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Quantitative uniqueness for zero-order perturbations of generalized Baouendi-Grushin operators

Agnid Banerjee and Nicola Garofalo

Dedicated to Giovanni Alessandrini, on his 60-th birthday, with great affection and admiration

Abstract. *Based on a variant of the frequency function approach of Almgren, we establish an optimal bound on the vanishing order of solutions to stationary Schrödinger equations associated to a class of subelliptic equations with variable coefficients whose model is the so-called Baouendi-Grushin operator. Such bound provides a quantitative form of strong unique continuation that can be thought of as an analogue of the recent results of Bakri and Zhu for the standard Laplacian.*

Keywords: quantitative uniqueness, Baouendi-Grushin operators.
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1. Introduction

In this note we study quantitative uniqueness for zero-order perturbations of variable coefficient subelliptic equations whose "constant coefficient" model is the so called Baouendi-Grushin operator. Precisely, in $\mathbb{R}^N$, with $N = m + k$, we analyze equations of the form

$$\sum_{i=1}^{N} X_i(a_{ij}(z,t)X_j u) = V(z,t)u, \tag{1}$$

where $z \in \mathbb{R}^m$, $t \in \mathbb{R}^k$, and the vector fields $X_1, ..., X_N$ are given by

$$X_i = \partial_{z_i}, \quad i = 1, ...m, \qquad X_{m+j} = |z|^\beta \partial_{t_j}, \quad j = 1, ...k, \quad \beta > 0. \tag{2}$$

Besides ellipticity, the $N \times N$ matrix-valued function $A(z,t) = [a_{ij}(z,t)]$ is requested to satisfy certain structural hypothesis that will be specified in (20), (21) in Section 2 below. These assumptions reduce to the standard Lipschitz continuity when the dimension $k = 0$, or the parameter $\beta \to 0$. The assumptions on the potential function $V(z,t)$ are specified in (22) below. They represent the counterpart, with respect to the non-isotropic dilations associated with the vector fields $X_1, ..., X_N$, of the requirements

$$|V(x)| \leq M, \qquad | < x, DV(x) > | \leq M, \tag{3}$$

for the classical Schrödinger equation $\Delta u = Vu$ in $\mathbb{R}^n$. To put this paper in the proper historical perspective we recall that for this operator, and under the hypothesis (3), quantitative unique continuation results akin to our have been recently obtained in [2], by Carleman estimates, and in [18], by means of a variant of Almgren's frequency function introduced in [17]. In these papers the authors established sharp estimates on the order of vanishing of solution to Schrödinger equations which generalized those in [6] and [7] for eigenvalues of the Laplacian on a compact manifold. Our results should be seen as a generalization of those in [2] and [18] to subelliptic equations such as (1) above. As the reader will realize such generalization is made possible by the combination of several quite non-trivial geometric facts that beautifully combine. Some of these facts are based on the previous work [13]. We also mention that the frequency approach in [17] and [18] has been recently extended in [3] to obtain sharp quantitative estimates at the boundary of Dini domains for more general elliptic equations with Lipschitz principal part.

When in (1) we take $[a_{ij}] = I_N$, the identity matrix in $\mathbb{R}^N$, then the operator in the left-hand side of (1) reduces to the well known Baouendi-Grushin operator

$$\mathcal{B}_\beta u = \sum_{i=1}^N X_i^2 u = \Delta_z u + |z|^{2\beta} \Delta_t u, \tag{4}$$

which is degenerate elliptic along the k-dimensional subspace $M = \{0\} \times \mathbb{R}^k$. We observe that $\mathcal{B}_\beta$ is not translation invariant in $\mathbb{R}^N$. However, it is invariant with respect to the translations along M. When $\beta = 1$ the operator $\mathcal{B}_\beta$ is intimately connected to the sub-Laplacians in groups of Heisenberg type. In such Lie groups, in fact, in the exponential coordinates with respect to a fixed orthonormal basis of the Lie algebra the sub-Laplacian is given by

$$\Delta_H = \Delta_z + \frac{|z|^2}{4} \Delta_t + \sum_{\ell=1}^k \partial_{t_\ell} \sum_{i<j} b_{ij}^\ell (z_i \partial_{z_j} - z_j \partial_{z_i}), \tag{5}$$

where b_{ij}^ℓ indicate the group constants. If u is a solution of Δ_H that further annihilates the symplectic vector field $\sum_{\ell=1}^k \partial_{t_\ell} \sum_{i<j} b_{ij}^\ell (z_i \partial_{z_j} - z_j \partial_{z_i})$, then we see that, in particular, u solves (up to a normalization factor of 4) the operator $\mathcal{B}_\beta$ obtained by letting $\beta = 1$ in (4) above.

We recall that a more general class of operators modeled on $\mathcal{B}_\beta$ was first introduced by Baouendi, who studied the Dirichlet problem in weighted Sobolev spaces in [4]. Subsequently, Grushin in [14, 15] studied the hypoelliptcity of the operator $\mathcal{B}_\beta$ when $\beta \in \mathbb{N}$, and showed that this property is drastically affected by addition of lower order terms.

In the paper [10] the second named author introduced a frequency function associated with $\mathcal{B}_\beta$, and proved that such frequency is monotone nondecreasing on solutions of $\mathcal{B}_\beta u = 0$. Such result, which generalized Almgren's in [1], was used to establish the strong unique continuation property for $\mathcal{B}_\beta$. The results in [10] were extended to more general equations of the form (1) by the second named author and Vassilev in [13], following the circle of ideas in the works [11, 12]. We mention that a version of the Almgren type monotonicity formula for $\mathcal{B}_\beta$ played an extensive role also in the recent work [5] on the obstacle problem for the fractional Laplacian. Remarkably, the operator $\mathcal{B}_\beta$ also played an important role in the recent work [16] on the higher regularity of the free boundary in the classical Signorini problem.

We can now state our main result.

THEOREM 1.1. *Let u be a solution to* (1) *in B_{10} such that (a_{ij}) satisfy* (20), (21) *and V satisfy* (22) *below. We furthermore assume that $X_iX_ju \in L^2_{loc}(B_{10})$ and $|u| \le C_0$. Then, there exist universal $R_1 > 0, a \in (0, 1/3)$, depending only on $\overline{R}, \Lambda$ in* (20), (21)*, and constants C_1, C_2 depending on $m, k, \beta, \lambda, \Lambda, C_0$ and $\int_{B_{\frac{R_1}{3}}} u^2\psi$, such that for all $0 < r < aR_1$ one has*

$$||u||_{L^\infty(B_r)} \ge C_1 \left(\frac{r}{R_1}\right)^{C_2\sqrt{K}}. \tag{6}$$

It is worth emphasizing that, when $k = 0$, we have $N = m$ and then (14) below gives $\psi \equiv 1$. In such a case the constant K in (22) below can be taken to be $||V||_{W^{1,\infty}} + 1$. We thus see that Theorem 1.1, when $A \equiv I_N$, reduces to the cited Euclidean result in [2] and [18]. Therefore, Theorem 1.1 can be thought of as a subelliptic generalization of this sharp quantitative uniqueness result for the standard Laplacian. We also would like to mention that, to the best of our knowledge, Theorem 1.1 is new even for $\mathcal{B}_\beta u = Vu$ where $\mathcal{B}_\beta$ is as in (4).

The present paper is organized as follows. In Section 2 we introduce the basic notations and gather some crucial preliminary results from [10] and [13]. In Section 3 we establish a monotonicity theorem for a generalized frequency. Such result plays a central role in this paper. In Section 4, we finally prove our main result, Theorem 1.1 above.

2. Notations and preliminary results

Henceforth in this paper we follow the notations adopted in [10] and [13], with one notable proviso: the parameter $\beta > 0$ in (2), (4), etc. in this paper plays the role of $\alpha > 0$ in [10] and [13]. The reason for this is that we have reserved the greek letter α for the powers of the weight $(r^2 - \rho)^\alpha$ in definitions (30), (31) and (32) below. Let $\{X_i\}$ for $i = 1, ...N$ be defined as in (2). We denote an arbitrary point in $\mathbb{R}^N$ as $(z, t) \in \mathbb{R}^m \times \mathbb{R}^k$. Given a function f, we denote

$$Xf = (X_1 f, ...X_N f), \qquad |Xf|^2 = \sum_{i=1}^{N} (X_i f)^2, \tag{7}$$

respectively the intrinsic gradient and the square of its length. We recall from [10] that the following family of anisotropic dilations are associated with the vector fields in (2)

$$\delta_a(z, t) = (az, a^{\beta+1} t), \qquad a > 0. \tag{8}$$

Let

$$Q = m + (\beta + 1)k. \tag{9}$$

Since denoting by $dzdt$ Lebesgue measure in $\mathbb{R}^N$ we have $d(\delta_a(z, t)) = a^Q dzdt$, the number Q plays the role of a dimension in the analysis of the operator $\mathcal{B}_\beta$. For instance, one has the following remarkable fact (see [10]) that the fundamental solution Γ of $\mathcal{B}_\beta$ with pole at the origin is given by the formula

$$\Gamma(z, t) = \frac{C}{\rho(z, t)^{Q-2}}, \qquad (z, t) \neq (0, 0),$$

where ρ is the pseudo-gauge

$$\rho(z, t) = (|z|^{2(\beta+1)} + (\beta + 1)^2 |t|^2)^{\frac{1}{2(\beta+1)}}. \tag{10}$$

We respectively denote by

$$B_r = \{(z, t) \in \mathbb{R}^N \mid \rho(z, t) < r\}, \qquad S_r = \{(z, t) \in \mathbb{R}^N \mid \rho(z, t) = r\},$$

the gauge pseudo-ball and sphere centered at 0 with radius r. The infinitesimal generator of the family of dilations (8) is given by the vector field

$$Z = \sum_{i=1}^{m} z_i \partial_{z_i} + (\beta + 1) \sum_{j=1}^{k} t_j \partial_{y_j}. \tag{11}$$

We note the important facts that

$$\operatorname{div} Z = Q, \qquad [X_i, Z] = X_i, \quad i = 1, ..., N. \tag{12}$$

A function v is δ_a-homogeneous of degree κ if and only if $Zv = \kappa v$. Since ρ in (10) is homogeneous of degree one, we have

$$Z\rho = \rho. \tag{13}$$

We also need the angle function ψ introduced in [10]

$$\psi = |X\rho|^2 = \frac{|z|^{2\beta}}{\rho^{2\beta}}. \tag{14}$$

The function ψ vanishes on the characteristic manifold $M = \mathbb{R}^n \times \{0\}$ and clearly satisfies $0 \leq \psi \leq 1$. Since ψ is homogeneous of degree zero with respect to (8), one has

$$Z\psi = 0. \tag{15}$$

A first basic assumption on the matrix-valued function $A = [a_{ij}]$ is that it be symmetric and uniformly elliptic. I.e., $a_{ij} = a_{ji}$, $i, j = 1, ..., N$, and there exists $\lambda > 0$ such that for every $(z,t) \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^N$ one has

$$\lambda|\eta|^2 \leq < A(z,t)\eta, \eta > \leq \lambda^{-1}|\eta|^2. \tag{16}$$

On the potential V we preliminarily assume that $V \in L^\infty_{loc}(\mathbb{R}^N)$. With these hypothesis in place we can introduce the notion of weak solution of (1).

Definition 2.1. *A weak solution to* (1) *in an open set* $\Omega \subset \mathbb{R}^N$ *is a function* $u \in L^2_{loc}(\Omega)$ *such that the distributional horizontal gradient* $Xu \in L^2_{loc}(\Omega)$, *and for which the following equality holds for all* $\varphi \in C^\infty_0(\Omega)$

$$\int_\Omega < AXu, X\varphi > = \int_\Omega Vu\varphi. \tag{17}$$

We note that when $A \equiv I_N$, and for a class of vector fields which are modeled on (2) above, in the pioneering paper [9] it was proved that a weak solution u to (1) is locally Hölder continuous in Ω with respect to the control metric associated with the vector fields (2). In particular, it is continuous with respect to the Euclidean topology of $\mathbb{R}^N$. For the general situation of (17) the local Hölder continuity of weak solutions can be proved essentially following [9], but see also [8] where such result is discussed for more general equations in the case in which $V = 0$ in (17) above. In this paper, however, all we need is the local boundedness of weak solutions of (17), and we do assume it a priori in Theorem 1.1 above, so we do not need to derive it.

Throughout the paper we assume that

$$A(0,0) = I_N, \tag{18}$$

where I_N indicates the identity matrix in $\mathbb{R}^N$. In order to state our main assumptions (H) on the matrix A it will be useful to represent the latter in the following block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

Here, the entries are respectively $m \times m,\ m \times k,\ k \times m$ and $k \times k$ matrices, and we assume that $A_{12}^t = A_{21}$. We shall denote by B the matrix

$$B = A - I_N,$$

and thus

$$B(0,0) = O_N, \tag{19}$$

thanks to (18). The proof of Theorem 1.1 relies crucially on the following assumptions on the matrix A. These will be our main hypothesis and, without further mention, will be assumed to hold throughout the paper.

HYPOTHESIS. *There exists a positive constant* Λ *such that, for some* $\overline{R} > 0$*, one has in* $B_{\overline{R}}$ *the following estimates*

$$|b_{ij}| = |a_{ij} - \delta_{ij}| \ \le \ \begin{cases} \Lambda\rho, & \textit{for} \ \ 1 \le i,\, j \le m, \\ \\ \Lambda\psi^{\frac{1}{2}+\frac{1}{2\beta}}\rho \ = \ \Lambda\frac{|z|^{\beta+1}}{\rho^{\beta}}, & \textit{otherwise}, \end{cases} \tag{20}$$

$$|X_k b_{ij}| = |X_k a_{ij}| \ \le \ \begin{cases} \Lambda, & \textit{for}\ 1 \le k \le m, \ \textit{and}\ 1 \le i,\, j \le m, \\ \\ \Lambda\psi^{\frac{1}{2}} \ = \ \Lambda\frac{|z|^{\beta}}{\rho^{\beta}}, & \textit{otherwise}. \end{cases} \tag{21}$$

REMARK 2.2. We note that in the situation when $k = 0$ the above hypothesis coincide with the usual Lipschitz continuity at the origin of the coefficients a_{ij}.

Now we assume that V in (1) satisfy the following hypothesis for some $K \ge 0$

$$|V| \le K\psi, \qquad |ZV| \le K\psi, \tag{22}$$

where ψ indicates the function introduced in (14) above. Without loss of generality we assume henceforth that $K \ge 1$.

We next collect several preliminary results established in [13] that will be important in the proof of Theorem 1.1. We consider the quantity

$$\mu =< AX\rho, X\rho > . \tag{23}$$

We note that, by the uniform ellipticity (16) of A, the function μ is comparable to ψ defined in (14), in the sense that

$$\lambda\psi \le \mu \le \lambda^{-1}\psi. \tag{24}$$

By (24) it is clear that, similarly to ψ, the function μ vanishes on the characteristic manifold $M = \{(0,t) \in \mathbb{R}^N \mid t \in \mathbb{R}^k\}$. The following vector field F

introduced in [13] will play an important role in this paper:

$$F = \frac{\rho}{\mu} \sum_{i,j=1}^{N} a_{ij} X_i \rho X_j. \tag{25}$$

It is clear that F is singular on M. However, using (29) below and the assumptions (20), (21) on the matrix A, it was shown in [13] that F can be extended to all of $\mathbb{R}^N$ to a continuous vector field that, near the characteristic manifold M, gives a small perturbation of the Euler vector field Z in (11) above, but see also the Remark 2.3 below. We note from (25) that

$$F\rho = \rho. \tag{26}$$

More in general, the action of F on a function u is given by

$$Fu = \frac{\rho}{\mu} < AX\rho, Xu > . \tag{27}$$

We also let

$$\sigma =< BX\rho, X\rho >= \mu - \psi. \tag{28}$$

As in (2.13) in [13], F can be represented in the following way

$$F = Z - \frac{\sigma}{\mu} Z + \frac{\rho}{\mu} \sum_{i,j=1}^{N} b_{ij} X_i \rho X_j. \tag{29}$$

Remark 2.3. We emphasize that when $A(z,t) \equiv I_N$, then $B(z,t) \equiv 0_N$. In such case we immediately see from (29) that $F \equiv Z$.

Henceforth, for any two vector fields U and W, $[U,W] = UW - WU$ denotes their commutator. In the next theorem we collect several important estimates that have been established in [10] and [13].

Theorem 2.4. *There exists a constant $C(\beta, \lambda, \Lambda, N) > 0$ such that for any function u one has:*

(i) $|Q - \operatorname{div} F| \leq C\rho$;

(ii) $|F\mu| \leq C\rho\psi$;

(iii) $\operatorname{div}(\frac{\sigma Z}{\mu}) \leq C\rho$;

(iv) $|X_i \rho| \leq \psi^{1+\frac{1}{2\beta}}, \quad i = 1, ..., m, \qquad |X_{m+j}\rho| \leq (\beta+1)\rho^{1/2}, \quad j = 1, ..., k$;

(v) $|F - Z| \leq C\rho^2$;

(vi) $| < FAXu, Xu > | \leq C\rho |Xu|^2$;

(vii) $|[X_i, F]u - X_i u| \leq C\rho |Xu|, \quad i = 1, ..., N;$

(viii) $|\sigma| \leq C\rho\psi^{3/2+\frac{1}{2\beta}} \ |X\sigma| \leq C\psi^{3/2};$

(ix) $|\frac{b_{ij}X_j\rho X_i}{\mu}| \leq C|z|;$

(x) $|X_i\psi| \leq \frac{C\beta\psi}{|z|}, i = 1, ..., m, \quad |X_{n+j}\psi| \leq \frac{C\beta\psi}{\rho}, j = 1, ..., k;$

(xi) $|\frac{\sigma}{\mu}| \leq C\rho\psi, \ |Z\sigma| \leq C\rho\psi, \ |X_k\sigma| \leq C\psi^{3/2};$

(xii) $|[X_i, -\frac{\sigma Z}{\mu}]u| \leq C\rho|Xu|,$ *(Lemma 2.7 in [13]);*

(xiii) $|[X_\ell, \frac{\rho}{\mu}\sum_{i,j=1}^N \frac{b_{ij}X_j\rho}{X}{}_i]u| \leq C\rho|Xu|, \ \ \ell = 1, ..., N.$

The properties expressed in (i) and (vii) should be compared with (12) above.

3. Monotonicity of a generalized frequency

Henceforth, we denote by u a weak solution to (1) in B_{10}. For the sake of brevity in all the integrals involved we will routinely omit the variable of integration $(z, t) \in \mathbb{R}^N$, as well as Lebesgue measure $dzdt$. When we say that a constant is universal, we mean that it depends exclusively on m, k, β, on the ellipticity bound λ on $A(z, t)$, see (16) above, and on the Lipschitz bound Λ in (20), (21). Likewise, we will say that $O(1)$, $O(r)$, etc. are universal if $|O(1)| \leq C$, $|O(r)| \leq Cr$, etc., with $C \geq 0$ universal.

For $0 < r < \overline{R}$, where $\overline{R}$ is as in the hypotheses (20), (21) above, we define the generalized *height function* of u in B_r as follows

$$H(r) = \int_{B_r} u^2(r^2 - \rho^2)^\alpha \mu, \tag{30}$$

where ρ is the pseudo-gauge in (10) above, the function μ is defined in (23), and $\alpha > -1$ is going to be fixed later (precisely, in passing from (55) to (56) below). We also introduce the *generalized energy* of u in B_r

$$I(r) = \int_{B_r} < AXu, Xu > (r^2 - \rho^2)^{\alpha+1} + \int_{B_r} Vu^2(r^2 - \rho^2)^{\alpha+1}, \tag{31}$$

where, besides (16), the $N \times N$ matrix-valued function $A(z, t)$ fulfills the requirements (20), (21) above, whereas the potential $V(z, t)$ satisfies the hypothesis (22) above. We define the *generalized frequency* of u as follows

$$N(r) = \frac{I(r)}{H(r)}. \tag{32}$$

The central result of this section is the following monotonicity result for the frequency $N(r)$.

THEOREM 3.1. *There exists $R_1 > 0$, depending only on $\overline{R}$ and Λ in* (20), (21), *such that the function*

$$r \to e^{C_1 r}(N(r) + C_2 K r^2),$$

is monotone non-decreasing on the interval $(0, R_1)$. *Here, C_1 and C_2 are two universal nonnegative numbers.*

The proof of Theorem 3.1 will be divided into several steps. We begin by noting that although the gauge ρ in (10) above is not smooth at the origin, nevertheless all subsequent calculations can be justified by integrating over the set $B_r - B_\varepsilon$, and then let $\varepsilon \to 0$. Moreover, by standard approximation type arguments as in [13] which crucially use the estimates in Theorem 2.4, we can assume that all the computations hereafter are classical. The initial step in the proof of Theorem 3.1 is the following result that provides a crucial alternative representation of the generalized energy (31).

LEMMA 3.2. *For every $0 < r < \overline{R}$ one has*

$$I(r) = 2(\alpha + 1) \int_{B_r} uFu(r^2 - \rho^2)^\alpha \mu. \tag{33}$$

Proof. Using the definition of F, the divergence theorem and (1), we find

$$\begin{aligned} 2(\alpha+1) \int_{B_r} uFu(r^2 - \rho^2)^\alpha \mu &= -\int_{B_r} u < AXu, X(r^2 - \rho^2)^{\alpha+1} > \\ &= \int_{B_r} < AXu, Xu > (r^2 - \rho^2)^{\alpha+1} + \int_{B_r} Vu^2 (r^2 - \rho^2)^{\alpha+1}, \end{aligned}$$

which proves (33) above. □

LEMMA 3.3 (First variation formula for $H(r)$). *There exists a universal $O(1)$ such that for every $r \in (0, \overline{R})$ one has*

$$H'(r) = \frac{2\alpha + Q}{r} H(r) + O(1) H(r) + \frac{1}{(\alpha+1)r} I(r). \tag{34}$$

Proof. Differentiating (30), and using the fact that $(r^2 - \rho^2)^\alpha$ vanishes on S_r, we find that

$$H'(r) = 2\alpha r \int_{B_r} u^2 (r^2 - \rho^2)^{\alpha - 1} \mu.$$

Using the identity

$$(r^2-\rho^2)^{\alpha-1} = \frac{1}{r^2}(r^2-\rho^2)^{\alpha} + \frac{\rho^2}{r^2}(r^2-\rho^2)^{\alpha-1},$$

the latter equation can be rewritten as

$$H'(r) = \frac{2\alpha}{r}H(r) + \frac{2\alpha}{r}\int_{B_r} u^2(r^2-\rho^2)^{\alpha-1}\rho^2\mu.$$

Recalling (26), we have

$$H'(r) = \frac{2\alpha}{r}H(r) - \frac{1}{r}\int_{B_r} u^2 F(r^2-\rho^2)^{\alpha}\mu.$$

Integrating by parts, we obtain

$$\begin{aligned} H'(r) &= \frac{2\alpha}{r}H(r) + \frac{1}{r}\int_{B_r} \operatorname{div}(\mu u^2 F)(r^2-\rho^2)^{\alpha} \\ &= \frac{2\alpha}{r}H(r) + \frac{2}{r}\int_{B_r} uFu(r^2-\rho^2)^{\alpha}\mu \\ &+ \frac{1}{r}\int_{B_r} u^2 \operatorname{div}(F)(r^2-\rho^2)^{\alpha}\mu + \frac{1}{r}\int_{B_r} u^2(r^2-\rho^2)^{\alpha}F\mu. \end{aligned}$$

Using (i) in Theorem 2.4 to estimate the third term in the right-hand side, and (ii) to estimate the forth one, we obtain

$$H'(r) = \frac{2\alpha+Q}{r}H(r) + O(1)H(r) + \frac{2}{r}\int_{B_r} uFu(r^2-\rho^2)^{\alpha}\mu. \tag{35}$$

Using (33) in (35) we conclude that (34) holds. □

Our next result is a basic first variation formula of the generalized energy $I(r)$. Its proof will be quite laborious, and it displays many of the beautiful geometric properties of the Baouendi-Grushin vector fields (2).

Lemma 3.4 (First variation formula for $I(r)$). *There exists a universal $O(1)$ and R_1 depending on $\overline{R}, \Lambda$ as in (20), (21) such that for every $r \in (0, R_1)$ one has*

$$\begin{aligned} I'(r) = &\frac{2\alpha+Q}{r}I(r) \\ &+ \frac{4(\alpha+1)}{r}\int_{B_r} (Fu)^2(r^2-\rho^2)^{\alpha}\mu + O(1)I(r) + O(1)KrH(r), \end{aligned} \tag{36}$$

where $K \geq 1$ is the constant in (22).

Proof. Differentiating the expression (31) of $I(r)$ we obtain,

$$I'(r) = 2(\alpha+1)r\int_{B_r} < AXu, Xu > (r^2-\rho^2)^\alpha \\ + 2(\alpha+1)r\int_{B_r} Vu^2(r^2-\rho^2)^\alpha.$$

Using the identity

$$(r^2-\rho^2)^\alpha = \frac{1}{r^2}(r^2-\rho^2)^{\alpha+1} + \frac{\rho^2}{r^2}(r^2-\rho^2)^\alpha,$$

we find

$$I'(r) = \frac{2(\alpha+1)}{r}\int_{B_r} < AXu, Xu > (r^2-\rho^2)^{\alpha+1} \\ + \frac{2(\alpha+1)}{r}\int_{B_r} < AXu, Xu > (r^2-\rho^2)^\alpha\rho^2 \\ + 2(\alpha+1)r\int_{B_r} Vu^2(r^2-\rho^2)^\alpha. \quad (37)$$

The second term in the right-hand side of (37) is dealt with as follows

$$\frac{2(\alpha+1)}{r}\int_{B_r} < AXu, Xu > (r^2-\rho^2)^\alpha\rho^2 \\ = -\frac{1}{r}\int_{B_r} < AXu, Xu > F(r^2-\rho^2)^{\alpha+1}. \quad (38)$$

To compute the integral in the right-hand side of (38) we now use the following Rellich type identity in Lemma 2.11 in [13]:

$$\int_{\partial B_r} < AXu, Xu >< G, \nu >= 2\int_{\partial B_r} a_{ij}X_iu < X_j, \nu > Gu \\ - 2\int_{B_r} a_{ij}(\operatorname{div} X_i)X_juGu - 2\int_{B_r} a_{ij}X_iu[X_j, G]u + \int_{B_r} \operatorname{div} G < AXu, Xu > \\ + \int_{B_r} < (GA)Xu, Xu > -2\int_{B_r} GuX_i(a_{ij}X_ju), \quad (39)$$

where G is a vector field, GA is the matrix with coefficients Ga_{ij}, ν denotes the outer unit normal to B_r, and the summation convention over repeated indices has been adopted. Since for the vector fields $X_1, ..., X_N$ in (2) above we have

$\operatorname{div} X_i = 0$, if in (39) we take a vector field such that $G \equiv 0$ on ∂B_r, we obtain

$$\int_{B_r} \operatorname{div} G < AXu, Xu >= 2 \int_{B_r} a_{ij} X_i u [X_j, G] u - \int_{B_r} < (GA) Xu, Xu > +2 \int_{B_r} G u X_i (a_{ij} X_j u). \quad (40)$$

In the identity (40) we now take $G = (r^2 - \rho^2)^{\alpha+1} F$. We remark that, while in our situation the vector fields X_i and G are not smooth, one can nonetheless rigorously justify the implementation of (40) as in [13] by standard approximation arguments based on the key estimates in Theorem 2.4 above. Now we look at each individual term in (40). We first note that from (1) the last integral in the right-hand side of (40) equals $-2 \int_{B_r} F u V u (r^2 - \rho^2)^{\alpha+1}$. For the left-hand side of (40) we have instead

$$\int_{B_r} \operatorname{div} G < AXu, Xu >= \int_{B_r} \operatorname{div} F < AXu, Xu > (r^2 - \rho^2)^{\alpha+1} + \int_{B_r} < AXu, Xu > F (r^2 - \rho^2)^{\alpha+1}. \quad (41)$$

Combining (40) and (41), we reach the conclusion

$$-\int_{B_r} < AXu, Xu > F (r^2 - \rho^2)^{\alpha+1} = \int_{B_r} \operatorname{div} F < AXu, Xu > (r^2 - \rho^2)^{\alpha+1} + \int_{B_r} < (FA) Xu, Xu > (r^2 - \rho^2)^{\alpha+1} - 2 \int_{B_r} a_{ij} X_i u [X_j, G] u - 2 \int_{B_r} F u V u (r^2 - \rho^2)^{\alpha+1}. \quad (42)$$

Using (i) in Theorem 2.4 we find

$$\int_{B_r} \operatorname{div} F < AXu, Xu > (r^2 - \rho^2)^{\alpha+1} = Q \int_{B_r} < AXu, Xu > (r^2 - \rho^2)^{\alpha+1} + O(r) \int_{B_r} < AXu, Xu > (r^2 - \rho^2)^{\alpha+1}. \quad (43)$$

Using (vi) in Theorem 2.4 we have

$$\int_{B_r} < (FA) Xu, Xu > (r^2 - \rho^2)^{\alpha+1} = O(r) \int_{B_r} < AXu, Xu > (r^2 - \rho^2)^{\alpha+1}. \quad (44)$$

We next keep in mind that

$$[X_j, G] = -2(\alpha+1)\rho(r^2-\rho^2)^{\alpha} X_j\rho F + (r^2-\rho^2)^{\alpha+1}[X_j, F].$$

This gives

$$\begin{aligned} a_{ij}X_iu[X_j, G]u &= -2(\alpha+1)(r^2-\rho^2)^{\alpha}\rho < AX\rho, Xu > Fu \\ &\quad + (r^2-\rho^2)^{\alpha+1} a_{ij}X_iu[X_i, F]u \\ &= -2(\alpha+1)(r^2-\rho^2)^{\alpha}(Fu)^2\mu \\ &\quad + (r^2-\rho^2)^{\alpha+1} a_{ij}X_iu\left([X_j, F]u - X_ju\right) \\ &\quad + (r^2-\rho^2)^{\alpha+1} < AXu, Xu >, \end{aligned}$$

where we have used the fact that

$$\rho < AX\rho, Xu >= \mu F u,$$

which follows from (27) above. We thus conclude that

$$\begin{aligned} -2\int_{B_r} a_{ij}X_iu[X_j, G]u = -2\int_{B_r} < AXu, Xu > (r^2-\rho^2)^{\alpha+1} \qquad (45) \\ + O(r)\int_{B_r} < AXu, Xu > (r^2-\rho^2)^{\alpha+1} + 4(\alpha+1)\int_{B_r}(Fu)^2(r^2-\rho^2)^{\alpha}\mu, \end{aligned}$$

where we have used the crucial estimate (vii) in Theorem 2.4 to control the integral

$$\int_{B_r} a_{ij}X_iu\left([X_j, F]u - X_ju\right)(r^2-\rho^2)^{\alpha+1}.$$

Using (43), (44) and (45) in (42), we conclude

$$\begin{aligned} -\int_{B_r} < AXu, Xu > F(r^2-\rho^2)^{\alpha+1} = (Q-2)\int_{B_r} < AXu, Xu > (r^2-\rho^2)^{\alpha+1} \\ + O(r)\int_{B_r} < AXu, Xu > (r^2-\rho^2)^{\alpha+1} + 4(\alpha+1)\int_{B_r}(Fu)^2(r^2-\rho^2)^{\alpha}\mu \\ -2\int_{B_r} FuVu(r^2-\rho^2)^{\alpha+1}. \qquad (46) \end{aligned}$$

With (46) in hands we now return to (38) to find

$$\begin{aligned} &\frac{2(\alpha+1)}{r}\int_{B_r} < AXu, Xu > (r^2-\rho^2)^{\alpha}\rho^2 \\ &= \frac{Q-2}{r}\int_{B_r} <AXu, Xu> (r^2-\rho^2)^{\alpha+1} + O(1)\int_{B_r} <AXu, Xu> (r^2-\rho^2)^{\alpha+1} \\ &\quad + \frac{4(\alpha+1)}{r}\int_{B_r}(Fu)^2(r^2-\rho^2)^{\alpha}\mu - \frac{2}{r}\int_{B_r} FuVu(r^2-\rho^2)^{\alpha+1}. \qquad (47) \end{aligned}$$

The equation (47) is the central one in the proof of the first variation of the energy. Such equation allows us to unravel the second term in the right-hand side of (38) above, to which we now return to find

$$I'(r) = \frac{2\alpha+Q}{r}\int_{B_r} < AXu, Xu > (r^2-\rho^2)^{\alpha+1}$$
$$+ \frac{4(\alpha+1)}{r}\int_{B_r}(Fu)^2(r^2-\rho^2)^{\alpha}\mu + O(1)\int_{B_r} < AXu, Xu > (r^2-\rho^2)^{\alpha+1}$$
$$- \frac{2}{r}\int_{B_r} FuVu(r^2-\rho^2)^{\alpha+1} + 2(\alpha+1)r\int_{B_r} Vu^2(r^2-\rho^2)^{\alpha}.$$

Recalling the definition (31) of $I(r)$ we see that we can rewrite the latter equation as follows

$$I'(r) = \frac{2\alpha+Q}{r}I(r) - \frac{2\alpha+Q}{r}\int_{B_r} Vu^2(r^2-\rho^2)^{\alpha+1}$$
$$+ \frac{4(\alpha+1)}{r}\int_{B_r}(Fu)^2(r^2-\rho^2)^{\alpha}\mu + O(1)I(r) - O(1)\int_{B_r} Vu^2(r^2-\rho^2)^{\alpha+1}$$
$$- \frac{2}{r}\int_{B_r} FuVu(r^2-\rho^2)^{\alpha+1} + 2(\alpha+1)r\int_{B_r} Vu^2(r^2-\rho^2)^{\alpha}. \quad (48)$$

An integration by parts now gives

$$- \frac{2}{r}\int_{B_r} FuVu(r^2-\rho^2)^{\alpha+1} = -\frac{1}{r}\int_{B_r} F(u^2/2)V(r^2-\rho^2)^{\alpha+1}$$
$$= \frac{1}{2r}\int_{B_r} u^2 \operatorname{div}((r^2-\rho^2)^{\alpha+1}VF) = \frac{1}{2r}\int_{B_r} Vu^2(r^2-\rho^2)^{\alpha+1}\operatorname{div} F$$
$$+ \frac{1}{2r}\int_{B_r} u^2FV(r^2-\rho^2)^{\alpha+1} - \frac{\alpha+1}{r}\int_{B_r} Vu^2\rho F\rho(r^2-\rho^2)^{\alpha}.$$

Since one has trivially $(r^2-\rho^2)^{\alpha+1} \le r^2(r^2-\rho^2)^{\alpha}$, from the assumptions (22) above, from (16) and from (i) in Theorem 2.4, we find

$$\left|\frac{1}{2r}\int_{B_r} Vu^2(r^2-\rho^2)^{\alpha+1}\operatorname{div} F\right| \le CKr\int_{B_r} u^2(r^2-\rho^2)^{\alpha}\mu = CKrH(r),$$

where $C = C(\beta, m, k, \lambda) > 0$ is universal. Similarly, one has

$$\left|\frac{1}{2r}\int_{B_r} u^2FV(r^2-\rho^2)^{\alpha+1}\right| \le CKrH(r).$$

Finally, since by (26) we have $F\rho = \rho$, we obtain

$$\left|-\frac{\alpha+1}{r}\int_{B_r} Vu^2\rho F\rho(r^2-\rho^2)^{\alpha}\right| \le CKrH(r).$$

In conclusion, we have for a universal $O(1)$

$$-\frac{2}{r}\int_{B_r} FuVu(r^2-\rho^2)^{\alpha+1} = O(1)KrH(r).$$

The other terms containing V in the right-hand side of (48) are estimated similarly. We thus conclude

$$I'(r) = \frac{2\alpha+Q}{r}I(r) + \frac{4(\alpha+1)}{r}\int_{B_r}(Fu)^2(r^2-\rho^2)^\alpha\mu + O(1)I(r) + O(1)KrH(r),$$

which is (36). □

We are now in a position to provide the

Proof of Theorem 3.1. Using (32), and the equations (34) in Lemma 3.3 and (36) in Lemma 3.4, we find for some universal $C_1, C_3 \geq 0$,

$$\begin{aligned} N'(r) = \frac{I'(r)}{H(r)} - \frac{H'(r)}{H(r)}N(r) &= O(1)N(r) + O(1)Kr \\ + \left(4(\alpha+1)\int_{B_r}(Fu)^2(r^2-\rho^2)^\alpha\mu - \frac{1}{(\alpha+1)}\frac{I(r)^2}{H(r)}\right)&\frac{1}{rH(r)} \\ &\geq -C_1N(r) - C_3Kr, \quad (49) \end{aligned}$$

where in the last inequality, we have used the fact that, in view of (33) in Lemma 3.2, the Cauchy-Schwarz inequality and the definition of $H(r)$, we have

$$\begin{aligned} I(r)^2 &= 4(\alpha+1)^2\left(\int_{B_r} uFu(r^2-\rho^2)^\alpha\mu\right)^2 \\ &\leq 4(\alpha+1)^2H(r)\int_{B_r}(Fu)^2(r^2-\rho^2)^\alpha\mu. \end{aligned}$$

The inequality (49) implies that, with $C_2 = C_3/2$, the function

$$r \to e^{C_1 r}(N(r) + C_2Kr^2)$$

is nondecreasing. □

4. Proof of Theorem 1.1

This final section is devoted to proving the main result in this paper, Theorem 1.1. We start from Theorem 3.1 which implies

$$e^{C_1 r}(N(r) + C_2Kr^2) \leq e^{C_1 s}(N(s) + C_2Ks^2), \qquad \text{for } 0 < r < s < R_1.$$

Henceforth, without loss of generality we assume that $R_1 \leq 1$. The latter monotonicity property implies, in particular, the existence of universal constants $C_2 > 0$ and $\overline{C} \geq 1$ such that

$$N(r) \leq \overline{C}(N(s) + C_2 K), \qquad \text{for } 0 < r < s < R_1. \tag{50}$$

Returning to (34) in Lemma 3.3, we rewrite it in the following form

$$\frac{d}{dr} \log\left(\frac{H(r)}{r^{2\alpha+Q}}\right) = O(1) + \frac{1}{(\alpha+1)r} N(r), \quad 0 < r < R_1, \tag{51}$$

where $|O(1)| \leq C$, with C universal.

Suppose now that $0 < r_1 < r_2 < 2r_2 < r_3 < R_1$. Integrating (51) between r_1 and $2r_2$, and using (50), we find

$$\frac{\log \frac{H(2r_2)}{H(r_1)} - C}{\log\left(\frac{2r_2}{r_1}\right)} - (2\alpha + Q) \leq \frac{\overline{C}}{\alpha+1}\left(N(2r_2) + C_2 K\right). \tag{52}$$

Next, we integrate (51) between $2r_2$ and r_3, and again using (50) we find

$$\frac{\overline{C}}{\alpha+1}\left(N(2r_2) - \overline{C} C_2 K\right) \leq \overline{C}^2 \left[\frac{\log \frac{H(r_3)}{H(2r_2)} + C}{\log\left(\frac{r_3}{2r_2}\right)} - (2\alpha + Q)\right]. \tag{53}$$

Combining (52) and (53) we conclude

$$\frac{\log \frac{H(2r_2)}{H(r_1)} - C}{\overline{C}^2 \log\left(\frac{2r_2}{r_1}\right)} \leq \frac{\log \frac{H(r_3)}{H(2r_2)} + C}{\log\left(\frac{r_3}{2r_2}\right)} + C' \frac{K}{\alpha+1} - \left(1 - \frac{1}{\overline{C}^2}\right)(2\alpha + Q),$$

where we have let $C' = (\overline{C} + 1)/\overline{C}$. Since $\overline{C} \geq 1$, if we now set

$$\alpha_0 = \log\left(\frac{r_3}{2r_2}\right), \quad \beta_0 = \overline{C}^2 \log\left(\frac{2r_2}{r_1}\right),$$

then we obtain

$$\alpha_0 \log \frac{H(2r_2)}{H(r_1)} \leq \beta_0 \log \frac{H(r_3)}{H(2r_2)} + C(\alpha_0 + \beta_0) + C' \frac{K}{\alpha+1} \alpha_0 \beta_0. \tag{54}$$

Dividing both sides of the latter inequality by the quantity $\alpha_0 + \beta_0$, we find

$$\log\left(\frac{H(2r_2)}{H(r_1)}\right)^{\frac{\alpha_0}{\alpha_0+\beta_0}} \leq \log\left(\frac{H(r_3)}{H(2r_2)}\right)^{\frac{\beta_0}{\alpha_0+\beta_0}} + C + C' \frac{K}{\alpha+1} \frac{\alpha\beta_0}{\alpha_0 + \beta_0}.$$

This gives

$$\log H(2r_2) \leq \log \left[H(r_3)^{\frac{\beta_0}{\alpha_0+\beta_0}} \, H(r_1)^{\frac{\alpha_0}{\alpha_0+\beta_0}} \right] + C + C' \frac{K}{\alpha+1}\alpha_0, \qquad (55)$$

where we have used the trivial estimate $\frac{\beta_0}{\alpha_0+\beta_0} \leq 1$. Exponentiating both sides of (55) and choosing $\alpha = \sqrt{K}$, we conclude

$$H(2r_2) \leq e^C \left(\frac{r_3}{2r_2} \right)^{C'\sqrt{K}} H(r_3)^{\frac{\beta_0}{\alpha_0+\beta_0}} H(r_1)^{\frac{\alpha_0}{\alpha_0+\beta_0}}. \qquad (56)$$

We now consider the quantity

$$h(r) = \int_{B_r} u^2 \mu. \qquad (57)$$

The following estimates are easily verified from (30) and (57)

$$H(r) \leq r^{2\alpha} h(r), \qquad \text{and} \qquad h(r) \leq \frac{H(s)}{(s^2 - r^2)^\alpha}, \; 0 < r < s < R_1.$$

From these estimates and (56) we obtain

$$h(r_2) \leq e^C (\frac{r_3}{2r_2})^{C''\sqrt{K}} \, h(r_3)^{\frac{\beta_0}{\alpha_0+\beta_0}} \, h(r_1)^{\frac{\alpha_0}{\alpha_0+\beta_0}}, \qquad (58)$$

for $r_1 < r_2 < 2r_2 < r_3 < R_1$. At this point, we take $r_2 = \frac{R_1}{3}$, $r_3 = R_1$. If

$$C_0 = ||u||^2_{L^\infty(B_{R_1})} \int_{B_{R_1}} \mu > 0,$$

then we clearly have $h(R_1) \leq C_0$, and we conclude from (58) that

$$h(R_1/3)^{1+\frac{\beta_0}{\alpha_0}} \leq e^{C(1+\frac{\beta_0}{\alpha_0})} \left(\frac{3}{2} \right)^{C''(1+\frac{\beta_0}{\alpha_0})\sqrt{K}} C_0^{\frac{\beta_0}{\alpha_0}} h(r), \quad 0 < r < R_1/3. \quad (59)$$

If we set $A = e^C$ and $\gamma = \frac{\overline{C}^2}{\log(3/2)}$, then $q = \beta_0/\alpha_0 = -\log(r/R_1)^\gamma - \overline{C}^2$, and recalling that $\overline{C} \geq 1$ we obtain from (59) for $0 < r < R_1/3$

$$h(r) \geq C_0 \left(\frac{h(R_1/3)}{AC_0} \right)^{1+q} \left(\frac{3}{2} \right)^{-C''(1+q)\sqrt{K}} \geq C_0 M_0^{1+q} \left(\frac{r}{R_1} \right)^{B\sqrt{K}},$$

where we have let $M_0 = \frac{h(R_1/3)}{AC_0}$, and $B = \gamma C'' \log(3/2)$. If $M_0 \geq 1$ this estimate implies in a trivial way for $0 < r < R_1/3$

$$h(r) \geq C_0 \left(\frac{r}{R_1} \right)^{B\sqrt{K}}.$$

If instead $0 < M_0 \leq 1$, keeping in mind that $\overline{C} \geq 1$, with $B' = \max\{B, \gamma \log(1/M_0)\}$ we obtain for $0 < r < R_1/3$

$$h(r) \geq C_0 \left(\frac{r}{R_1}\right)^{B\sqrt{K}+\gamma\log(1/M_0)} \geq C_0 \left(\frac{r}{R_1}\right)^{B'(1+\sqrt{K})} \geq C_0 \left(\frac{r}{R_1}\right)^{2B'\sqrt{K}},$$

where the last inequality follows by remembering that $K \geq 1$. In either case, the desired conclusion of Theorem 1.1 follows by noticing that $h(r) \leq ||u||^2_{L^\infty(B_r)} \int_{B_r} \mu$, and that $\int_{B_r} \mu \leq \lambda^{-1} \int_{B_r} \psi = \lambda^{-1} \omega r^Q$, where we have let $\omega = \int_{B_1} \psi$. In fact, we would find

$$||u||_{L^\infty(B_r)} \geq C_3 \left(\frac{r}{R_1}\right)^{C_4\sqrt{K}},$$

with $C_3 = C_0\sqrt{\frac{\lambda}{\omega R_1^Q}}$ and $C_4 = 2B'$. This finishes the proof of Theorem 1.1.

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Authors' addresses:

Agnid Banerjee
Department of Mathematics
University of California
Irvine-92697, U.S.A.
E-mail: agnidban@gmail.com

Nicola Garofalo
Dipartimento di Ingegneria Civile, Edile e Ambientale
Università di Padova
35131 Padova, Italy
E-mail: rembrandt54@gmail.com

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