $B(\delta)$, $supp\, \chi_\delta \subset B(2\delta)$ and $|\partial^\alpha \chi_\delta| \le C(\delta)$

when $|\alpha| \leq 2$. Due to the definition of $K$,

$$\begin{aligned}\Lambda^{-1}_{\sigma(t)} w(x,t) &= \int_{B(\delta)} K(x-y;\sigma(t)) w(y,t) dy \\ &= \int_{B(2\delta)} \chi_\delta(y) K(x-y;\sigma(t))(-\Delta+\sigma(t)) W(y,t) dy \\ &= \int_{B(2\delta)} (-\Delta+\sigma(t))(\chi_\delta(y) K(x-y;\sigma(t))) W(y,t) dy.\end{aligned}$$

Therefore, by Lemma 2.2 and (32)

$$\begin{aligned}|\Lambda^{-1}_{\sigma(t)} w(x,t)| &\leq C(\gamma,\delta)\tau^{-1} \int_{B(2\delta)} |x-y|^{-8} |W(y,t)| dy \\ &\leq C(\gamma,\delta)\tau^{-1}(1+|x|)^{-8} \Big( \int_{B(2\delta)} |W(y,t)|^2 dy \Big)^{\frac{1}{2}}.\end{aligned}$$

This combined with (34) completes the proof of (33) and hence of Lemma 2.3. □

Since $\psi \in C^2$, using (8) we will choose $\delta(\gamma)$ so that

$$\frac{\sigma(t)}{2} \leq \sigma \leq 2\sigma(t) \tag{35}$$

on $Cyl(x^0; 4\delta(\gamma))$.

Lemma 2.4. *There is $C$ such that*

$$\int_{\mathbf{R}^4} |\partial_j \Lambda^{-1}_{\sigma(t)} v|^2 + \int_{\mathbf{R}^4} |\Lambda^{-1}_{\sigma(t)}(a\sigma v)|^2 \leq C \int_{\mathbf{R}^4} |v|^2,\ j=1,2,3, \tag{36}$$

*for all $v \in H^2_0(Cyl(x^0;4\delta(\gamma))), x^0 \in \bar{\Omega}'$, provided $|a|_1(\mathbf{R}^4) < C$ and $a$ is constant outside $\Omega^*$.*

*Proof.* As above we let $x^0 = 0$ and drop it. Due to the Parseval identity

$$\begin{aligned}&\int_{\mathbf{R}^4} |\partial^\alpha \Lambda^{-1}_{\sigma(t)} v|^2 \leq \int_{\mathbf{R}} \left( \int_{\mathbf{R}^3} \frac{|\xi|^2}{(<\xi> + \sigma(t))^2} |\mathcal{F}v(\xi,t)|^2 d\xi \right) dt \\ &\leq \int_{\mathbf{R}} \left( \int_{\mathbf{R}^3} |\mathcal{F}v(\xi,t)|^2 d\xi \right) dt = \int_{\mathbf{R}} \left( \int_{\mathbf{R}^3} |v(x,t)|^2 dx \right) dt = \int_{Cyl(\delta(\gamma))} |v|^2\end{aligned}$$

when $|\alpha| = 1, \alpha_4 = 0$.

Similarly,

$$\int_{\mathbf{R}^4} |\Lambda^{-1}_{\sigma(t)}(a\sigma v)|^2 = \int_{\mathbf{R}} \left( \int_{\mathbf{R}^3} \frac{1}{(<\xi> + \sigma(t))^2} |\mathcal{F}(a\sigma v)(\xi,t)|^2 d\xi \right) dt$$
$$\leq \int_{\mathbf{R}} \frac{1}{\sigma(t)^2} \left( \int_{\mathbf{R}^3} |\mathcal{F}(a\sigma v)(\xi,t)|^2 d\xi \right) dt$$
$$= \int_{\mathbf{R}} \frac{1}{\sigma(t)^2} \left( \int_{\mathbf{R}^3} |(a\sigma v)(x,t)|^2 dx \right) dt$$
$$= \int_{Cyl(\delta(\gamma))} \left( \frac{\sigma}{\sigma(t)} \right)^2 |(av)|^2 \leq C \int_{Cyl(\delta(\gamma))} |v|^2$$

since, due to the definition of $\delta(\gamma)$, we have (35). □

LEMMA 2.5. *Let $\psi$ be $K$ pseudo-convex with respect to $P$ on $\overline{\Omega}$.*
*Then there is $C$ such that*

$$\int_{\mathbf{R}^4} \sigma\big(|v|^2 + |\Lambda^{-1}_{\sigma(t)} \partial_t v|^2\big) \leq C \int_{\mathbf{R}^4} |\Lambda^{-1}_{\sigma(t)} P_\varphi v|^2$$

*for all $v \in H^2_0(Cyl(x^0; \delta(\gamma)))$ provided $\tau > C, x^0 \in \bar{\Omega}'$.*

*Proof.* We can assume that $x^0 = 0$ and we let $Cyl(\delta) = Cyl(x^0; \delta(\gamma))$. By Theorem 1.1 in [11] there exists $C$ such that the following Carleman estimate holds

$$\sum_{|\alpha|=0}^{1} \int_{Cyl(4\delta)} \sigma^{3-2|\alpha|} |\partial^\alpha v_0|^2 \leq C \int_{Cyl(4\delta)} |P_\varphi v_0|^2 \quad \text{for all } v_0 \in H^2_0(Cyl(4\delta))$$

provided $C < \gamma, C(\gamma) < \tau$.

Let $\chi \in C^\infty_0(Cyl(4\delta))$, is determined only by $\gamma$, $0 \leq \chi \leq 1$, annd $\chi = 1$ on $Cyl(3\delta)$. Using this Carleman type estimate for $v_0 = \chi \Lambda^{-1}_{\sigma(t)} v$, we obtain

$$\int_{Cyl(4\delta)} \Big( \sigma^3 \chi^2 |\Lambda^{-1}_{\sigma(t)} v|^2 + \sigma \sum_{|\alpha|=1} |\chi \partial^\alpha (\Lambda^{-1}_{\sigma(t)} v) + \partial^\alpha \chi \Lambda^{-1}_{\sigma(t)} v|^2 \Big)$$
$$\leq C \int_{Cyl(4\delta)} |P_\varphi(\chi \Lambda^{-1}_{\sigma(t)} v)|^2$$
$$\leq C \int_{Cyl(4\delta)} \Big( |P_\varphi(\Lambda^{-1}_{\sigma(t)} v)|^2 + C(\gamma) \Big( \tau^2 |\Lambda^{-1}_{\sigma(t)} v|^2 + \sum_{|\alpha|=1} |\partial^\alpha (\Lambda^{-1}_{\sigma(t)} v)|^2 \Big) \Big). \quad (37)$$

where we used (20), the Leibniz' formulas

$$P(\chi w) = \chi P w + P_1(;\chi) w + P(\chi) w,\ P_1(\chi w; \varphi) = \chi P_1(w; \varphi) + P_1(\chi; \varphi) w,$$

and the triangle inequality.

Using these inequalities, Lemma 2.4, and recalling that $\chi = 1$ on $Cyl(3\delta)$ we derive from the bound (37) that

$$\int_{Cyl(3\delta)} \Big(\sigma^3|\Lambda^{-1}_{\sigma(t)}v|^2 + \sigma \sum_{|\alpha|=1} |\partial^\alpha(\Lambda^{-1}_{\sigma(t)}v)|^2\Big) - C(\gamma)\int_{Cyl(\delta)} |v|^2$$
$$\le C\int_{Cyl(4\delta)} \Big(|P_\varphi(\Lambda^{-1}_{\sigma(t)}v)|^2 + C(\gamma)(|v|^2) + |\partial_t\Lambda^{-1}_{\sigma(t)}v|^2\Big). \quad (38)$$

The Parseval identity, (35), and the definition of $\Lambda_\sigma$ yield

$$\int_{Cyl(\delta)} \sigma v^2 \le\ 2\int_{Cyl(\delta)} \sigma(t)v^2$$
$$= \int_{\mathbf{R}} \sigma(0)\int_{\mathbf{R}^3} \frac{\sigma(t)^2+1}{<\xi>^2+\sigma(t)^2}|\hat{v}(\xi,t)|^2 d\xi dt$$
$$+ \int_{\mathbf{R}} \sigma(t)\int_{\mathbf{R}^3} \frac{|\xi|^2}{<\xi>^2+\sigma(t)^2}|\hat{v}(\xi,t)|^2 d\xi dt$$
$$= C\int_{\mathbf{R}} (\sigma(t))^3 \int_{\mathbf{R}^3} |\Lambda^{-1}_{\sigma(t)}v|^2 + \sum_{|\alpha|=1,\alpha_4=0} \int_{\mathbf{R}} \sigma(t)\int_{\mathbf{R}^3} |\partial^\alpha(\Lambda^{-1}_{\sigma(t)}v)|^2$$
$$\le C\int_{Cyl(3\delta)} \sigma^3|\Lambda^{-1}_{\sigma(t)}v|^2 + C\sum_{|\alpha|=1,\alpha_4=0} \int_{Cyl(3\delta)} \sigma|\partial^\alpha(\Lambda^{-1}_{\sigma(t)}v)|^2$$
$$+C\int_{\mathbf{R}^4\setminus Cyl(3\delta)} ((\sigma(t))^3|\Lambda^{-1}_{\sigma(t)}v|^2 + \sigma(t)\sum_{|\alpha|=1,\alpha_4=0} |\partial^\alpha(\Lambda^{-1}_{\sigma(t)}v)|^2).$$

Choosing $\tau > C(\gamma)$ and using Lemma 2.3 we will have from (38)

$$\int_{\mathbf{R}^4} \sigma\big(|v|^2 + |\partial_t\Lambda^{-1}_{\sigma(t)}v|^2\big) \le C\int_{Cyl(4\delta)} |\Lambda^{-1}_{\sigma(t)}P_\varphi v|^2$$
$$+ C(\gamma)\int_{\mathbf{R}^4\setminus Cyl(3\delta)} \Big(\tau^3|\Lambda^{-1}_{\sigma(t)}v|^2 + \tau\sum_{|\alpha|=1} |\partial^\alpha\Lambda^{-1}_{\sigma(t)}v|^2\Big). \quad (39)$$

Now, by using Lemma 2.3 and choosing again $\tau > C(\gamma)$ we will eliminate the last integral in this bound and complete the proof. □

## 3. Proof of Theorem 1.1

LEMMA 3.1. *Let* $|\nabla\psi| > 0$ *on* $\bar{\Omega}$.

*Then for any* $x \in \bar{\Omega}$ *there are* $\delta(\gamma)$ *and* $C$ *such that*

$$\gamma\int_{B(\delta)} (\sigma^2|\mathbf{v}^*|^2 + |\partial_j\mathbf{v}^*|^2) \le C\int_{B(\delta)} \sigma(|div_{T,\varphi}\mathbf{v}^*|^2 + |curl_{T,\varphi}\mathbf{v}^*|^2),\ j = 1,2,3,$$

*for all* $\mathbf{v}^* \in H_0^1(Cyl(x^0;\delta(\gamma))$ *provided* $\tau > C$.

Proof is available in [10], Lemma 5, where the spatial bound need to be integrated with respect to $t$ like in the proof of Lemma 2.1.

*Proof of Theorem 1.1.* In [10], (25) it was shown that the system (10) implies

$$P(1)\mathbf{u} = \frac{\mathbf{f}}{\rho} + \mathbf{A}(1)\mathbf{u},$$

$$P(2)v = div_T\frac{\mathbf{f}}{\rho} + A(2)\mathbf{u},$$

$$P(1)\mathbf{w} = curl_T\frac{\mathbf{f}}{\rho} + \mathbf{A}(3)\mathbf{u},$$

where

$$P(1) = \partial_t^2 - \rho^{-1}(A+R), P(2) = \partial_t^2 - \rho^{-1}(A+R+\partial_1^2+\partial_2^2+a_3a_4\partial_3^2),$$

$\mathbf{A}(j)$ are sums of $\partial_k(\mathbf{A}_1\partial_t\mathbf{u})$, $\partial_m(\mathbf{A}_1\partial_k\mathbf{u})$, $\mathbf{A}\partial_k\mathbf{u}$, $\mathbf{A}\partial_t\mathbf{u}$ $\mathbf{A}\mathbf{u}$ with the (matrix) coefficients $\mathbf{A}, \mathbf{A}_1$, $|\mathbf{A}|_1(\Omega) + |\mathbf{A}_1|_0(\Omega) \le C$, $j,k,m = 1,2,3$.

Using the the substitution $\mathbf{u}^* = e^{\tau\varphi}\mathbf{u}, v^* = e^{\tau\varphi}v, \mathbf{w}^* = e^{\tau\varphi}\mathbf{w}, \mathbf{f}^* = e^{\tau\varphi}\mathbf{f}$ this system is transformed into

$$\begin{aligned} P_\varphi(1)\mathbf{u}^* &= \frac{\mathbf{f}^*}{\rho} + \mathbf{A}_\varphi(1)\mathbf{u}^*, \\ P_\varphi(2)v^* &= div_{T,\varphi}\frac{\mathbf{f}^*}{\rho} + A_\varphi(2)\mathbf{u}^*, \\ P_\varphi(1)\mathbf{w}^* &= curl_{T,\varphi}\frac{\mathbf{f}^*}{\rho} + \mathbf{A}_\varphi(3)\mathbf{u}^*. \end{aligned} \tag{40}$$

Let $x^0 \in \bar\Omega'$ and $Cyl(\delta) = Cyl(x^0;\delta(\gamma))$ with $\delta(\gamma)$ defined in (35). Let a cut off function $\chi = 1$ on $Cyl(\frac{\delta}{2})$, $supp\chi \subset Cyl(\delta)'$, $0 \le \chi \le 1, |\chi|_2|(\mathbf{R}^4) \le C(\gamma), \partial_t\chi = 0$, then the system (40) implies

$$\begin{aligned} P_\varphi(1)(\chi\mathbf{u}^*) &= \chi\big(\frac{\mathbf{f}^*}{\rho} + \mathbf{A}_\varphi(1)\mathbf{u}^*\big) + \mathbf{A}(1,1)(\mathbf{u}^*) \\ P_\varphi(2)(\chi v^*) &= \chi\big(div_{T,\varphi}\frac{\mathbf{f}^*}{\rho} + A_\varphi(2)\mathbf{u}^*\big) + A(2,1)(v^*), \\ P_\varphi(1)(\chi\mathbf{w}^*) &= \chi\big(curl_{T,\varphi}\frac{\mathbf{f}^*}{\rho} + \mathbf{A}_\varphi(3)\mathbf{u}^*\big) + \mathbf{A}(3,1)(\mathbf{w}^*), \end{aligned} \tag{41}$$

where $\mathbf{A}(j,1)$ are sums of the terms $a(\gamma)\partial_j\mathbf{u}, a(\gamma)\partial_j v, a(\gamma)\partial_j\mathbf{w}$, $\sigma a(\gamma)\mathbf{u}, \sigma a(\gamma)v$, $\sigma a(\gamma)\mathbf{w}$ with $|a(\gamma)|_2(\Omega) < C(\gamma)$.

Using that $v^* = div_{T,\varphi}\mathbf{u}^*, \mathbf{w}^* = curl_{T,\varphi}\mathbf{u}^*$, applying Lemma 2.5 to each of 7 scalar equations in this system, and adding the resulting inequalities we yield

$$\begin{aligned}
\int_{\mathbf{R}^4} \sigma\Big(&|\chi\mathbf{u}^*|^2 + |\Lambda^{-1}_{\sigma(t)}(\chi\partial_t\mathbf{u}^*)|^2 + |\chi v^*|^2 + |\chi div_{T,\varphi}(\mathbf{u}^*)|^2 \\
&+ |\Lambda^{-1}_{\sigma(t)}(\chi\partial_t div_{T,\varphi}(\mathbf{u}^*))|^2 + |\chi\mathbf{w}^*|^2 + |\chi curl_{T,\varphi}(\mathbf{u}^*)|^2 \\
&+ |\Lambda^{-1}_{\sigma(t)}(\chi\partial_t curl_{T,\varphi}(\mathbf{u}^*))|^2\Big) \\
&\leq C\int_{\mathbf{R}^4}\chi^2\Big(|\mathbf{f}^*|^2 + \sum_{j=1}^{4}|\partial_j\mathbf{u}|^2 + \sigma^2|\mathbf{u}^*|^2\Big) \\
&+ C(\gamma)\tau^{-2}\int_{\mathbf{R}^4}\Big(|\mathbf{f}^*|^2 + \sum_{j=1}^{4}|\partial_j\mathbf{u}|^2 + \sigma^2|\mathbf{u}^*|^2\Big) \\
&+ C(\gamma)\int_{\mathbf{R}^4}(|\mathbf{u}^*|^2 + |v^*|^2 + |\mathbf{w}^*|^2). \qquad (42)
\end{aligned}$$

Observe that for a first order operator $P_1 v = \sum_{j=1}^{3} b_j\partial_j v$ we have $\chi P_{1,\varphi}v = P_{1,\varphi}(\chi v) - P_1(\chi)v$. Since $div_{T,\varphi}(\rho^{-1}\mathbf{f})$ is the sum of terms $a\partial_j f_j, \sigma a f_j$ with $|a|_1(\Omega^*) < C$, by using Lemma 2.4 we will have the terms with $\mathbf{f}^*$ on the right side of (42). Moreover, $\chi\mathbf{A}_\varphi(m)u$ is the sum of terms $(\partial_k - \sigma\partial_k\psi)(\chi(\mathbf{A}_1(\partial_1 - \sigma\partial_j\psi)\mathbf{u}^*)$ and of $\partial_k(\mathbf{A}_1(\partial_j - \sigma\partial_j\psi)\mathbf{u}^*)$, so again using Lemma 2.4 we will have remaining terms of the first two integrals on the right side of (42).

By standard calculations $\partial_t div_{T,\varphi}\mathbf{u}^* = div_{T,\varphi}\partial_t\mathbf{u}^* + r(1)$ where $r(1)$ is the sum of terms $a(\gamma)\sigma u_j^*$ and $a\partial_k u^*$ with $|a(\gamma)|_1(\Omega) < C(\gamma), |a|_1(\Omega) < C$ and $\chi div_{T,\varphi}\partial_t\mathbf{u}^* = div_{T,\varphi}(\chi\partial_t\mathbf{u}^*) + r(2)$ where $r(2)$ is the sum of terms $a(\gamma)\partial_t u_j^*$ with $|a(\gamma)|_1(\Omega) < C(\delta)$. Hence

$$\begin{aligned}
&\Lambda^{-1}_{\sigma(t)}(\chi\partial_t(div_{T,\varphi}\mathbf{u}^*)) - div_{T,\varphi}(\Lambda^{-1}_{\sigma(t)}(\chi\partial_t\mathbf{u}^*)) \\
&\quad = \Lambda^{-1}_{\sigma(t)}(\chi div_{T,\varphi}\partial_t\mathbf{u}^*) + \Lambda^{-1}_{\sigma(t)}(\chi r(1)) - div_{T,\varphi}(\Lambda^{-1}_{\sigma(t)}(\chi\partial_t\mathbf{u}^*)) \\
&\quad = \Lambda^{-1}_{\sigma(t)}(div_{T,\varphi}(\chi\partial_t\mathbf{u}^*)) - div_{T,\varphi}(\Lambda^{-1}_{\sigma(t)}(\chi\partial_t\mathbf{u}^*)) + \Lambda^{-1}_{\sigma(t)}(\chi r(1) + r(2))).
\end{aligned}$$

So using Lemma 2.1 we yield

$$\begin{aligned}
\|\sigma^{\frac{1}{2}}(\Lambda^{-1}_{\sigma(t)}&(\chi\partial_t(div_{T,\varphi}\mathbf{u}^*)) - div_{T,\varphi}(\Lambda^{-1}_{\sigma(t)}(\chi\partial_t\mathbf{u}^*)))\|_{(0)}(\mathbf{R}^4) \\
&\leq C(\gamma)\tau^{-\frac{1}{2}}\|\chi\partial_t\mathbf{u}^*\|_{(0)}(\mathbf{R}^4) + C(\gamma)(\tau^{-\frac{1}{2}}\|\partial_t\mathbf{u}^*\|_{(0)}(\mathbf{R}^4) \\
&\qquad + \tau^{-\frac{1}{2}}\|\chi\nabla\mathbf{u}^*\|_{(0)}(\mathbf{R}^4) + \tau^{\frac{1}{2}}\|\chi\mathbf{u}^*\|_{(0)}(\mathbf{R}^4)).
\end{aligned}$$

Therefore from (42) we obtain

$$\int_{\mathbf{R}^4} \sigma(|\chi \mathbf{u}^*|^2 + |\chi v^*|^2 + |\chi \mathbf{u}^*|^2 + |\Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)|^2 + |div_{T,\varphi}(\chi \mathbf{u}^*)|^2$$
$$+ |curl_{T,\varphi}(\chi \mathbf{u}^*)|^2 + |div_{T,\varphi}\Lambda^{-1}_{\sigma(t)}(\chi \partial_t(\mathbf{u}^*))|^2 + |curl_{T,\varphi}\Lambda^{-1}_{\sigma(t)}(\chi \partial_t(\mathbf{u}^*))|^2)$$
$$\leq C \int_{\mathbf{R}^4} \chi^2 \Big(|\mathbf{f}^*|^2 + \sum_{j=1}^{4} |\partial_j \mathbf{u}^*|^2 + \sigma^2 |\mathbf{u}^*|^2\Big)$$
$$+ C(\gamma)\tau^{-1} \int_{\mathbf{R}^4} \Big(|\mathbf{f}^*|^2 + \sum_{j=1}^{4} |\partial_j \mathbf{u}^*|^2 + \sigma^2 |\mathbf{u}^*|^2\Big)$$
$$+ C(\gamma) \int_{\mathbf{R}^4} (|\mathbf{u}^*|^2 + |v^*|^2 + |\mathbf{w}^*|^2). \quad (43)$$

Introducing another cut off function $\chi_1, \partial_t \chi_1 = 0$, supported in $B'(4\delta) \times \mathbf{R}$ with $\chi_1 = 1$ on $B'(3\delta) \times \mathbf{R}$, $|\chi_1|_1(\mathbf{R}^4 < C(\gamma)$, and applying Lemma 3.1 we yield

$$\int_{\mathbf{R}^4} \sigma \Big(|div_{T,\varphi}(\chi_1(\Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)))|^2 + |curl_{T,\varphi}(\chi_1(\Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)))|^2\Big)$$
$$\geq C^{-1}\gamma \int_{\mathbf{R}^4} \Big(\sigma^2 |\chi_1 \Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)|^2 + \sum_{j=1}^{3} |\partial_j(\chi_1(\Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)))|^2\Big)$$
$$\geq C^{-1}\gamma \int_{\mathbf{R}^4} \sigma^2 \Big(|\Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)|^2 + \sum_{j=1}^{3} |\partial_j \Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)|^2\Big)$$
$$- C(\gamma)\tau^{-2} \int_{\mathbf{R}^4} \sigma^2 |\chi \partial_t \mathbf{u}^*)|^2,$$

because

$$\int_{\mathbf{R}^4} \Big(\sigma^2 |(1-\chi_1)\Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)|^2 + |\partial_j((1-\chi_1)(\Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)))|^2\Big)$$
$$\leq C(\gamma)\tau^{-2} \int_{\mathbf{R}^4} \sigma^2 |\chi \partial_t \mathbf{u}^*|^2$$

due to Lemma 2.1.

As above, by using the basic Fourier analysis we yield

$$C \int_{\mathbf{R}^4} \sigma^2 \Big(|\Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)|^2 + \sum_{j=1}^{3} |\partial_j \Lambda^{-1}_{\sigma(t)}(\chi \partial_t \mathbf{u}^*)|^2\Big) \geq \int_{\mathbf{R}^4} \sigma^2 |\chi \partial_t \mathbf{u}^*|^2.$$

Using the two previous inequalities, from (43) we obtain

$$\int_{\mathbf{R}^4}\Big(\gamma\big(|\partial_j(\chi\mathbf{u}^*)|^2+\sigma^2|\chi\mathbf{u}^*|^2\big)+\sigma\chi^2\big(|v^*|^2+|\mathbf{w}^*|^2\big)\Big)$$
$$\le C\int_{\mathbf{R}^4}\chi^2\Big(|\mathbf{f}^*|^2+\sum_{j=1}^4|\partial_j\mathbf{u}^*|^2+\sigma^2|\mathbf{u}^*|^2\Big)$$
$$+C(\gamma)\tau^{-2}\int_{\mathbf{R}^4}\Big(|\mathbf{f}^*|^2+\sum_{j=1}^4|\partial_j\mathbf{u}^*|^2+\sigma^2|\mathbf{u}^*|^2\Big)$$
$$+C(\gamma)\int_{\mathbf{R}^4}\Big(|\mathbf{u}^*|^2+|v^*|^2+|\mathbf{w}^*|^2\Big). \quad (44)$$

Now the claim follows by partition of the unity argument. Since our choice of $\delta$ depends on $\gamma$ we give this argument in some detail.

The balls $B'(x^0;\delta(\gamma))$ form an open covering of the compact set $\bar{\Omega}'$, so we can find a finite covering of $\bar{\Omega}'$ by balls $B'(x(k),\delta(\gamma)), k=1,...,K(\gamma)$. Let $\chi(;k)$ be a $C^\infty$- partition of the unity subordinated to this covering, i.e. $supp\chi(;k)\subset B'(x(k);\delta(\gamma)$ with $\sum_{k=1}^K\chi^2(;k)=1$ on $\Omega$.

Summing (44) with $x=x(k),\delta=\delta(\gamma,k)$ over $k=1,...,K$ and choosing $\tau>C(\gamma)$ we get

$$\int_\Omega\left(\gamma\Big(\sum_{j=1}^4|\partial_j\mathbf{u}^*|^2+\sigma^2|\mathbf{u}^*|^2\Big)+\sigma(|v^*|^2+|\mathbf{w}^*|^2)\right)$$
$$\le(C+C(\gamma)\tau^{-2})\int_\Omega|\mathbf{f}^*|^2\;+\;C\int_\Omega\Big(\sum_{j=1}^4|\partial_j\mathbf{u}^*|^2+\sigma^2|\mathbf{u}^*|^2\Big)$$
$$+C(\gamma)\tau^{-1}\int_\Omega\Big(\sum_{j=1}^4|\partial_j\mathbf{u}^*|^2+\sigma^2|\mathbf{u}^*|^2\Big)+C(\gamma)\int_\Omega\big(|\mathbf{u}^*|^2+v^{*2}+|\mathbf{w}^*|^2\big).$$

By choosing $\gamma>2C$ we can absorb the second integral in the right side by the left side. Then we fix $\gamma$ and choosing $\tau>C(\gamma)$ absorb the third and the fourth integral by the left side and complete the proof of (9). $\square$

## 4. Proofs of stability estimates

In this section we will prove Theorems 1.3, 1.4.

*Proof of Theorem 1.3.* By extension theorems for Sobolev spaces we can find $\mathbf{u}^*\in H^2(\Omega)$ so that

$$\mathbf{u}^*=\mathbf{g}_0,\partial_\nu\mathbf{u}^*=\mathbf{g}_1 \text{ on } \Gamma$$

and

$$\|\mathbf{u}^*\|_{(2)}(\Omega) \le CF. \tag{45}$$

Let

$$\mathbf{v} = \mathbf{u} - \mathbf{u}^*. \tag{46}$$

The function $\mathbf{v}$ solves the Cauchy problem

$$(\rho\partial_t^2 - (\mathbf{A}_T + R))\mathbf{v} = \mathbf{f}^* \text{ in } \Omega,\ \mathbf{v} = 0, \quad \partial_\nu \mathbf{v} = 0 \quad \text{on} \quad \partial G \times (-T, T), \tag{47}$$

where $\mathbf{f}^* = \mathbf{f} - (\rho\partial_t^2 - (\mathbf{A}_T + R))\mathbf{u}^*$.

Due to the strict positivity condition (4) by standard energy estimates for hyperbolic systems (i.e. [3], p. 128 ) we have

$$C^{-1}\big(E(0;\mathbf{v}) - ||\mathbf{f}^*||_{(0)}(\Omega)\big) \le E(t;\mathbf{v}) \le C\big(E(0;\mathbf{v}) + ||\mathbf{f}^*||_{(0)}(\Omega)\big), \tag{48}$$

when $t \in (-T, T)$.

Let us fix $\gamma$ in Theorem 1.1. By using (12) we choose $\delta_0$ depending on the same parameters as $C$ so that $\varphi < 1 - 2\delta_0$ on $\{t : T - \delta_0 < |t| < T\}$ and $1 - \delta_0 < \varphi$ on $(-\delta_0, \delta_0)$. We choose a smooth cut-off function $0 \le \chi_0(t) \le 1$ such that $\chi_0(t) = 1$ when $|t| < T - 2\delta_0$ and $\chi_0(t) = 0$ when $|t| > T - \delta_0$. It is clear that

$$(\rho\partial_t^2 - (\mathbf{A}_T + R))(\chi_0\mathbf{v}) = \chi_0\mathbf{f}^* + 2\rho\partial_t\chi_0\partial_t\mathbf{v} + \rho\partial_t^2\chi_0\mathbf{v}. \tag{49}$$

Obviously, $\chi_0\mathbf{v} \in H_0^2(\Omega)$, hence by Theorem 1.1

$$\begin{aligned}
&\int_\Omega \big(|\partial_t^2(\chi_0\mathbf{v})|^2 + |\nabla(\chi_0\mathbf{v})|^2 + |\chi_0\mathbf{v}|^2\big)e^{2\tau\varphi} \\
&\qquad \le C\int_\Omega \big(|(\rho\partial_t^2 - (\mathbf{A}_T + R))(\chi_0\mathbf{v})|^2\big)e^{2\tau\varphi} \\
&\qquad\quad \le C\left(\int_\Omega |\mathbf{f}^*|^2 e^{2\tau\varphi} + \int_{G\times\{T-2\delta_0<|t|<T\}} (|\partial_t\mathbf{v}|^2 + |\mathbf{v}|^2)e^{2\tau\varphi}\right)
\end{aligned}$$

by (47).

Shrinking the integration domain $\Omega$ on the left side to $G \times (0, \delta_0)$ and using our choice of $\delta_0$ we yield

$$\begin{aligned}
e^{2\tau(1-\delta_0)}\int_0^{\delta_0} E(t;\mathbf{v})dt &\le C\int_{G\times(-\delta_0,\delta_0)} (|\partial_t\mathbf{v}|^2 + |\nabla\mathbf{v}|^2 + |\mathbf{v}|^2)e^{2\tau\varphi} \\
\le C\int_\Omega |\mathbf{f}^*|^2 e^{2\tau\varphi} &+ Ce^{2\tau(1-2\delta_0)}\int_{\{T-2\delta_0<|t|<T\}}\int_G (|\partial_t\mathbf{v}|^2 + |\nabla\mathbf{v}|^2 + |\mathbf{v}|^2) \\
&\le C\int_\Omega |\mathbf{f}^*|^2 e^{2\tau\varphi} + Ce^{2\tau(1-2\delta)}\int_{T-2\delta_0}^{T} E(t;\mathbf{v})dt.
\end{aligned}$$

Choosing $\Phi = \sup_{\Omega} \varphi$ and using (48)

$$e^{2\tau(1-\delta_0)}\frac{\delta}{C}E(0;\mathbf{v}) - Ce^{2\tau\Phi}||\mathbf{f}^*||^2_{(1)}(\Omega) \leq Ce^{2\tau(1-2\delta_0)}E(0;\mathbf{v}) + Ce^{2\tau\Phi}||\mathbf{f}^*||^2_{(0)}(\Omega)$$

To eliminate the first term on the right side we choose $\tau$ (depending on $C$) so large that $e^{-2\tau\delta_0} < \frac{1}{C^2}$ and by using energy estimates (48) we finally get

$$E(t;\mathbf{v}) \leq C||\mathbf{f}^*||_{(0)}(\Omega)$$

and

$$\begin{aligned} E(t;\mathbf{u}) &\leq C\big(||\mathbf{f}^*||_{(0)}(\Omega) + E(t;\mathbf{u}^*)\big) \\ &\leq C\big(||\mathbf{f}^*||_{(0)}(\Omega) + ||\mathbf{u}^*||_{(\frac{3}{2})}(\Gamma) + ||\partial_\nu \mathbf{u}^*||_{(\frac{1}{2})}(\Gamma)\big) \\ &\leq C\big(||\mathbf{f}||_{(0)}(\Omega) + ||\mathbf{g}_0||_{(\frac{3}{2})}(\Gamma) + ||\mathbf{g}_1||_{(\frac{1}{2})}(\Gamma)\big). \end{aligned}$$

The proof is complete. □

*Proof of Theorem 1.4.* By extension theorems for Sobolev spaces we can find $\mathbf{U}^* \in H^2(\Omega)$ so that

$$\mathbf{U}^* = \mathbf{0}, \partial_\nu \mathbf{U}^* = \partial_t^2 \partial_\nu \mathbf{u} \text{ on } \Gamma$$

and

$$\|\mathbf{U}^*\|_{(2)}(\Omega) \leq CF. \tag{50}$$

Let

$$\mathbf{V} = \partial_t^2 \mathbf{u} - \mathbf{U}^*. \tag{51}$$

Differentiating (14) in $t$ and using time-independence of the coefficients of the system, we get

$$\begin{aligned} &(\rho\partial_t^2 - (\mathbf{A}_T + R))\mathbf{V} = \partial_t^2 \mathcal{A}\mathbf{f} - \mathbf{F}^* && \text{in } \Omega, \\ &\mathbf{v} = 0, \quad \partial_\nu \mathbf{v} = 0 && \text{on } \partial G \times (-T,T), \end{aligned} \tag{52}$$

where $\mathbf{F}^* = (\rho\partial_t^2 - (\mathbf{A}_T + R))\mathbf{U}^*$.

By standard energy estimates for hyperbolic systems (i.e. [3])

$$C^{-1}E(0;\mathbf{V}) - C\Big(\int_G |\mathbf{f}|^2 + \int_\Omega |\mathbf{F}^*|^2\Big) \leq E(t;\mathbf{V}) \leq CE(0;\mathbf{V}) + C\Big(\int_G |\mathbf{f}|^2 + \int_\Omega |\mathbf{F}^*|^2\Big), \tag{53}$$

when $t \in (-T,T)$.

Using (12) we choose $\delta_0$ depending on the same parameters as $C$ so that $\psi < -\delta_0$ on $G \times \{t : T - \delta_0 < |t| < T\}$ and $0 < \psi$ on $G \times (-\delta_0, \delta_0)$. Then we fix a smooth cut-off function $\chi_0, 0 \leq \chi_0(t) \leq 1$ such that $\chi_0(t) = 1$ when $|t| < T - 2\delta_0$ and $\chi_0(t) = 0$ when $|t| > T - \delta_0$, $0 \leq \chi_0 \leq 1$, $|\partial_t^j \chi_0| \leq C, j = 0, 1, 2$. By the Leibniz formula

$$(\rho\partial_t^2 - (\mathbf{A}_T + R))(\chi_0 \mathbf{V}) = \chi_0(\partial_t^2 \mathcal{A}\mathbf{f} - \partial_t^j \mathbf{f}^*) + 2\rho\partial_t\chi_0\partial_t\mathbf{V} + \rho\partial_t^2\chi_0\mathbf{V}.$$

Obviously, $\chi_0\mathbf{V} \in H_0^2(\Omega)$, hence by Theorem 1.1

$$\int_\Omega \gamma\Big(|\partial_t(\chi_0\mathbf{V})|^2 + |\nabla(\chi_0\mathbf{V})|^2 + \sigma^2|(\chi_0\mathbf{V})|^2\Big)e^{2\tau\varphi}$$
$$\leq C\Big(\int_\Omega (|\mathbf{f}|^2 + |\mathbf{F}^*|^2)e^{2\tau\varphi} + \int_{G\times\{T-2\delta_0<|t|<T\}} (|\partial_t\mathbf{V}|^2 + |\mathbf{V}|^2)e^{2\tau\varphi}\Big). \quad (54)$$

We have

$$\mathbf{V}(,0)e^{\tau\varphi(,0)} = -\int_0^T \partial_s\Big((\chi_0\mathbf{V}(,s))e^{\tau\varphi(,s)}\Big)ds$$
$$= -\int_0^T \Big(\partial_s(\chi_0\mathbf{V}(,s)) + \sigma\partial_s\psi(,s)\chi_0\mathbf{V}(,s)\Big)e^{\tau\varphi(,s)}ds.$$

So by splitting the left side in (54) into two equal terms and using the Cauchy-Schwarz inequality we obtain

$$\gamma\int_G |\mathbf{V}(,0))|^2 e^{2\tau\varphi(,0)} + e^{2\tau}\int_{-\delta_0}^{\delta_0} E(,t;\mathbf{V})dt$$
$$\leq C\Big(\int_\Omega (|\mathbf{f}|^2 + |\mathbf{F}^*|^2)e^{2\tau\varphi} + e^{2\tau\theta}\int_{\{T-2\delta_0<|t|<T\}} E(t;\mathbf{V})dt\Big),$$

where $\theta = e^{-\gamma\delta_0} < 1$. Using (53) and the inequality

$$|f| \leq C|\partial_t^2\mathbf{u}(,0)| \leq C(|\mathbf{U}^*(,0)| + |\mathbf{V}(,0)|)$$

(due to (52) at $t = 0$, the condition (16), and to (51) ) we yield

$$\gamma\int_G |\mathbf{f}|^2 e^{2\tau\varphi(,0)} - C\gamma\int_G |\mathbf{U}^*(,0)|^2 e^{2\tau\varphi(,0)}$$
$$+ e^{2\tau}E(0;\mathbf{V}) - Ce^{2\tau}\Big(\int_G |\mathbf{f}|^2 + \int_\Omega |\mathbf{F}^*|^2\Big)$$
$$\leq C\int_G |\mathbf{f}|^2 e^{2\tau\varphi(,0)} + Ce^{2\tau\theta}\int_\Omega |\mathbf{F}^*|^2 + Ce^{2\tau\theta}E(0;\mathbf{V}) + Ce^{2\tau\theta}\int_G |\mathbf{f}|^2.$$

We choose and fix large $\gamma$ (depending on $C$ only) to absorb three other terms with $\mathbf{f}$ by the first term on the left side. Then we choose and fix $\tau$ (depending on $C$) so large that $Ce^{2\tau\theta} < e^{2\tau}$ to absorb the term with $\mathbf{V}$ on the right by the term with $\mathbf{V}$ on the left and arrive at

$$\int_G |\mathbf{f}|^2 \leq C \int_G |\mathbf{U}^*(,0)|^2 + C \int_\Omega |\mathbf{F}^*|^2.$$

Using the bound (50), Trace theorems and the definition of $\mathbf{F}^*$, we complete the proof of Theorem 1.4. □

## 5. Conclusion

One can use Carleman estimates of Theorem 1.1 for coefficients identification as in [7, 11]. However, for systems in divergent form (like the elasticity system) most precise results need a weak form of Carleman estimate which is expected to follow from Theorem 1.1 by using smoothing operators similar to $\Lambda_\sigma^{-1}$. At present, the condition (7) is essential for our proofs. Geometrical or mechanical meaning of this condition is not clear. A challenge is to obtain Carleman estimates and identification results for the system of transversely isotropic elasticity, without (or with relaxed) condition (7).

### Acknowledgments

This work was supported in part by the Emylou Keith and Betty Dutcher Distinguished Professorship and the NSF grant DMS 15-14886.

### References

[1] R. Coifman and Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[2] H. Ding, W. Chen, and L. Zhang, *Elasticity of transversely isotropic materials*, Solid mechanics and its applications, no. 126, Springer, Dordrecht, 2006.

[3] G. Duvaut and J.-L. Lions, *Inequalities in mechanics and physics*, Grundlagen der mathematischen Wissenschaften, no. 219, Springer, Berlin-Heidelberg-New York, 1976.

[4] M. Eller and V. Isakov, *Carleman estimates with two large parameters and applications*, Contemp. Math., AMS **268** (2000), 117–137.

[5] M. Eller, V. Isakov, G. Nakamura, and D. Tataru, *Uniqueness and stability in the Cauchy problem for Maxwell's and the elasticity system*, "Nonlinear Partial Differential Equations", Collège de France Seminar **14** (2002), 329–349.

[6] L. Hörmander, *Linear partial differential operators*, Grundlagen der mathematischen Wissenschaften, no. 116, Springer, Berlin-Heidelberg-New York, 1963.

[7] O. Imanuvilov, V. Isakov, and M. Yamamoto, *An inverse problem for the dynamical Lamé system with two sets of boundary data*, Comm. Pure Appl. Math. **56** (2003), 117–137.

[8] V. Isakov, *A nonhyperbolic Cauchy problem for $\Box_b \Box_c$ and its applications to elasticity theory*, Comm. Pure Appl. Math. **39** (1986), 747–767.

[9] V. Isakov, *Inverse problems for partial differential equations*, Applied Mathematical Sciences, no. 127, Springer, New York, 2006.

[10] V. Isakov, *Carleman estimates for some anisotropic elasticity systems and applications*, Evolution Equations and Control Theory **1** (2012), 141–154.

[11] V. Isakov and N. Kim, *Carleman estimates with two large parameters for second order operators and applications to elasticity with residual stress*, Discr. Cont. Dyn. Syst. A **27** (2010), 799–827.

[12] V. Isakov, J.-N. Wang, and M. Yamamoto, *Uniqueness and stability of determining the residual stress by one measurement*, Comm. Part. Diff. Equat. **23** (2007), 833–848.

[13] A. Khaidarov, *Carleman estimates and inverse problems for second order hyperbolic equations*, Math. USSR Sbornik **58** (1987), 267–277.

[14] C.-S. Lin, G. Nakamura, and M. Sini, *Unique continuation for the elastic transversely isotropic dynamical systems and its application*, J. Diff. Equat. **245** (2008), 3008–3024.

[15] C.-S. Man, *Hartig's law and linear elasticity with initial stress*, Inverse Problems **14** (1998), 313–320.



Author's address:

Victor Isakov
Department of Mathematics, Statistics and Physics
Wichita State University
Wichita, KS 67260, USA
E-mail: `victor.isakov@wichita.edu`



Received January 6, 2016
Revised May 6, 2016
Accepted May 12, 2016