

An introduction to the study of critical points of solutions of elliptic and parabolic equations

ROLANDO MAGNANINI

Dedicated to Giovanni Alessandrini in the occasion of his 60th birthday

ABSTRACT. *We give a survey at an introductory level of old and recent results in the study of critical points of solutions of elliptic and parabolic partial differential equations. To keep the presentation simple, we mainly consider four exemplary boundary value problems: the Dirichlet problem for the Laplace's equation; the torsional creep problem; the case of Dirichlet eigenfunctions for the Laplace's equation; the initial-boundary value problem for the heat equation. We shall mostly address three issues: the estimation of the local size of the critical set; the dependence of the number of critical points on the boundary values and the geometry of the domain; the location of critical points in the domain.*

Keywords: Elliptic partial differential equations, parabolic partial differential equations, critical points of solutions, hot spots.
MS Classification 2010: 35B38, 35J05, 35J08, 35J15, 35J25, 35K10, 35K20.

1. Introduction

Let Ω be a domain in the Euclidean space \mathbb{R}^N , Γ be its boundary and $u : \Omega \rightarrow \mathbb{R}$ be a differentiable function. A *critical point* of u is a point in Ω at which the gradient ∇u of u is the zero vector. The importance of critical points is evident. At an elementary level, they help us to visualize the graph of u , since they are some of its notable points (they are local maximum, minimum, or inflection/saddle points of u). At a more sophisticated level, if we interpret u and ∇u as a gravitational, electrostatic or velocity potential and its underlying field of force or flow, the critical points are the positions of equilibrium for the field of force or stagnation points for the flow and give information on the topology of the equipotential lines or of the curves of steepest descent (or stream lines) related to u .

A merely differentiable function can be very complicated. For instance, Whitney [88] constructed a non-constant function of class C^1 on the plane

with a connected set of critical values (the images of critical points). If we allow enough smoothness, this is no longer possible as Morse-Sard's lemma informs us: indeed, if u is at least of class C^N , the set of its critical values must have zero Lebesgue measure and hence the regular values of u must be dense in the image of u (see [8] for a proof).

When the function u is the solution of some partial differential equation, the situation improves. In this survey, we shall consider the four archetypical equations:

$$\Delta u = 0, \quad \Delta u = -1, \quad \Delta u + \lambda u = 0, \quad u_t = \Delta u,$$

that is the *Laplace's equation*, the *torsional creep equation*, the *eigenfunction equation* and the *heat equation*.

It should be noticed at this point some important differences between the first and the remaining three equations.

One is that the critical points of harmonic functions — the solutions of the Laplace's equation — are always “saddle points” as it is suggested by the maximum and minimum principles and the fact that Δu is the sum of the eigenvalues of the hessian matrix $\nabla^2 u$. The other three equations instead admit solutions with maximum or minimum points.

Also, we know that the critical points of a non-constant harmonic function u on an open set of \mathbb{R}^2 are isolated and can be assigned a sort of finite multiplicity, for they are the zeroes of the holomorphic function $f = u_x - iu_y$. By means of the theory of quasi-conformal mappings and generalized analytic functions, this result can be extended to solutions of the elliptic equation

$$(a u_x + b u_y)_x + (b u_x + c u_y)_y + d u_x + e u_y = 0 \quad (1)$$

(with suitable smoothness assumptions on the coefficients) or even to weak solutions of the elliptic equation in divergence form,

$$(a u_x + b u_y)_x + (b u_x + c u_y)_y = 0 \quad \text{in } \Omega, \quad (2)$$

even allowing discontinuous coefficients.

Instead, solutions of the other three equations can show curves of critical points in \mathbb{R}^2 , as one can be persuaded by looking at the solution of the torsional creep equation in a circular annulus with zero boundary values.

These discrepancies extend to any dimension $N \geq 2$, in the sense that it has been shown that the set of the critical points of a non-constant harmonic function (or of a solution of an elliptic equation with smooth coefficients modeled on the Laplace equation) has at most locally finite $(N - 2)$ -dimensional Hausdorff measure, while solutions of equations fashioned on the other three equations have at most locally finite $(N - 1)$ -dimensional Hausdorff measure.

Further assumptions on solutions of a partial differential equation, such as their behaviour on the boundary and the shape of the boundary itself, can

give more detailed information on the number and location of critical points. In these notes, we shall consider the case of harmonic functions with various boundary behaviors and the solutions τ , ϕ and h of the following three problems:

$$-\Delta\tau = 1 \text{ in } \Omega, \quad \tau = 0 \text{ on } \Gamma; \quad (3)$$

$$\Delta\phi + \lambda\phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \Gamma; \quad (4)$$

$$h_t = \Delta h \text{ in } \Omega \times (0, \infty), \quad (5)$$

$$h = 0 \text{ on } \Gamma \times (0, \infty), \quad h = \varphi \text{ on } \Omega \times \{0\}, \quad (6)$$

where φ is a given function. We will refer to (3), (4), (5)-(6), as the *torsional creep problem*, the *Dirichlet eigenvalue problem*, and the *initial-boundary value problem for the heat equation*, respectively.

A typical situation is that considered in Theorem 3.2: a harmonic function u on a planar domain Ω is given together with a vector field ℓ on Γ of assigned topological degree D ; the number of critical points in Ω then is bounded in terms of D , the Euler characteristic of Ω and the number of proper connected components of the set $\{z \in \Gamma : \ell(z) \cdot \nabla u(z) > 0\}$ (see Theorem 3.2 for the exact statement). We shall also see how this type of theorem has recently been extended to obtain a bound for the number of critical points of the Li-Tam Green's function of a non-compact Riemannian surface of finite type in terms of its genus and the number of its ends.

Owing to the theory of quasi-conformal mappings, Theorem 3.2 can be extended to solutions of quite general elliptic equations and, thanks to the work of G. Alessandrini and co-authors, has found effective applications to the study of inverse problems that have as a common denominator the reconstruction of the coefficients of an elliptic equation in a domain from measurements on the boundary of a set of its solutions.

A paradigmatic example is that of Electric Impedance Tomography (EIT) in which a conductivity γ is reconstructed, as the coefficient of the elliptic equation

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega,$$

from the so-called Neumann-to-Dirichlet (or Dirichlet-to-Neumann) operator on Γ . In physical terms, an electrical current (represented by the co-normal derivative γu_ν) is applied on Γ generating a potential u , that is measured on Γ within a certain error. One wants to reconstruct the conductivity γ from some of these measurements. Roughly speaking, one has to solve for the unknown γ the first order differential equation

$$\nabla u \cdot \nabla \gamma + (\Delta u) \gamma = 0 \text{ in } \Omega,$$

once the information about u has been extended from Γ to Ω . It is clear that such an equation is singular at the critical points of u . Thus, it is helpful to

know *a priori* that ∇u does not vanish and this can be done via (appropriate generalizations of) Theorem 3.2 by choosing suitable currents on Γ .

The possible presence of maximum and/or minimum points for the solutions of (3), (4), or (5)-(6) makes the search for an estimate of the number of critical points a difficult task (even in the planar case). In fact, the mere topological information only results in an estimate of the *signed sum* of the critical points, the sign depending on whether the relevant critical point is an extremal or saddle point. For example, for the solution of (3) or (4), we only know that the difference between the number of its (isolated) maximum and saddle points (minimum points are not allowed) must equal $\chi(\Omega)$, the *Euler characteristic* of Ω — a Morse-type theorem. Thus, further assumptions, such as geometric information on Ω , are needed. More information is also necessary even if we consider the case of harmonic functions in dimension $N \geq 3$.

In the author's knowledge, results on the number of critical points of solutions of (3), (4), or (5)-(6) reduce to deduction that their solutions admit a *unique* critical point if Ω is *convex*. Moreover, the proof of such results is somewhat indirect: the solution is shown to be *quasi-concave* — indeed, log-concave for the cases of (4) and (5)-(6), and 1/2-concave for the case (3) — and then its analyticity completes the argument. Estimates of the number of critical points when the domain Ω has more complex geometries would be a significant advance. In this survey, we will propose and justify some conjectures.

The problem of locating critical points is also an interesting issue. The first work on this subject dates back to Gauss [36], who proved that the critical points of a complex polynomial are its zeroes, if they are multiple, and the equilibrium points of the gravitational field of force generated by particles placed at the zeroes and with masses proportional to the zeroes' multiplicities (see Section 4). Later refinements are due to Jensen [47] and Lucas [62], but the first treatises on this matter are Marden's book [68] and, primarily, Walsh's monograph [87] that collects most of the results on the number and location of critical points of complex polynomials and harmonic functions known at that date. In general dimension, even for harmonic functions, results are sporadic and rely on explicit formulae or symmetry arguments.

Two well known questions in this context concern the location of the *hot spot* in a heat conductor — a hot spot is a point of (absolute or relative) maximum temperature in the conductor. The situation described by (5)-(6) corresponds with the case of a *grounded* conductor. By some asymptotic analysis, under appropriate assumptions on φ , one can show that the hot spots *originate* from the set of maximum points of the function $d_\Omega(x)$ — the distance of $x \in \Omega$ from Γ — and tend to the maximum points of the unique positive solution of (4), as $t \rightarrow \infty$. In the case Ω is convex, we have only one hot spot, as already observed. In Section 4, we will describe three techniques to locate it; some of them extend their validity to locate the maximum points of the solutions to

(3) and (4). We will also give an account of what it is known about convex conductors that admit a stationary hot spot (that is the hot spot does not move with time).

It has also been considered the case in which the homogeneous Dirichlet boundary condition in (6) is replaced by the homogeneous Neumann condition:

$$u_\nu = 0 \quad \text{on } \Gamma \times (0, \infty). \quad (7)$$

These settings describe the evolution of temperature in an *insulated* conductor of given constant initial temperature and has been made popular by a conjecture of J. Rauch [76] that would imply that the hot spot must tend to a boundary point. Even if we now know that it is false for a general domain, the conjecture holds true for certain planar convex domains but it is still standing for unrestrained convex domains.

The remainder of the paper is divided into three sections that reflect the aforementioned features. In Section 2, we shall describe the local properties of critical points of harmonic functions or, more generally, of solutions of elliptic equations, that lead to estimates of the size of critical sets. In Section 3, we shall focus on bounds for the number of critical points that depend on the boundary behavior of the relevant solutions and/or the geometry of Γ . Finally, in Section 4, we shall address the problem of locating the possible critical points. As customary for a survey, our presentation will stress ideas rather than proofs.

This paper is dedicated with sincere gratitude to Giovanni Alessandrini — an inspiring mentor, a supportive colleague and a genuine friend — on the occasion of his 60th birthday. Much of the material presented here was either inspired by his ideas or actually carried out in his research with the author.

2. The size of the critical set of a harmonic function

A harmonic function in a domain Ω is a solution of the Laplace's equation

$$\Delta u = u_{x_1 x_1} + \cdots + u_{x_N x_N} = 0 \quad \text{in } \Omega.$$

It is well known that harmonic functions are analytic, so there is no difficulty to define their critical points or the *critical set*

$$\mathcal{C}(u) = \{x \in \Omega : \nabla u(x) = 0\}.$$

Before getting into the heart of the matter, we present a relevant example.

2.1. Harmonic polynomials

In dimension two, we have a powerful tool since we know that a harmonic function is (locally) the real or imaginary part of a holomorphic function. This

remark provides our imagination with a reach set of examples on which we can speculate. For instance, the harmonic function

$$u = \operatorname{Re}(z^n) = \operatorname{Re}[(x + iy)^n], \quad n \in \mathbb{N},$$

already gives some insight on the properties of harmonic functions we are interested in. In fact, we have that

$$u_x - iu_y = nz^{n-1};$$

thus, u has only one distinct critical point, $z = 0$, but it is more convenient to say that u has $n - 1$ critical points at $z = 0$ or that $z = 0$ is a critical point with *multiplicity* m with $m = n - 1$. By virtue of this choice, we can give a topological meaning to m .

To see that, it is advantageous to represent u in polar coordinates:

$$u = r^n \cos(n\theta);$$

here, $r = |z|$ and θ is the principal branch of $\arg z$, that is we are assuming that $-\pi \leq \theta < \pi$. Thus, the topological meaning of m is manifest when we look at the level “curve” $\{z : u(z) = u(0)\}$: it is made of $m + 1 = n$ straight lines passing through the critical point $z = 0$, divides the plane into $2n$ *cones* (angles), each of amplitude π/n and the sign of u changes across those lines (see Fig. 1). One can also show that the signed angle ω formed by ∇u and the direction of the positive real semi-axis, since it equals $-(n - 1) \arg z$, increases by $2\pi m$ while z makes a complete loop clockwise around $z = 0$; thus, m is a sort of *winding number* for ∇u .

The critical set of a homogeneous polynomial $P : \mathbb{R}^N \rightarrow \mathbb{R}$ is a cone in \mathbb{R}^N . Moreover, if P is also harmonic (and non-constant) one can show that

$$\text{dimension of } \mathcal{C}(u) \leq N - 2. \quad (8)$$

2.2. Harmonic functions

If $N = 2$ and u is any harmonic function, the picture is similar to that outlined in the example. In fact, we can again consider the “complex gradient” of u ,

$$g = u_x - iu_y,$$

and observe that g is *holomorphic* in Ω , since $\partial_{\bar{z}}g = 0$, and hence analytic. Thus, the zeroes of g (and hence the critical points of u) in Ω are *isolated* and have *finite multiplicity*. If z_0 is a zero with multiplicity m of g , then we can write that

$$g(z) = (z - z_0)^m h(z),$$

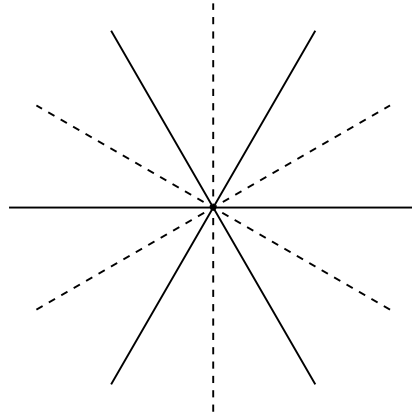


Figure 1: Level set diagram of $u = r^6 \cos(6\theta)$ at the critical point $z = 0$; u changes sign from positive to negative at dashed lines and from negative to positive at solid lines.

where h is holomorphic in Ω and $h(z_0) \neq 0$.

On the other hand, we also know that u is locally the real part of a holomorphic function f and hence, since $f' = g$, by an obvious normalization, it is not difficult to infer that

$$f(z) = \frac{1}{n} (z - z_0)^n k(z),$$

where $n = m + 1$ and k is holomorphic and $k(z_0) = h(z_0) \neq 0$. Passing to polar coordinates by $z = z_0 + re^{i\theta}$ tells us that

$$f(z_0 + re^{i\theta}) = \frac{|h(z_0)|}{n} r^n e^{i(n\theta + \theta_0)} + O(r^{n+1}) \text{ as } r \rightarrow 0,$$

where $\theta_0 = \arg h(z_0)$. Thus, we have that

$$u = \frac{|h(z_0)|}{n} r^n \cos(n\theta + \theta_0) + O(r^{n+1}) \text{ as } r \rightarrow 0,$$

and hence, modulo a rotation by the angle θ_0 , in a small neighborhood of z_0 , we can say that the critical level curve $\{z : u(z) = u(z_0)\}$ is very similar to that described in the example with 0 replaced by z_0 . In particular, it is made of n simple curves passing through z_0 and any two adjacent curves meet at z_0 with an angle that equals π/n (see Fig. 2).

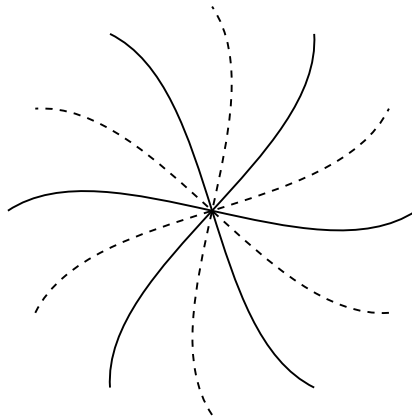


Figure 2: Level set diagram of a harmonic function at a critical point with multiplicity $m = 5$. The curves meet with equal angles at the critical point.

If $N \geq 3$, similarly, a harmonic function can be approximated near a point 0 at which vanishes by a homogeneous harmonic polynomial of some degree n :

$$u(x) = P_n(x) + O(|x|^{n+1}) \quad \text{as } |x| \rightarrow 0. \quad (9)$$

However, the structure of the set $\mathcal{C}(u)$ depends on whether 0 is an isolated critical point of P_n or not. In fact, if 0 is not isolated, then $\mathcal{C}(u)$ and $\mathcal{C}(P_n)$ could not be diffeomorphic in general, as shown by the harmonic function

$$u(x, y, z) = x^2 - y^2 + (x^2 + y^2)z - \frac{2}{3}z^3, \quad (x, y, z) \in \mathbb{R}^3.$$

Indeed, if $P_2(x, y, z) = x^2 - y^2$, $\mathcal{C}(P_2)$ is the z -axis, while $\mathcal{C}(u)$ is made of 5 isolated points ([74]).

2.3. Elliptic equations in the plane

These arguments can be repeated with some necessary modifications for solutions of uniformly elliptic equations of the type (1), where the variable coefficients a, b, c are Lipschitz continuous and d, e are bounded measurable on Ω and the uniform ellipticity is assumed to take the following form:

$$ac - b^2 = 1 \quad \text{in } \Omega.$$

Now, the classical theory of *quasi-conformal* mappings comes in our aid (see [14, 86] and also [4, 5]). By the *uniformization theorem* (see [86]), there exists

a quasi-conformal mapping $\zeta(z) = \xi(z) + i\eta(z)$, satisfying the equation

$$\zeta_{\bar{z}} = \kappa(z)\zeta_z \quad \text{with} \quad |\kappa(z)| = \frac{a+c-2}{a+c+2} < 1,$$

such that the function U defined by $U(\zeta) = u(z)$ satisfies the equation

$$\Delta U + P U_\xi + Q U_\eta = 0 \quad \text{in} \quad \zeta(\Omega),$$

where P and Q are real-valued functions depending on the coefficients in (1) and are essentially bounded on $\zeta(\Omega)$. Notice that, since the composition of ζ with a conformal mapping is still quasi-conformal, if it is convenient, by the Riemann mapping theorem, we can choose $\zeta(\Omega)$ to be the unit disk \mathbb{D} .

By setting $G = U_\xi - iU_\eta$, simple computations give that

$$G_{\bar{\zeta}} = R G + \overline{R} \overline{G} \quad \text{in} \quad \mathbb{D},$$

where $R = (P + iQ)/4$ is essentially bounded. This equation tells us that G is a *pseudo-analytic* function for which the following *similarity principle* holds (see [86]): there exist two functions, $H(\zeta)$ holomorphic in \mathbb{D} and $s(\zeta)$ Hölder continuous on the whole \mathbb{C} , such that

$$G(\zeta) = e^{s(\zeta)} H(\zeta) \quad \text{for} \quad \zeta \in \mathbb{D}. \tag{10}$$

Owing to (10), it is clear that the critical points of u , by means of the mapping $\zeta(z)$, correspond to the zeroes of $G(\zeta)$ or, which is the same, of $H(\zeta)$ and hence we can claim that they are isolated and have a finite multiplicity.

This analysis can be further extended if the coefficients d and e are zero, that is for the solutions of (2). In this case, we can even assume that the coefficients a, b, c be merely essentially bounded on Ω , provided that we agree that u is a non-constant *weak* solution of (1). It is well known that, with these assumptions, solutions of (1) are in general only Hölder continuous and the usual definition of critical point is no longer possible. However, in [5] we got around this difficulty by introducing a different notion of critical point, that is still consistent with the topological structure of the level curves of u at its critical values.

To see this, we look for a surrogate of the harmonic conjugate for u . In fact, (1) implies that the 1-form

$$\omega = -(b u_x + c u_y) dx + (a u_x + b u_y) dy$$

is closed (in the weak sense) in Ω and hence, thanks to the theory developed in [15], we can find a so-called *stream function* $v \in W^{1,2}(\Omega)$ whose differential dv equals ω , in analogy with the theory of gas dynamics (see [13]).

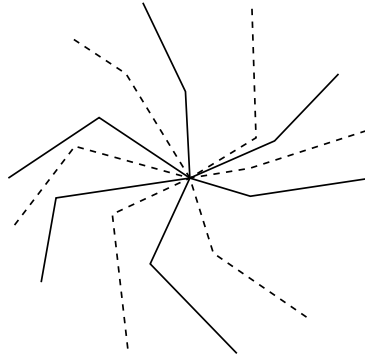


Figure 3: Level set diagram of a solution of an elliptic equation with discontinuous coefficients at a geometric critical point with multiplicity $m = 5$. At that point, any two consecutive curves meet with positive angles, possibly not equal to one another.

Thus, in analogy with what we have done in Subsection 2.2, we find out that the function $f = u + iv$ satisfies the equation

$$f_{\bar{z}} = \mu f_z \quad (11)$$

where

$$\mu = \frac{c - a - 2ib}{2 + a + c} \quad \text{and} \quad |\mu| \leq \frac{1 - \lambda}{1 + \lambda} < 1 \quad \text{in } \Omega,$$

and $\lambda > 0$ is a lower bound for the smaller eigenvalue of the matrix of the coefficients:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

The fact that $f \in W^{1,2}(\Omega, \mathbb{C})$ implies that f is a *quasi-regular* mapping that can be factored as

$$f = F \circ \chi \quad \text{in } \Omega,$$

where $\chi : \Omega \rightarrow \mathbb{D}$ is a *quasi-conformal homeomorphism* and F is holomorphic in \mathbb{D} (see [60]). Therefore, the following *representation formula* holds:

$$u = U(\chi(z)) \quad \text{for } z \in \Omega,$$

where U is the real part of F .

This formula informs us that the level curves of u can possibly be distorted by the homeomorphism χ , but preserve the topological structure of a harmonic function (see Fig. 3). This remark gives grounds to the definition introduced

in [5]: $z_0 \in \Omega$ is a *geometric critical point* of u if the gradient of U vanishes at $\chi(z_0) \in \mathbb{D}$. In particular, geometric critical points are isolated and can be classified by a sort of multiplicity.

2.4. Quasilinear elliptic equations in the plane

A similar local analysis can be replicated when $N = 2$ for quasilinear equations of type

$$\operatorname{div}\{A(|\nabla u|)\nabla u\} = 0,$$

where $A(s) > 0$ and $0 < \lambda \leq 1 + sA'(s)/A(s) \leq \Lambda$ for every $s > 0$ and some constants λ and Λ .

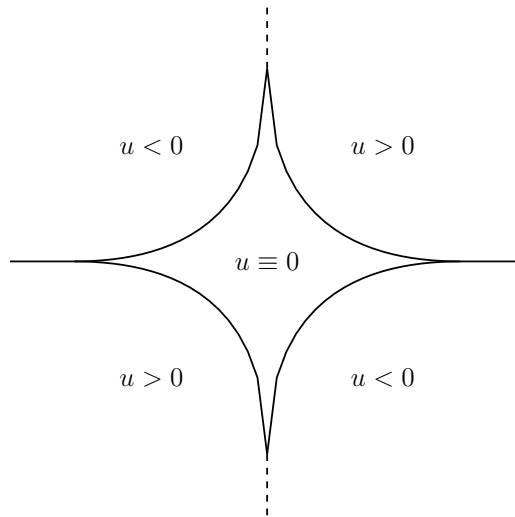


Figure 4: Level set diagram of a solution of a degenerate quasilinear elliptic equation with $B(s) = \sqrt{1 + s^2}$ at a critical value.

These equations can be even degenerate, such as the p -Laplace equation with $1 < p < \infty$ (see [8]). It is worth mentioning that also the case in which $A(s) = B(s)/s$, where B is increasing, with $B(0) > 0$, and superlinear and growing polynomially at infinity (e.g. $B(s) = \sqrt{1 + s^2}$), has been studied in [23]. In this case the function $1 + sA'(s)/A(s)$ vanishes at $s = 0$ and it turns out that the critical points of a solution u (if any) are *never* isolated (Fig. 4).

2.5. The case $N \geq 3$

As already observed, critical points of harmonic functions in dimension $N \geq 3$ may not be isolated. Besides the example given in Section 2.2, another concrete example is given by the function

$$u(x, y, z) = J_0\left(\sqrt{x^2 + y^2}\right) \cosh(z), \quad (x, y, z) \in \mathbb{R}^3,$$

where J_0 is the first Bessel function: the gradient of u vanishes at the origin and on the circles on the plane $z = 0$ having radii equal to the zeroes of the second Bessel function J_1 . It is clear that a region Ω can be found such that $\mathcal{C}(u) \cap \Omega$ is a *bounded continuum*.

Nevertheless, it can be proved that $\mathcal{C}(u)$ always has locally finite $(N - 2)$ -dimensional Hausdorff measure \mathcal{H}^{N-2} . A nice argument to see this was suggested to me by D. Peralta-Salas [74]. If u is a non-constant harmonic function and we suppose that $\mathcal{C}(u)$ has dimension $N - 1$, then the general theory of analytic sets implies that there is an open and dense subset of $\mathcal{C}(u)$ which is an analytic sub-manifold (see [59]). Since u is constant on a connected component of the critical set, it is constant on $\mathcal{C}(u)$, and its gradient vanishes. Thus, by the Cauchy-Kowalewski theorem u must be constant in a neighborhood of $\mathcal{C}(u)$, and hence everywhere by unique continuation. Of course, this argument would also work for solutions of an elliptic equation of type

$$\sum_{i,j=1}^N a_{ij}(x) u_{x_i x_j} + \sum_{j=1}^N b_j(x) u_{x_j} = 0 \quad \text{in } \Omega, \quad (12)$$

with analytic coefficients.

When the coefficients a_{ij}, b_j in (12) are of class $C^\infty(\Omega)$, the result has been proved in [39] (see also [38]): if u is a non-constant solution of (12), then for any compact subset K of Ω it holds that

$$\mathcal{H}^{N-2}(\mathcal{C}(u) \cap K) < \infty. \quad (13)$$

The proof is based on an estimate similar to (8) for the complex dimension of the singular set in \mathbb{C}^N of the complexification of the polynomial P_n in the approximation (9).

The same result does not hold for solutions of equation

$$\sum_{i,j=1}^N a_{ij}(x) u_{x_i x_j} + \sum_{j=1}^N b_j(x) u_{x_j} + c(x) u = 0 \quad \text{in } \Omega, \quad (14)$$

with $c \in C^\infty(\Omega)$. For instance the gradient of the first Laplace-Dirichlet eigenfunction for a spherical annulus vanishes exactly on a $(N - 1)$ -dimensional

sphere. A more general counterexample is the following (see [39, Remark p. 362]): let v be of class C^∞ and with non-vanishing gradient in the unit ball B in \mathbb{R}^N ; the function $u = 1 + v^2$ satisfies the equation

$$\Delta u - cu = 0 \quad \text{with} \quad c = \frac{\Delta v^2}{1 + v^2} \in C^\infty(B);$$

we have that $\mathcal{C}(u) = \{x \in B : v(x) = 0\}$ and it has been proved that any closed subset of \mathbb{R}^N can be the zero set of a function of class C^∞ (see [84]).

However, once (13) is settled, it is rather easy to show that the *singular set*

$$\mathcal{S}(u) = \mathcal{C}(u) \cap u^{-1}(0) = \{x \in \Omega : u(x) = 0, \nabla u(x) = 0\}$$

of a non-constant solution of (14) also has locally finite $(N - 2)$ -dimensional Hausdorff measure [39, Corollary 1.1]. This can be done by a trick, since around any point in Ω there always exists a *positive* solution u_0 of (14) and it turns out that the function $w = u/u_0$ is a solution of an equation like (12) and that $\mathcal{S}(u) \subseteq \mathcal{C}(w)$. In particular the set of critical points on the nodal line of an eigenfunction of the Laplace operator has locally finite $(N - 2)$ -dimensional Hausdorff measure.

Nevertheless, for a solution of (12) the set $\mathcal{S}(u)$ can be very complicated, as a simple example in [39, p. 361] shows: the function $u(x, y, z) = xy + f(z)^2$, where f is a smooth function with $|ff''| + (f')^2 < 1/4$ that vanishes exactly on an arbitrary given closed subset K of \mathbb{R} , is a solution of

$$u_{xx} + u_{yy} + u_{zz} - (f^2)''(z) u_{xy} = 0 \quad \text{and} \quad \mathcal{S}(u) = \{(0, 0)\} \times K.$$

Heuristically, as in the 2-dimensional case, the proof of (13) is essentially based on the observation that, by Taylor's expansion, a harmonic function u can be approximated near any of its zeroes by a homogeneous harmonic polynomial $P_m(x_1, \dots, x_n)$ of degree $m \geq 1$. Technically, the authors use the fact that the complex dimension of the critical set in \mathbb{C}^N of the complexified polynomial $P_m(z_1, \dots, z_N)$ is bounded by $N - 2$. A C^∞ -perturbation argument and an inequality from geometric measure theory then yield that, near a zero of u , the \mathcal{H}^{N-2} -measure of $\mathcal{C}(u)$ can be bounded in terms of N and m . The extension of these arguments to the case of a solution of (12) is then straightforward. Recently in [26], (13) has been extended to the case of solutions of elliptic equations of type

$$\sum_{i,j=1}^N \{a_{ij}(x) u_x\}_{x_j} + \sum_{j=1}^N b_j(x) u_{x_j},$$

where the coefficients $a_{ij}(x)$ and $b_j(x)$ are assumed to be Lipschitz continuous and essentially bounded, respectively.

3. The number of critical points

A more detailed description of the critical set $\mathcal{C}(u)$ of a harmonic function u can be obtained if we assume to have some information on its behavior on the boundary Γ of Ω . While in Section 2 the focus was on a qualitative description of the set $\mathcal{C}(u)$, here we are concerned with establishing bounds on the number of critical points.

3.1. Counting the critical points of a harmonic function in the plane

An exact counting formula is given by the following result.

THEOREM 3.1 ([4]). *Let Ω be a bounded domain in the plane and let*

$$\Gamma = \bigcup_{j=1}^J \Gamma_j,$$

where $\Gamma_j, j = 1, \dots, J$ are simple closed curves of class $C^{1,\alpha}$. Consider a harmonic function $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ that satisfies the Dirichlet boundary condition

$$u = a_j \text{ on } \Gamma_j, j = 1, \dots, J, \quad (15)$$

where a_1, \dots, a_J are given real numbers, not all equal.

Then u has in $\bar{\Omega}$ a finite number of critical points z_1, \dots, z_K ; if $m(z_1), \dots, m(z_K)$ denote their multiplicities, then the following identity holds:

$$\sum_{z_k \in \Omega} m(z_k) + \frac{1}{2} \sum_{z_k \in \Gamma} m(z_k) = J - 2. \quad (16)$$

Thanks to the analysis presented in Subsection 2.3, this theorem still holds if we replace the Laplace equation in (15) by the general elliptic equation (1). In fact, modulo a suitable change of variables, we can use (10) with $\text{Im}(s) = 0$ on the boundary.

The function considered in Theorem 3.1 can be interpreted in physical terms as the potential in an electrical capacitor and hence its critical points are the points of equilibrium of the electrical field (Fig. 5).

The proof of Theorem 3.1 relies on the fact that the critical points of u are the zeroes of the holomorphic function $f = u_x - i u_y$ and hence they can be counted with their multiplicities by applying the classical *argument principle* to f with some necessary modifications. The important remark is that, since the boundary components are level curves for u , the gradient of u is parallel on them to the (exterior) unit normal ν to the boundary, and hence $\arg f = -\arg \nu$.

Thus, the situation is clear if u does not have critical points on Γ : the argument principle gives at once that

$$\begin{aligned} \sum_{z_k \in \Omega} m(z_k) &= \frac{1}{2\pi i} \int_{+\Gamma} \frac{f'(z)}{f(z)} dz \\ &= \frac{1}{2\pi} \operatorname{Incr}(\arg f, +\Gamma) = \frac{1}{2\pi} \operatorname{Incr}(-\arg \nu, +\Gamma) = -[1 - (J - 1)] = J - 2, \end{aligned}$$

where by $\operatorname{Incr}(\cdot, +\gamma)$ we intend the *increment* of an angle on an oriented curve $+\gamma$ and by $+\Gamma$ we mean that Γ is trodden in such a way that Ω is on the left-hand side.

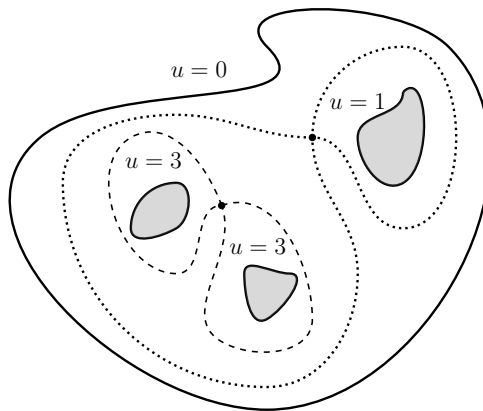


Figure 5: An illustration of Theorem 3.1: the domain Ω has 3 holes; u has exactly 2 critical points; dashed and dotted are the level curves at critical values.

If Γ contains critical points, we must first prove that they are also isolated. This is done, by observing that, if z_0 is a critical point belonging to some component Γ_j , since u is constant on Γ_j , by the *Schwarz's reflection principle* (modulo a conformal transformation of Ω), u can be extended to a function \tilde{u} which is harmonic in a whole neighborhood of z_0 . Thus, z_0 is a zero of the holomorphic function $\tilde{f} = \tilde{u}_x - i\tilde{u}_y$ and hence is isolated and with finite multiplicity. Moreover, the increment of $\arg \tilde{f}$ on an oriented closed simple curve $+\gamma$ around z_0 is exactly twice as much as that of $\arg f$ on the part of $+\gamma$ inside Ω . This explains the second addendum in (16).

Notice that condition (15) can be re-written as

$$u_\tau = 0 \quad \text{on } \Gamma,$$

where $\tau : \Gamma \rightarrow \mathbb{S}^1$ is the *tangential* unit vector field on Γ . We cannot hope to obtain an identity as (16) if u_τ is not constant. However, a bound for the number of critical points of a harmonic function (or a solution of (1)) can be derived in a quite general setting.

In what follows, we assume that Ω is as in Theorem 3.1 and that $\ell : \Gamma \rightarrow \mathbb{S}^1$ denotes a (unitary) vector field of class $C^1(\Gamma, \mathbb{S}^1)$ of given topological degree D , that can be defined as

$$2\pi D = \text{Incr}(\arg(\ell), +\Gamma). \quad (17)$$

Also, we will use the following definitions:

- (i) if $(\mathcal{J}^+, \mathcal{J}^-)$ is a decomposition of Γ into two disjoint subsets such that $u_\ell \geq 0$ on \mathcal{J}^+ and $u_\ell \leq 0$ on \mathcal{J}^- , we denote by $M(\mathcal{J}^+)$ the number of connected components of \mathcal{J}^+ which are *proper subsets* of some component Γ_j of Γ and set:

$$M = \min\{M(\mathcal{J}^+) : (\mathcal{J}^+, \mathcal{J}^-) \text{ decomposes } \Gamma\};$$

- (ii) if $\mathcal{I}^\pm = \{z \in \Gamma : \pm u_\ell(z) > 0\}$, by M^\pm we denote the number of connected components of \mathcal{I}^\pm which are *proper subsets* of some component Γ_j of Γ .

Notice that in (i) the definition of M does not change if we replace \mathcal{J}^+ by \mathcal{J}^- .

THEOREM 3.2 ([4]). *Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be harmonic in Ω and denote by $m(z_j)$ the multiplicity of a zero z_j of $f = u_x - i u_y$.*

- (a) *If M is finite and u has no critical point in Γ , then*

$$\sum_{z_j \in \Omega} m(z_j) \leq M - D;$$

- (b) *if $M^+ + M^-$ is finite, then*

$$\sum_{z_j \in \Omega} m(z_j) \leq \left[\frac{M^+ + M^-}{2} \right] - D,$$

where $[x]$ is the greatest integer $\leq x$.

This theorem is clearly less sharp than Theorem 3.1 since, in that setting, it does not give information about critical points on the boundary. However, it gives the same information on the number of interior critical points, since in the setting of Theorem 3.1 the degree of the field τ on $+\Gamma$ equals $2 - J$ and $M = 0$.

The possibility of choosing the vector field ℓ arbitrarily makes Theorem 3.2 a very flexible tool: for instance, the number of critical points in Ω can be

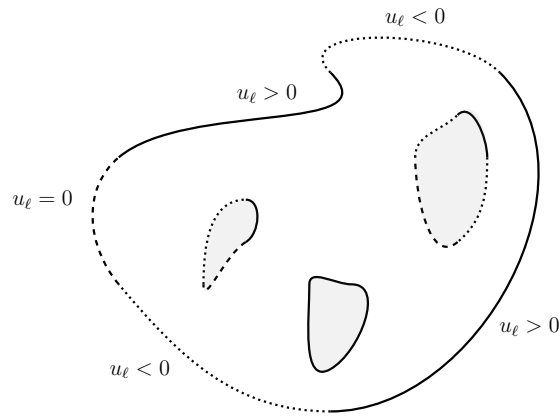


Figure 6: An illustration of Theorem 3.2. Here, $M = M^+ = 4$; $M^- = 4$; $D = -2$ if $\ell = \nu$ or τ ; $D = 1$ if $\ell = z/|z|$ and the origin is in Ω ; $D = 0$ if $\ell = (1, 0)$ or $(0, 1)$.

estimated from information on the tangential, normal, co-normal, partial, or radial (with respect to some origin) derivatives (see Fig. 6).

As an illustration, it says that in a domain topologically equivalent to a disk, in order to have n interior critical point the normal (or tangential, or co-normal) derivative of a harmonic function must change sign at least $n + 1$ times and a partial derivative at least n times. Thus, Theorem 3.2 helps to choose Neumann data that insures the absence of critical points in Ω . For this reason, in its general form for elliptic operators, it has been useful in the study of EIT and other similar inverse problems.

We give a sketch of the proof of (a) of Theorem 3.2, that hinges on the simple fact that, if we set $\theta = \arg(\ell)$ and $\omega = \arg(u_x - i u_y)$, then

$$u_\ell = \ell \cdot \nabla u = |\nabla u| \cos(\theta + \omega).$$

Hence, if $(\mathcal{J}^+, \mathcal{J}^-)$ is a minimizing decomposition of Γ as in (i), then

$$|\omega + \theta| \leq \frac{\pi}{2} \text{ on } \mathcal{J}^+ \text{ and } |\omega + \theta - \pi| \leq \frac{\pi}{2} \text{ on } \mathcal{J}^-.$$

Thus, two occurrences must be checked. If a component Γ_j is contained in \mathcal{J}^+ or \mathcal{J}^- , then

$$\left| \frac{1}{2\pi} \text{Incr}(\omega + \theta, +\Gamma_j) \right| \leq \frac{1}{2},$$

that implies that ω and $-\theta$ must have the same increment, being the right-hand side an integer. If Γ_j contains points of both \mathcal{J}^+ and \mathcal{J}^- , instead, if $\sigma^+ \subset \mathcal{J}^+$ and $\sigma^- \subset \mathcal{J}^-$ are two consecutive components on Γ_j , then

$$\frac{1}{2\pi} \operatorname{Incr}(\omega + \theta, +(\sigma^+ \cap \sigma^-)) \leq 1.$$

Therefore, if M_j is the number of connected components of $\mathcal{J}^+ \cap \Gamma_j$ (which equals that of $\mathcal{J}^+ \cap \Gamma_j$), then

$$\frac{1}{2\pi} \operatorname{Incr}(\omega + \theta, +\Gamma_j) \leq M_j,$$

and hence

$$\begin{aligned} \sum_{z_k \in \Omega} m(z_k) &= \frac{1}{2\pi} \operatorname{Incr}(\omega, +\Gamma) = \frac{1}{2\pi} \operatorname{Incr}(\omega + \theta, +\Gamma) - D \\ &= \sum_{j=1}^J \frac{1}{2\pi} \operatorname{Incr}(\omega + \theta, +\Gamma_j) \leq \sum_{j=1}^J M_j - D = M - D. \end{aligned}$$

The obstacle problem. An estimate similar to that of Theorem 3.2 has been obtained also for $N = 2$ by Sakaguchi [78] for the *obstacle problem*. Let Ω be bounded and simply connected and let ψ be a given function in $C^2(\bar{\Omega})$ — the obstacle. There exists a unique solution $u \in H_0^1(\Omega)$ such that $u \geq \psi$ in Ω of the obstacle problem

$$\int_{\Omega} \nabla u \cdot \nabla(v - u) dx \geq 0 \quad \text{for every } v \in H_0^1(\Omega) \text{ such that } u \geq \psi.$$

It turns out that $u \in C^{1,1}(\bar{\Omega})$ and u is harmonic outside of the *contact set* $I = \{x \in \Omega : u(x) = \psi(x)\}$. In [78] it is proved that, if the number of connected components of local maximum points of ψ equals J , then

$$\sum_{z_k \in \Omega \setminus I} m(z_k) \leq J - 1,$$

with the usual meaning for z_k and $m(z_k)$. In [78], this result is also shown to hold for a more general class of quasi-linear equations. The proof of this result is based on the analysis of the level sets of u at critical values, in the wake of [1] and [40].

Topological bounds as in Theorems 3.1 or 3.2 are not possible in dimension greater than 2. We give two examples.

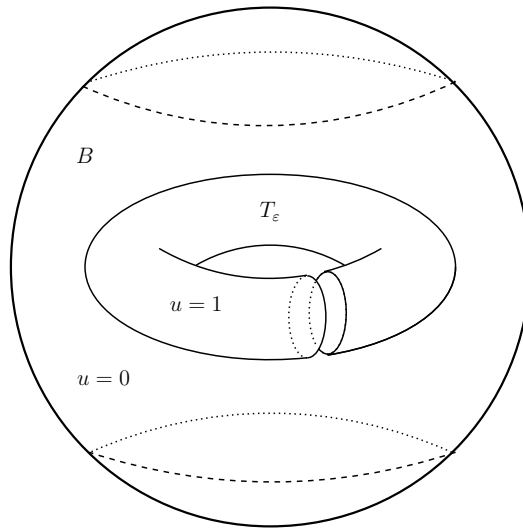


Figure 7: The broken doughnut in a ball: u must have a critical point near the center of B and one between the ends of T .

The broken doughnut in a ball. The first is an adaptation of one contained in [28] and reproduces the situation of Theorem 3.1 (see Fig. 7). Let B be the unit ball centered at the origin in \mathbb{R}^3 and T an open torus with center of symmetry at the origin and such that $\overline{T} \subset B$. We can always choose coordinate axes in such a way that the x_3 -axis is the axis of revolution for T and hence define the set $T_\varepsilon = \{x \in T : x_2 < \varepsilon^{-1}|x_1|\}$. \overline{T}_ε is simply connected and tends to T as $\varepsilon \rightarrow 0^+$. Now, set $\Omega_\varepsilon = B \setminus \overline{T}_\varepsilon$ and consider a capacity potential for Ω , that is the harmonic function in Ω_ε with the following boundary values

$$u = 0 \text{ on } \partial B, \quad u = 1 \text{ on } \partial T_\varepsilon.$$

Since Ω_ε has 2 planes of symmetry (the x_1x_2 and x_2x_3 planes), the partial derivatives u_{x_1} and u_{x_3} must be zero on the two segments that are the intersection of Ω_ε with the x_2 -axis. If σ is the segment that contains the origin, the restriction of u to $\overline{\sigma}$ equals 1 at the point $\overline{\sigma} \cap \partial T_\varepsilon$, is 0 at the point $\overline{\sigma} \cap \partial B$, is bounded at the origin by a constant < 1 independent of ε , and can be made arbitrarily close to 1 between the “ends” of T_ε , when $\varepsilon \rightarrow 0^+$. It follows that, if ε is sufficiently small, u_{x_2} (and hence ∇u) must vanish twice on σ .

It is clear that this argument does not depend on the size or on small deformations of T . Thus, we can construct in B a (simply connected) “chain” C_ε of an arbitrary number n of such tori, by gluing them together: the solution in

the domain obtained by replacing T_ε by C_ε will then have at least $2n$ critical points.

Circles of critical points. The second example shows that, in general dimension, a finite number of sign changes of some derivative of a harmonic function u on the boundary does not even imply that u has a finite number of critical points.

To see this, consider the harmonic function in Subsection 2.5:

$$u(x, y, z) = J_0(\sqrt{x^2 + y^2}) \cosh(z).$$

It is easy to see that, for instance, on any sphere centered at the origin the normal derivative u_ν changes its sign a finite number of times. However, if the radius of the sphere is larger than the first positive zero of $J_1 = 0$, the corresponding ball contains at least one circle of critical points.

Star-shaped annuli. Nevertheless, if some additional geometric information is added, something can be done. Suppose that $\Omega = D_0 \setminus \overline{D_1}$, where D_0 and D_1 are two domains in \mathbb{R}^N , with boundaries of class C^1 and such that $\overline{D_1} \subset D_0$. Suppose that D_0 and D_1 are *star-shaped* with respect to the same origin O placed in D_1 , that is the segment OP is contained in the domain for every point P chosen in it. Then, the *capacity potential* u defined as the solution of the Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial D_0, \quad u = 1 \text{ on } \partial D_1,$$

does not have critical points in $\overline{\Omega}$. This is easily proved by considering the harmonic function

$$w(x) = x \cdot \nabla u(x), \quad x \in \Omega.$$

Since D_0 and D_1 are starshaped and of class C^1 , $w \geq 0$ on $\partial\Omega$. By the strong maximum principle, then $w > 0$ in Ω ; in particular, ∇u does not vanish in Ω and all the sets $D_1 \cup \{x \in \overline{\Omega} : u(x) > s\}$ turn out to be star shaped too (see [33]). This theorem can be extended to the capacity potential defined in $\Omega = \mathbb{R}^N \setminus \overline{D_1}$ as the solution of

$$\Delta u = 0 \text{ in } \Omega, \quad u = 1 \text{ on } \partial\Omega, \quad u \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

Such results have been extended in [35, 75, 81] to a very general class of non-linear elliptic equations.

3.2. Counting the critical points of Green's functions on manifolds

With suitable restrictions on the coefficients, (2) can be regarded as the *Laplace-Beltrami equation* on the Riemannian surface \mathbb{R}^2 equipped with the metric

$$c(dx)^2 - 2b(dx)(dy) + a(dy)^2.$$

Theorems 3.1 and 3.2 can then be interpreted accordingly.

This point of view has been considered in a more general context in [29, 30], where the focus is on Green's functions of a 2-dimensional complete Riemannian surface (M, g) of finite topological type (that is, the first fundamental group of M is finitely generated). A Green's function is a symmetric function $\mathcal{G}(x, y)$ that satisfies in M the equation

$$-\Delta_g \mathcal{G}(\cdot, y) = \delta_y, \tag{18}$$

where Δ_g is the Laplace-Beltrami operator induced by the metric g and δ_y is the *Dirac delta* centered at a point $y \in M$.

A symmetric Green's function \mathcal{G} can always be constructed by an approximation argument introduced in [61]: an increasing sequence of compact subsets Ω_n containing y and exhausting M is introduced and \mathcal{G} is then defined as the limit on compact subsets of $M \setminus \{y\}$ of the sequence $\mathcal{G}_n - a_n$, where \mathcal{G}_n is the solution of (18) such that $\mathcal{G}_n = 0$ on Γ_n and a_n is a suitable constant. A Green's function defined in this way is generally not unique, but has many properties in common with the fundamental solution for Laplace's equation in the Euclidean plane.

With these premises, in [29, 30] it has been proved the following notable topological bound:

$$\text{number of critical points of } \mathcal{G} \leq 2\mathfrak{g} + \mathfrak{e} - 1,$$

where \mathfrak{g} and \mathfrak{e} are the *genus* and the *number of ends* of M ; the number $2\mathfrak{g} + \mathfrak{e} - 1$ is known as the first *Betti number* of M . Moreover, if the Betti number is attained, then \mathcal{G} is *Morse*, that is at its critical points the Hessian matrix is non-degenerate. In [29], it is also shown that, in dimensions greater than two, an upper bound by topological invariants is impossible.

Two different proofs are constructed in [29] and [30], respectively. Both proofs are based on the following *uniformization principle*: since (M, g) is a smooth manifold of finite topological type, it is well known (see [54]) that there exists a compact surface Σ endowed with a metric g' of constant curvature, a finite number $J \geq 0$ of isolated points $p_j \in \Sigma$ and a finite number $K \geq 0$ of (analytic) topological disks $D_k \subset \Sigma$ such that (M, g) is conformally isometric to the manifold (M', g') , where M' is interior of

$$M' = \Sigma \setminus \left(\bigcup_{j=1}^J \{p_j\} \cup \bigcup_{k=1}^K D_k \right).$$

That means that there exist a diffeomorphism $\Phi : M \rightarrow M'$ and a positive function f on M such that $\Phi^* g' = fg$; it turns out that the genus \mathfrak{g} of Σ and the number $J + K$ — that equals the number \mathfrak{e} ends of M — determine M up to diffeomorphisms.

The proof in [29] then proceeds by analyzing the transformed Green's function $\mathcal{G}' = \mathcal{G} \circ \Phi^{-1}$. It is proved that \mathcal{G}' satisfies the problem

$$-\Delta_{g'} \mathcal{G}'(\cdot, y') = \delta_{y'} - \sum_{j=1}^J c_j \delta_{p_j} \quad \text{in the interior of } M', \quad \mathcal{G}' = 0 \quad \text{on } \bigcup_{k=1}^K \partial D_k,$$

where $y' = \Phi(y)$ and the constants c_j , possibly zero (in which case \mathcal{G}' would be g' -harmonic near p_j), sum up to 1. Thus, a local blow up analysis of the *Hopf index* $\mathfrak{J}(z_n)$, $j = 1, \dots, N$, of the gradient of \mathcal{G}' at the critical points z_1, \dots, z_N (isolated and with finite multiplicity), together with the *Hopf Index Theorem* ([70, 71]), yield the formula

$$\sum_{n=1}^N \mathfrak{J}(z_n) + \sum_{c_j \neq 0} \mathfrak{J}(p_j) = \chi(\Sigma^*),$$

where $\chi(\Sigma^*)$ is the Euler characteristic of the manifold

$$\Sigma^* = \Sigma \setminus \left(D_{y'} \cup \bigcup_{k=1}^K D_k \right)$$

and $D_{y'}$ is a sufficiently small disk around y' . Since $\chi(\Sigma^*)$ is readily computed as $1 - 2g - K$ and $\mathfrak{J}(z_n) \leq -1$, one then obtains that

$$\begin{aligned} \text{number of critical points of } \mathcal{G}' &= - \sum_{n=1}^N \mathfrak{J}(z_n) = \\ &= 2g + K - 1 + \sum_{c_j \neq 0} \mathfrak{J}(p_j) \leq 2g + J + K - 1 = 2g + \epsilon - 1. \end{aligned}$$

Of course, the gradient of \mathcal{G}' vanishes if and only if that of \mathcal{G} does.

The proof contained in [30] has a more geometrical flavor and focuses on the study of the integral curves of the gradient of \mathcal{G} . This point of view is motivated by the fact that in Euclidean space the Green's function (the fundamental solution) arises as the electric potential of a charged particle at y , so that its critical points correspond to equilibria and the integral curves of its gradient field are the lines of force classically studied in the XIX century. Such a description relies on techniques of dynamical systems rather than on the toolkit of partial differential equations.

We shall not get into the details of this proof, but we just mention that it gives a more satisfactory portrait of the integral curves connecting the various critical points of \mathcal{G} — an issue that has rarely been studied.

3.3. Counting the critical points of eigenfunctions

The bounds and identities on the critical points that we considered so far are based on a crucial topological tool: the index $\mathfrak{J}(z_0)$ of a critical point z_0 .

For a function $u \in C^1(\Omega)$, the integer $\mathfrak{J}(z_0)$ is the *winding number or degree* of the vector field ∇u around z_0 and is related to the portrait of the set $\mathcal{N}_u = \{z \in \mathcal{U} : u(z) = u(z_0)\}$ for a sufficiently small neighborhood \mathcal{U} of z_0 . As a matter of fact, if z_0 is an isolated critical point of u , one can distinguish two situations (see [4, 77]):

- (I) if \mathcal{U} is sufficiently small, $\mathcal{N}_u = \{z_0\}$ and $\mathfrak{J}(z_0) = 1$;
- (II) if \mathcal{U} is sufficiently small, \mathcal{N}_u consists of n simple curves and, if $n \geq 2$, each pair of such curves crosses at z_0 only; it turns out that $\mathfrak{J}(z_0) = 1 - n$.

Critical points with index \mathfrak{J} equal to 1, 0, or negative are called *extremal*, *trivial*, or *saddle* points, respectively (see [4]). A saddle point is *simple* or *Morse* if the hessian matrix of u at that point is not trivial.

In the cases we examined so far, we always have that $\mathfrak{J}(z_0) \leq -1$, that is z_0 is a saddle point, since (I) and (II) with $n = 1$ cannot occur, by the maximum principle.

The situation considerably changes when u is a solution of (3), (4), or (5). Here, we shall give an account of what can be said for solutions of (4). The same ideas can be used for solutions of the semilinear equation

$$-\Delta u = f(u) \quad \text{in } \Omega,$$

subject to a homogeneous Dirichlet boundary condition, where the non-linearity $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the assumptions:

$$f(t) > 0 \quad \text{if } t > 0 \quad \text{or} \quad f(t)/t > 0 \quad \text{for } t \neq 0$$

(see [4] for details). We present here the following result that is in the spirit of Theorem 3.1.

THEOREM 3.3 ([4]). *Let Ω be as in Theorem 3.1 and $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of (4). If $z_0 \in \overline{\Omega}$ is an isolated critical point of u in $\overline{\Omega}$, then*

- (A) *either z_0 is a nodal critical point, that is $z_0 \in \mathcal{S}(u)$, and the function $u_x - i u_y$ is asymptotic to $c(z - z_0)^m$, as $z \rightarrow z_0$, for some $c \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{N}$,*
- (B) *or z_0 is an extremal, trivial, or simple saddle critical point.*

Finally, if all the critical points of u in $\bar{\Omega}$ are isolated¹, the following identity holds:

$$\sum_{z_k \in \Omega} m(z_k) + \frac{1}{2} \sum_{z_k \in \Gamma} m(z_k) + n_S - n_E = J - 2. \quad (19)$$

Here, n_S and n_E denote the number of the simple saddle and extremal points of u .

Thus, a bound on the number of critical points in topological terms is not possible — additional information of different nature should be added.

The proof of this theorem can be outlined as follows.

First, one observes that, at a nodal critical point $z_0 \in \Omega$, Δu vanishes, and hence the situation described in Subsection 2.2 is in order, that is $u_x - i u_y$ actually behaves as specified in (A) and the index $\mathfrak{J}(z_0)$ equals $-m$. If $z_0 \in \Gamma$, a reflection argument like the one used for Theorem 3.1 can be used, so that z_0 can be treated as an interior nodal critical point of an extended function with vanishing laplacian at z_0 and (A) holds; in this case, however, as done for Theorem 3.1, the contribution of z_0 must be counted as $-m/2$.

Secondly, one examines non-nodal critical points. At these points Δu is either positive or negative. If, say, $\Delta u(z_0) < 0$, then at least one eigenvalue of the hessian matrix of u must be negative and the remaining eigenvalue is either positive (and hence a simple saddle point arises), negative (and hence a maximum point arises) or zero (and hence, with a little more effort, either a trivial or a simple saddle point arises). Thus, the total index of these points sums up to $n_E - n_S$.

Finally, identity (19) is obtained by applying Hopf's index theorem in a suitable manner.

3.4. Extra assumptions: the emergence of geometry

As emerged in the previous subsection, topology is not enough to control the number of critical points of an eigenfunction or a torsion function. Here, we will explain how some geometrical information about Ω can be helpful.

Convexity is a useful information. If the domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is convex, one can expect that the solution τ of (3) and the only *positive* solution ϕ_1 of (4) — it exists and, as is well known, corresponds to the first Dirichlet eigenvalue λ_1 — have only one critical point (the maximum point). This expectation is realistic, but a rigorous proof is not straightforward.

¹This assumption can be removed when Ω is simply connected, by using the analyticity of u (see [4])

In fact, one has to first show that τ and ϕ_1 are *quasi-concave*, that is one shows the convexity of the level sets

$$\{x \in \Omega : u(x) \geq s\} \text{ for every } 0 \leq s \leq \max_{\bar{\Omega}} u,$$

for $u = \tau$ or $u = \phi_1$. It should be noted that ϕ_1 is *never concave* and examples of convex domains Ω can be constructed such that τ is not concave (see [57]).

The quasi-concavity of τ and ϕ_1 can be proved in several different ways (see [16, 17, 21, 46, 56, 57, 82]). Here, we present the argument used in [57]. There, the desired quasi-convexity is obtained by showing that the functions $\sigma = \sqrt{\tau}$ and $\psi = \log \phi_1$ are concave functions (τ and ϕ_1 are then said *1/2-concave* and *log-concave*, respectively).

In fact, one shows that σ and ψ satisfy the conditions

$$\Delta\sigma = -\frac{1 + 2|\nabla\sigma|^2}{2\sigma} \text{ in } \Omega, \quad \sigma = 0 \text{ on } \Gamma,$$

and

$$\Delta\psi = -(\lambda_1 + |\nabla\psi|^2) \text{ in } \Omega, \quad \psi = -\infty \text{ on } \Gamma.$$

The concavity test established by Korevaar in [57], based on a maximum principle for the so-called *concavity function* (see also [55]), applies to these two problems and guarantees that both σ and ψ are concave. With similar arguments, one can also prove that the solution of (5)-(6) is log-concave in x for any fixed time t .

The obtained quasi-concavity implies in particular that, for $u = \tau$ or ϕ_1 , the set of critical points $\mathcal{C}(u)$, that here coincides with the set

$$\mathcal{M}(u) = \left\{x \in \Omega : u(x) = \max_{\bar{\Omega}} u\right\},$$

is convex. This set cannot contain more than one point, due to the analyticity of u . In fact, if it contained a segment, being the restriction of u analytic on the chord of $\bar{\Omega}$ containing that segment, u would be a *positive* constant on this chord and this is impossible, since $u = 0$ at the endpoints of this chord.

This same argument makes sure that, if $\varphi \equiv 1$ in a convex domain Ω , then for any fixed $t > 0$ there is a *unique* point $x(t) \in \Omega$ — the so-called *hot spot* — at which the solution of (5)-(6) attains its maximum in $\bar{\Omega}$, that is

$$h(x(t), t) = \max_{x \in \bar{\Omega}} h(x, t) \text{ for } t > 0.$$

The location of $x(t)$ in Ω will be one of the issues in the next section.

A conjecture. Counting (or estimating the number of) the critical points of τ , ϕ_1 , or h when Ω is not convex seems a difficult task. For instance, to the

author's knowledge, it is not even known whether or not the uniqueness of the maximum point holds true if Ω is assumed to be *star-shaped* with respect to some origin.

We conclude this subsection by offering and justifying a conjecture on the number of hot spots in a bounded simply connected domain Ω in \mathbb{R}^2 . To this aim, we define for $t > 0$ the set of hot spots as

$$\mathcal{H}(t) = \{x \in \Omega : x \text{ is a local maximum point of } h(\cdot, t)\}.$$

We shall suppose that the function φ in (6) is continuous, non-negative and not identically equal to zero in Ω , so that, by *Hopf's boundary point lemma*, $\mathcal{H}(t) \cap \Gamma = \emptyset$. Also, by an argument based on the analyticity of h similar to that used for the uniqueness of the maximum point in a convex domain, we can be sure that $\mathcal{H}(t)$ is made of isolated points (see [4] for details). (A parabolic version of) Theorem 3.3 then yields that

$$n_E(t) - n_S(t) = 1,$$

where $n_E(t)$ and $n_S(t)$ are the number extremal and simple saddle points of $h(\cdot, t)$; clearly $n_E(t)$ is the cardinality of $\mathcal{H}(t)$. An estimate on the total number of critical points of $h(\cdot, t)$ will then follow from one on $n_E(t)$.

Notice that, if λ_n and ϕ_n , $n \in \mathbb{N}$, are Dirichlet eigenvalues (arranged in increasing order) and eigenfunctions (normalized in $L^2(\Omega)$) of the Laplace's operator in Ω , then the following *spectral formula*

$$h(x, t) = \sum_{n=1}^{\infty} \widehat{\varphi}(n) \phi_n(x) e^{-\lambda_n t} \text{ holds for } x \in \overline{\Omega} \text{ and } t > 0, \quad (20)$$

where $\widehat{\varphi}(n)$ is the Fourier coefficient of φ corresponding to ϕ_n . Then we can infer that $e^{\lambda_1 t} h(x, t) \rightarrow \widehat{\varphi}(1) \phi_1(x)$ as $t \rightarrow \infty$, with

$$\widehat{\varphi}(1) = \int_{\Omega} \varphi(x) \phi_1(x) dx > 0,$$

and the convergence is uniform on $\overline{\Omega}$ under sufficient assumptions on φ and Ω . This information implies that, if $x(t) \in \mathcal{H}(t)$, then

$$\text{dist}(x(t), \mathcal{H}_{\infty}) \rightarrow 0 \text{ as } t \rightarrow \infty, \quad (21)$$

where \mathcal{H}_{∞} is the set of local maximum points of ϕ_1 .

Now, our conjecture concerns the influence of the shape of Ω on the number $n_E(t)$. To rule out the possible influence of the values of φ , we assume that $\varphi \equiv 1$: then we know that there holds the following asymptotic formula (see [85]):

$$\lim_{t \rightarrow 0^+} 4t \log[1 - h(x, t)] = -d_{\Gamma}(x)^2 \text{ for } x \in \overline{\Omega}; \quad (22)$$

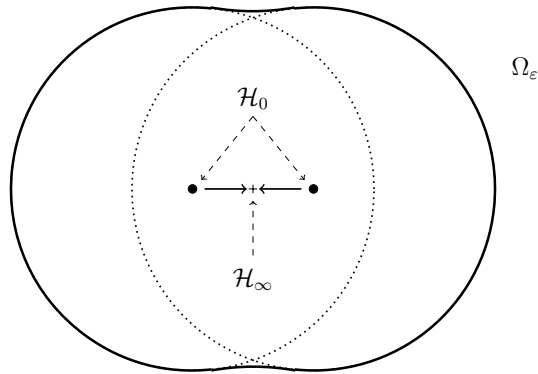


Figure 8: As time t increases, $\mathcal{H}(t)$ goes from \mathcal{H}_0 , the set of maximum points of d_Γ , to \mathcal{H}_∞ , the set of maximum points of ϕ_1 .

here, $d_\Gamma(x)$ is the *distance* of a point $x \in \overline{\Omega}$ from the boundary Γ . The convergence in (22) is uniform on $\overline{\Omega}$ under suitable regularity assumptions on Γ .

Now, suppose that d_Γ has *exactly* m distinct local (strict) maximum points in Ω . Formula (22) suggests that, when t is sufficiently small, $h(\cdot, t)$ has the same number m of maximum points in Ω . As time t increases, one expects that the maximum points of $h(\cdot, t)$ do not increase in number. Therefore, the following bounds should hold:

$$n_E(t) \leq m \text{ and hence } n_E(t) + n_S(t) \leq 2m - 1 \text{ for every } t > 0. \quad (23)$$

From the asymptotic analysis performed on (20), we also derive that the *total number of critical points of ϕ_1* does not exceeds $2m - 1$.

We stress that (23) cannot always hold with the equality sign. In fact, if D_ε^\pm denotes the unit disk centered at $(\pm\varepsilon, 0)$ and we consider the domain Ω_ε obtained from $D_\varepsilon^+ \cup D_\varepsilon^-$ by “smoothing out the corners” (see Fig. 8), we notice that $m = 2$ for every $0 < \varepsilon < 1$, while Ω_ε tends to the unit ball centered at the origin and hence, if ε is small enough, ϕ_1 has only one critical point, being Ω_ε “almost convex”.

Based on a similar argument, inequalities like (23) should also hold for the number of critical points of the torsion function τ . In fact, if U_s is the solution of the one-parameter family of problems

$$-\Delta U_s + sU_s = 1 \text{ in } \Omega, \quad U_s = 0 \text{ on } \Gamma,$$

where s is a positive parameter, we have that

$$\lim_{s \rightarrow 0^+} U_s = \tau \quad \text{and} \quad \lim_{s \rightarrow \infty} \frac{1}{\sqrt{s}} \log[1 - sU_s] = -d_\Gamma,$$

uniformly on $\overline{\Omega}$ (see again [85]).

We finally point out that the asymptotic formulas presented here hold in any dimension; thus, the bounds in (23) may be generalized in some way.

3.5. A conjecture by S. T. Yau

To conclude this section about the number of critical points of solutions of partial differential equations, we cannot help mentioning a conjecture proposed in [89] (also see [32, 48, 49]). This is motivated by the study of eigenfunctions of the Laplace-Beltrami operator Δ_g in a compact Riemannian manifold (M, g) .

Let $\{\phi_k\}_{k \in \mathbb{N}}$ be a sequence of eigenfunctions,

$$\Delta_g \phi_k + \lambda_k \phi_k = 0 \quad \text{in } M.$$

Let $x_k \in M$ be a point of maximum for ϕ_k in M and B_k a geodesic ball centered at x_k and with radius $C/\sqrt{\lambda_k}$. If we blow up B_k to the unit disk in \mathbb{R}^2 and let $u_k/\max \phi_k$ be the eigenfunction after that change of variables, then a subsequence of $\{u_k\}_{k \in \mathbb{N}}$ will converge to a solution u of

$$\Delta u + u = 0, \quad |u| < 1 \quad \text{in } \mathbb{R}^2. \quad (24)$$

If we can prove that u has *infinitely many* isolated critical points, then we can expect that their number be unbounded also for the sequence $\{\phi_k\}_{k \in \mathbb{N}}$.

A naive insight built up upon the available concrete examples of entire eigenfunctions (the *separated* eigenfunctions in rectangular or polar coordinates) may suggest that it would be enough to prove that any solution of (24) has infinitely many nodal domains. It turns out that this is not always true, as a clever counterexample obtained in [32, Theorem 3.2] shows: *there exists a solution of (24) with exactly two nodal domains*.

The counterexample is constructed by perturbing the solution of (24)

$$f = J_1(r) \sin \theta,$$

where (r, θ) are the usual polar coordinates and J_1 is the second Bessel's function; f has infinitely many nodal domains. The desired example is thus obtained by the perturbation $h = f + \varepsilon g$, where $g(x, y) = f(x - \delta_x, y - \delta_y)$ and (δ_x, δ_y) is suitably chosen. As a result, if ε is sufficiently small, the set $\{(x, y) \in \mathbb{R}^2 : h(x, y) \neq 0\}$ is made of two interlocked spiral-like domains (see [32, Figure 3.1]).

A related result was proved in [31], where it is shown that there is no topological upper bound for the number of critical points of the first eigenfunction on Riemannian manifolds (possibly with boundary) of dimension larger than two. In fact, with no restriction on the topology of the manifold, it is possible to construct metrics whose first eigenfunction has as many isolated critical points as one wishes.

Recently, it has been proved in [52] that, if (M, g) is a non-positively curved surface with concave boundary, the number of nodal domains of ϕ_k diverges along a subsequence of eigenvalues of density 1 (see also [53] for related results). The surface needs not have any symmetries. The number can also be shown to grow like $\log \lambda_k$ ([91]). In light of such results, Yau's conjecture was updated as follows: show that, for any (generic) (M, g) there exists at least one subsequence of eigenfunctions for which the number of nodal domains (and hence of the critical points) tends to infinity ([90, 91]).

4. The location of critical points

4.1. A little history

The first result that studies the critical points of a function is probably *Rolle's theorem*: between two zeroes of a differentiable real-valued function there is *at least one* critical point. Thus, a function that has n distinct zeroes also has at least $n - 1$ critical points — an estimate from below — and we roughly know where they are located.

After Rolle's theorem, the first general result concerning the zeroes of the derivative of a general polynomial is *Gauss's theorem*: if

$$P(z) = a_n (z - z_1)^{m_1} \cdots (z - z_K)^{m_K}, \quad \text{with } m_1 + \cdots + m_K = n,$$

is a polynomial of degree n , then

$$\frac{P'(z)}{P(z)} = \frac{m_1}{z - z_1} + \cdots + \frac{m_K}{z - z_K}$$

and hence the zeroes of $P'(z)$ are, in addition to the multiple zeroes of $P(z)$ themselves, the roots of

$$\frac{m_1}{z - z_1} + \cdots + \frac{m_K}{z - z_K} = 0.$$

These roots can be interpreted as the *equilibrium points* of the gravitational field generated by the masses m_1, \dots, m_K placed at the points z_1, \dots, z_K , respectively.

If the zeroes of $P(z)$ are placed on the real line then, by Rolle's theorem, it is not difficult to convince oneself that the zeroes of $P'(z)$ lie in the smallest

interval of the real axis that contains the zeroes of $P(z)$. This simple result has a geometrically expressive generalization in *Lucas's theorem*: the zeroes of $P'(z)$ lie in the *convex hull* Π of the set $\{z_1, \dots, z_K\}$ — named *Lucas's polygon* — and no such zero lies on $\partial\Pi$ unless it is a multiple zero z_k of $P(z)$ or all the zeroes of $P(z)$ are collinear (see Fig. 9).

In fact, it is enough to observe that, if $z \notin \Pi$ or $z \in \partial\Pi$, then all the z_k lie in the closed half-plane H containing them and the side of Π which is the closest to z . Thus, if $\ell = \ell_x + i\ell_y$ is an outward direction to ∂H , we have that

$$\operatorname{Re} \left[\left(\sum_{k=1}^K \frac{m_k}{z - z_k} \right) \ell \right] = \sum_{k=1}^K m_k \frac{\operatorname{Re}[(z - z_k) \ell]}{|z - z_k|^2} > 0,$$

since all the addenda are non-negative and not all equal to zero, unless the z_k 's are collinear.

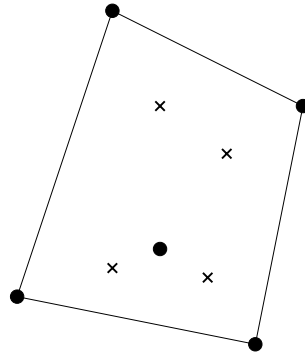


Figure 9: Lucas's theorem: the zeroes of $P'(z)$ must fall in the convex envelope of those of $P(z)$.

If $P(z)$ has real coefficients, we know that its non-real zeroes occur in conjugate pairs. Using the circle whose diameter is the segment joining such a pair — this is called a *Jensen's circle* of $P(z)$ — one can obtain a sharper estimate of the location of the zeroes of $P'(z)$: each non-real zero of $P'(z)$ lies on or within a Jensen's circle of $P(z)$. This result goes under the name of Jensen's theorem (see [87] for a proof).

All these results can be found in Walsh's treatise [87], that contains many other results about zeroes of complex polynomials or rational functions and their extensions to critical points of harmonic functions: among them restricted

versions of Theorem 3.1 give information (i) on the critical points of the Green's function of an infinite region delimited by a finite collection of simple closed curves and (ii) of harmonic measures generated by collections of Jordan arcs. Besides the *argument's principle* already presented in these notes, a useful ingredient used in those extensions is a *Hurwitz's theorem* (based on the classical *Rouché's theorem*): if $f_n(z)$ and $f(z)$ are holomorphic in a domain Ω , continuous on $\bar{\Omega}$, $f(z)$ is non-zero on Γ and $f_n(z)$ converges uniformly to $f(z)$ on $\bar{\Omega}$, then there is a $n_0 \in \mathbb{N}$ such that, for $n > n_0$, $f_n(z)$ and $f(z)$ have the same number of zeroes in Ω .

4.2. Location of critical points of harmonic functions in space

The following result is somewhat an analog of Lucas's theorem and is related to [87, Theorem 1, p. 249], which holds in the plane.

THEOREM 4.1 ([28]). *Let D_1, \dots, D_J be bounded domains in \mathbb{R}^N , $N \geq 3$, with boundaries of class $C^{1,\alpha}$ and with mutually disjoint closures, and set*

$$\Omega = \mathbb{R}^N \setminus \bigcup_{j=1}^J \bar{D}_j.$$

Let $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ be the solution of the boundary value problem

$$\Delta u = 0 \text{ in } \Omega, \quad u = 1 \text{ on } \Gamma, \quad u(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty. \quad (25)$$

If \mathcal{K} denotes the convex hull of

$$\bigcup_{j=1}^J D_j,$$

then u does not have critical points in $\overline{\mathbb{R}^N \setminus \mathcal{K}}$ (see Fig. 10).

This theorem admits at least two proofs and it is worth to present both of them. The former is somewhat reminiscent of Lucas's proof and is based on an explicit formula for u ,

$$u(x) = \frac{1}{(N-2)\omega_N} \int_{\Gamma} \frac{u_\nu(y)}{|x-y|^{N-2}} dS_y, \quad x \in \Omega,$$

that can be derived as a consequence of *Stokes's formula*. Here, ω_N is the surface area of a unit sphere in \mathbb{R}^N , dS_y denotes the $(N-1)$ -dimensional surface measure, and u_ν is the (outward) normal derivative of u .

By the Hopf's boundary point lemma, $u_\nu > 0$ on Γ . Also, if $x \in \overline{\mathbb{R}^N \setminus \mathcal{K}}$, we can choose a hyperplane π passing through x and supporting \mathcal{K} (at some point). If ℓ is the unit vector orthogonal to π at x and pointing into the half-space containing \mathcal{K} , we have that $(x-y) \cdot \ell$ is non-negative and is not identically zero for $y \in \Gamma$. Therefore,

$$u_\ell(x) = -\frac{1}{\omega_N} \int_{\Gamma} \frac{u_\nu(y) (x-y) \cdot \ell}{|x-y|^N} dS_y < 0,$$

which means that $\nabla u(x) \neq 0$.

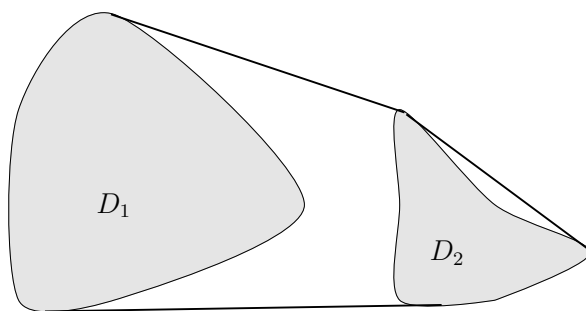


Figure 10: No critical points outside of the convex envelope.

The latter proof is based on a symmetry argument ([79]) and, as it will be clear, can also be extended to more general non-linear equations. Let π be any hyperplane contained in $\overline{\Omega}$ and let H be the open half-space containing \mathcal{K} and such that $\partial H = \pi$. Let x' be the mirror reflection in π of any point $x \in H \cap \Omega$. Then the function defined by

$$u'(x) = u(x') \quad \text{for } x \in H \cap \Omega$$

is harmonic in $H \cap \Omega$, tends to 0 as $|x| \rightarrow \infty$ and

$$u' < u \quad \text{in } H \cap \Omega, \quad u' = u \quad \text{on } \pi \setminus \Gamma.$$

Therefore, by the Hopf's boundary point lemma, $u_\ell(x) \neq 0$ at any $x \in \pi \setminus \Gamma$ for any direction ℓ not parallel to π . Of course, if $x \in \Gamma \cap \pi$, we obtain that $u_\nu(x) > 0$ by directly using the Hopf's boundary point lemma.

Generalizations of Lucas's theorem hold for other problems. Here, we mention the well known result of Chavel and Karp [25] for the minimal solution of the Cauchy problem for the heat equation in a Riemannian manifold (M, g) :

$$u_t = \Delta_g u \quad \text{in } M \times (0, \infty), \quad u = \varphi \quad \text{on } M \times \{0\}, \quad (26)$$

where φ is a bounded initial data with compact support in M . In [23], it is shown that, if M is complete, simply connected and of constant curvature, then the set of the *hot spots* of u ,

$$\mathcal{H}(t) = \left\{ x \in M : u(x, t) = \max_{y \in M} u(y, t) \right\},$$

is contained in the convex hull of the support of φ . The proof is based on an explicit formula for u in terms of the initial values φ . For instance, when $M = \mathbb{R}^N$, we have the formula

$$u(x, t) = (4\pi t)^{-N/2} \int_{\mathbb{R}^N} e^{-|x-y|^2} \varphi(y) dy \quad \text{for } (x, t) \in \mathbb{R}^N \times (0, \infty).$$

With this formula in hand, by looking at the second derivatives of u , one can also prove that there is a time $T > 0$ such that, for $t > T$, $\mathcal{H}(t)$ reduces to the single point

$$\frac{\int_{\mathbb{R}^N} y \varphi(y) dy}{\int_{\mathbb{R}^N} \varphi(y) dy},$$

which is the center of mass of the measure space $(\mathbb{R}^N, \varphi(y)dy)$ (see [51]).

We also mention here the work of Ishige and Kabeya ([43, 44, 45]) on the large time behavior of hot spots for solutions of the heat equation with a rapidly decaying potential and for the Schrödinger equation.

4.3. Hot spots in a grounded conductor

From a physical point of view, the solution (26) describes the evolution of the temperature of M when its initial value distribution is known on M . The situation is more difficult if ∂M is not empty. We shall consider here the case of a *grounded* heat conductor, that is we will study the solution h of the Cauchy-Dirichlet problem (5)-(6).

Bounded conductor. As already seen, if $\varphi \geq 0$, (20) implies (21). For an arbitrary continuous function φ , from (20) we can infer that, if m is the first integer such that $\widehat{\varphi}(n) \neq 0$ and $m + 1, \dots, m + k - 1$ are all the integers such that $\lambda_m = \lambda_{m+1} = \dots = \lambda_{m+k-1}$, then

$$e^{\lambda_m t} h(x, t) \rightarrow \sum_{n=m}^{m+k-1} \widehat{\varphi}(n) \phi_n(x) \quad \text{if } t \rightarrow \infty.$$

Also, when $\varphi \equiv 1$, (22) holds and hence

$$\text{dist}(x(t), \mathcal{H}_0) \rightarrow 0 \quad \text{as } t \rightarrow 0, \tag{27}$$

where \mathcal{H}_0 is the set of local (strict) maximum points of d_Γ . These informations give a rough picture of the *set of trajectories* of the hot spots:

$$\mathcal{T} = \bigcup_{t>0} \mathcal{H}(t).$$

Notice in passing that, if Ω is convex and has N distinct hyperplanes of symmetry, it is clear that \mathcal{T} is made of the same single point — the intersection of the hyperplanes — that is the hot spot *does not move* or is *stationary*. Also, it is not difficult to show (see [24]) that the hot spot does not move if Ω is invariant under an *essential* group G of orthogonal transformations (that is for every $x \neq 0$ there is $A \in G$ such that $Ax \neq 0$). Characterizing the class \mathcal{P} of convex domains that admit a stationary hot spot seems to be a difficult task: some partial results about convex polygons can be found in [64, 65] (see also [63]). There it is proved that: (i) the equilateral triangle and the parallelogram are the only polygons with 3 or 4 sides in \mathcal{P} ; (ii) the equilateral pentagon and the hexagons invariant under rotations of angles $\pi/3, 2\pi/3$, or π are the only polygons with 5 or 6 sides *all* touching the inscribed circle centered at the hot spot.

The analysis of the behavior of $\mathcal{H}(t)$ for $t \rightarrow 0^+$ and $t \rightarrow \infty$ helps us to show that hot spots *do move* in general.

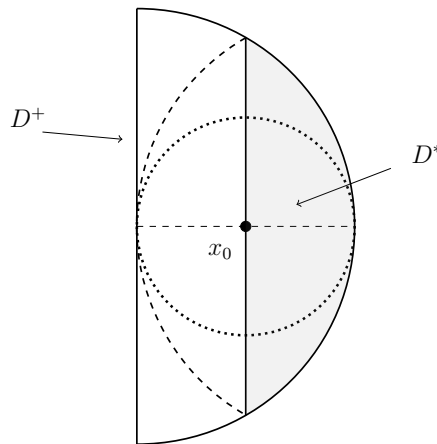


Figure 11: The reflected D^* is contained in D^+ , hence h' can be defined in D^* .

To see this, it is enough to consider the *half-disk* (see Fig. 11)

$$D^+ = \{(x, y) \in \mathbb{R}^2 : |x| < 1, x_1 > 0\};$$

being D^+ convex, for each $t > 0$, there is a unique hot spot that, as $t \rightarrow 0^+$, tends to the maximum point $x_0 = (1/2, 0)$ of d_Γ . Thus, it is enough to show that x_0 is not a spatial critical point of $h(x, t)$ for some $t > 0$ or, if you like, for ϕ_1 .

This is readily seen by *Alexandrov's reflection principle*. Let $D^* = \{x \in D^+ : x_1 > 1/2\}$ and define

$$h'(x_1, x_2, t) = h(1 - x_1, x_2, t) \text{ for } (x_1, x_2, t) \in \overline{D^*} \times (0, \infty);$$

h' is the reflection of h in the line $x_1 = 1/2$. We clearly have that

$$\begin{aligned} (h' - h)_t &= \Delta(h - h') \text{ in } D^* \times (0, \infty), \quad h' - h = 0 \text{ on } D^* \times \{0\}, \\ h' - h > 0 &\text{ on } (\partial D^* \cap \partial^+) \times (0, \infty), \quad h' - h = 0 \text{ on } (\partial D^* \cap D^+) \times (0, \infty). \end{aligned}$$

Thus, the strong maximum principle and the Hopf's boundary point lemma imply that

$$-2h_{x_1}(1/2, x_2, t) = h'_{x_1}(1/2, x_2, t) - h_{x_1}(1/2, x_2, t) > 0$$

for $(1/2, x_2, t) \in (\partial D^* \cap D^+) \times (0, \infty)$, and hence x_0 cannot be a critical point of h .

The Alexandrov's principle just mentioned can also be employed to estimate the location of a hot spot. In fact, as shown in [18], by the same arguments one can prove that hot spots must belong to the subset $\heartsuit(\Omega)$ of Ω defined as follows. Let π_ω be a hyperplane orthogonal to the direction $\omega \in \mathbb{S}^{N-1}$ and let H_ω^+ and H_ω^- be the two half-spaces defined by π_ω ; let $\mathcal{R}_\omega(x)$ denote the mirror reflection of a point x in π_ω . Then, the *heart*² of Ω is defined by

$$\heartsuit(\Omega) = \bigcap_{\omega \in \mathbb{S}^{N-1}} \{H_\omega^- \cap \Omega : \mathcal{R}_\omega(H_\omega^+ \cap \Omega) \subset \Omega\}.$$

When Ω is convex, then $\heartsuit(\Omega)$ is also convex and, if Γ is of class C^1 , we are sure that its distance from Γ is positive (see [34]). Also, we know that $\mathcal{H}(t)$ is made of only one point $x(t)$, so that

$$\text{dist}(x(t), \Gamma) \geq \text{dist}(\heartsuit(\Omega), \Gamma).$$

The set $\heartsuit(\Omega)$ contains many notable geometric points of the set Ω , such as the *center of mass*, the *incenter*, the *circumcenter*, and others; see [19], where

² $\heartsuit(\Omega)$ has also been considered in [72] under the name of *minimal unfolded region*.

further properties of the heart of a convex body are presented. See also [80] for related research on this issue.

As clear from [18], the estimate just presented is of purely geometric nature, that is it only depends on the lack of symmetry of Ω and does not depend on the particular equation we are considering in Ω , as long as the equation is invariant by reflections.

A different way to estimate the location of the hot spot of a grounded convex heat conductor or the maximum point of the solution of certain elliptic equations is based on ideas related to Alexandrov-Bakelman-Pucci's maximum principle and does take into account the information that comes from the relevant equation. For instance, in [18] it is proved that the maximum point x_∞ of ϕ_1 in $\bar{\Omega}$ is such that

$$\text{dist}(x_\infty, \Gamma) \geq C_N r_\Omega \left(\frac{r_\Omega}{\text{diam}(\Omega)} \right)^{N^2-1}, \tag{28}$$

where C_N is a constant only depending on N , r_Ω is the *inradius* of Ω (the radius of a largest ball contained in Ω) and $\text{diam}(\Omega)$ is the *diameter* of Ω .

The idea of the proof of (28) is to compare the *concave envelope* f of ϕ_1 — the smallest concave function above ϕ_1 — and the function g whose graph is the surface of the (truncated) cone based on Ω and having its tip at the point $(x_\infty, \phi(x_\infty))$ (see Fig. 12).

Since $f \geq g$ and $f(x_\infty) = g(x_\infty)$, we can compare their respective *subdifferential images*:

$$\begin{aligned} \partial f(\Omega) &= \bigcup_{x \in \bar{\Omega}} \{p \in \mathbb{R}^N : f(x) + p \cdot (y - x) \geq f(y) \text{ for } y \in \bar{\Omega}\}, \\ \partial g(\Omega) &= \bigcup_{x \in \bar{\Omega}} \{p \in \mathbb{R}^N : g(x) + p \cdot (y - x) \geq g(y) \text{ for } y \in \bar{\Omega}\}; \end{aligned}$$

in fact, it holds that $\partial g(\Omega) \subseteq \partial f(\Omega)$.

Now, $\partial g(\Omega)$ has a precise geometrical meaning: it is the set $\phi_1(x_\infty) \Omega^*$, that is a multiple of the *polar set* of Ω with respect to x_∞ defined by

$$\Omega^* = \{y \in \mathbb{R}^N : (x - x_\infty) \cdot (y - x_\infty) \leq 1 \text{ for every } x \in \bar{\Omega}\}.$$

The volume $|\partial f(\Omega)|$ can be estimated by the formula of change of variables to obtain:

$$\phi_1(x_\infty)^N |\Omega^*| = |\partial g(\Omega)| \leq |\partial f(\Omega)| \leq \int_C |\det D^2 f| dx = \int_C |\det D^2 \phi_1| dx,$$

where $C = \{x \in \bar{\Omega} : f(x) = \phi_1(x)\}$ is the *contact set*. Since the determinant and the trace of a matrix are the product and the sum of the eigenvalues of the

matrix, by the *arithmetic-geometric mean inequality*, we have that $|\det D^2\phi_1| \leq (-\Delta\phi_1/N)^N$, and hence we can infer that

$$|\Omega^*| \leq \int_C \left[\frac{-\Delta\phi_1}{N\phi_1(x_\infty)} \right]^N dx = \int_C \left[\frac{\lambda_1(\Omega)\phi_1}{N\phi_1(x_\infty)} \right]^N dx \leq \left[\frac{\lambda_1(\Omega)}{N} \right]^N |\Omega|,$$

being $\phi_1 \leq \phi_1(x_\infty)$ in $\bar{\Omega}$. Finally, in order to get (28) explicitly, one has to bound $|\Omega^*|$ from below by the volume of the polar set of a suitable half-ball containing Ω , and $\lambda_1(\Omega)$ from above by the *isodiametric inequality* (see [18] for details).

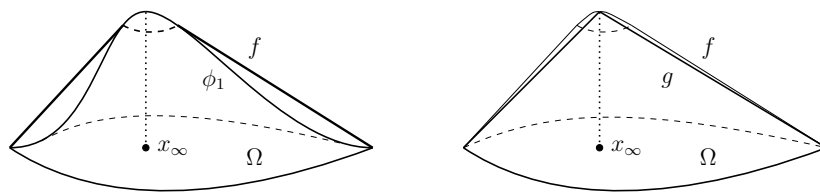


Figure 12: The concave envelope of ϕ_1 and the cone g . The dashed cap is the image $f(C) = \phi_1(C)$ of the contact set C .

The two methods we have seen so far, give estimates of how far the hot spot must be from the boundary. We now present a method, due to Grieser and Jerison [37], that gives an estimate of how far the hot spot can be from a specific point in the domain. The idea is to adapt the classical *method of separation of variables* to construct a suitable approximation u of the first Dirichlet eigenfunction ϕ_1 in a planar convex domain. Clearly, if Ω were a rectangle, say $[a, b] \times [0, 1]$, then that approximation would be exact: in fact

$$u(x, y) = \phi_1(x, y) = \sin[\pi(x - a)/(b - a)].$$

If Ω is not a rectangle, after some manipulations, we can suppose that

$$\Omega = \{(x, y) : a < x < b, f_1(x) < y < f_2(x)\}$$

where, in $[a, b]$, f_1 is convex, f_2 is concave and

$$0 \leq f_1 \leq f_2 \leq 1 \quad \text{and} \quad \min_{[a,b]} f_1 = 0, \quad \max_{[a,b]} f_2 = 1$$

(see Fig. 13).

The geometry of Ω does not allow to find a solution by separation of variables as in the case of the rectangle. However, one can operate “as if” that

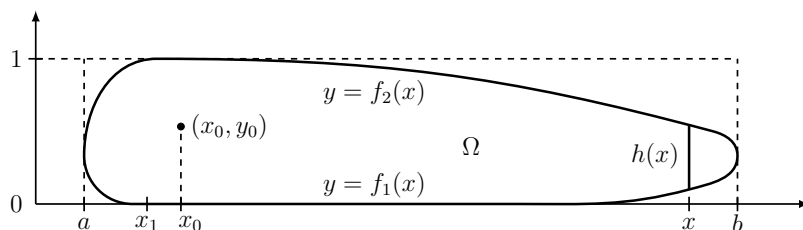


Figure 13: Estimating the hot spot in the “long” convex set Ω .

separation were possible. To understand that, consider the length of the section of foot x , parallel to the y -axis, by

$$h(x) = f_2(x) - f_1(x) \quad \text{for } a \leq x \leq b,$$

and notice that, if we set

$$\alpha(x, y) = \pi \frac{y - f_1(x)}{h(x)},$$

the function

$$e(x, y) = \sqrt{2/h(x)} \sin \alpha(x, y) \quad \text{for } f_1(x) \leq y \leq f_2(x),$$

satisfies for fixed x the problem

$$e_{yy} + \pi^2 e = 0 \quad \text{in } (f_1(x), f_2(x)), \quad e(x, f_1(x)) = e(x, f_2(x)) = 0$$

— thus, it is the first Dirichlet eigenfunction in the interval $(f_1(x), f_2(x))$, normalized in the space $L^2([f_1(x), f_2(x)])$. The basic idea is then that $\phi_1(x, y)$ should be (and in fact it is) well approximated by its lowest Fourier mode in the y -direction, computed for each fixed x , that is by the projection of ϕ_1 along e :

$$\psi(x) e(x, y) \quad \text{where } \psi(x) = \int_{f_1(x)}^{f_2(x)} \phi_1(x, \eta) e(x, \eta) d\eta.$$

To simplify matters, a further approximation is needed: it turns out that ψ and its first derivative can be well approximated by $\phi/\sqrt{2}$ and its derivative, where ϕ is the first eigenfunction of the problem

$$\phi''(x) + \left[\mu - \frac{\pi^2}{h(x)^2} \right] \phi(x) = 0 \quad \text{for } a < x < b, \quad \phi(a) = \phi(b) = 0.$$

Since near the maximum point x_1 of ϕ , $|\phi'(x)|$ can be bounded from below by a constant times $|x - x_1|$, the constructed chain of approximations gives that, if (x_0, y_0) is the maximum point of ϕ_1 on $\bar{\Omega}$, then there is an absolute constant C such that

$$|x_1 - x_0| \leq C.$$

C is independent of Ω , but the result has clearly no content unless $b - a > C$.

Unbounded conductor. If Ω is unbounded, by working with suitable barriers, one can still prove formula (22) when $\varphi \equiv 1$ (see [66, 67]), the convergence holding uniformly on compact subsets of Ω . Thus, any hot spot $x(t)$ will again satisfy (27).

To the author's knowledge, [51] is the only reference in which the behavior of hot spots for large times has been studied for some grounded unbounded conductors. There, the cases of a half-space $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}$ and the exterior of a ball $B^c = \{x \in \mathbb{R}^N : |x| > 1\}$ are considered. It is shown that there is a time $T > 0$ such that for $t > T$ the set $\mathcal{H}(t)$ is made of only one hot spot $x(t) = (x_1(t), \dots, x_N(t))$ and

$$x_j(t) \rightarrow \frac{\int_{\mathbb{R}^{N-1}} y_j y_N \varphi(y') dy'}{\int_{\mathbb{R}^{N-1}} y_N \varphi(y') dy'}, \quad 1 \leq j \leq N - 1, \quad \frac{x_N(t)}{\sqrt{2t}} \rightarrow 1 \quad \text{as } t \rightarrow \infty,$$

if $\Omega = \mathbb{R}_+^N$, while for $\Omega = B^c$, if φ is radially symmetric, then there is a time $T > 0$ such that $\mathcal{H}(t) = \{x \in \mathbb{R}^N : |x| = r(t)\}$, for $t > T$, where $r(t)$ is some smooth function of t such that

$$\limsup_{t \rightarrow \infty} r(t) = \infty.$$

Upper bounds for $\mathcal{H}(t)$ are also given in [51] for the case of the exterior of a smooth bounded domain.

4.4. Hot spots in an insulated conductor

We conclude this survey by giving an account on the so-called *hot spot conjecture* by J. Rauch [76]. This is related to the asymptotic behavior of hot spots in a *perfectly insulated* heat conductor modeled by the following initial-boundary value problem:

$$h_t = \Delta h \quad \text{in } \Omega \times (0, \infty), \quad h = \varphi \quad \text{on } \Omega \times \{0\}, \quad \partial_\nu u = 0 \quad \text{on } \Gamma \times (0, \infty). \quad (29)$$

Observe that, similarly to (20), a spectral formula also holds for the solution of (29):

$$h(x, t) = \sum_{n=1}^{\infty} \hat{\varphi}(n) \psi_n(x) e^{-\mu_n t}, \quad \text{for } x \in \bar{\Omega} \quad \text{and } t > 0. \quad (30)$$

Here $\{\mu_n\}_{n \in \mathbb{N}}$ is the increasing sequence of Neumann eigenvalues and $\{\psi_n\}_{n \in \mathbb{N}}$ is a complete orthonormal system in $L^2(\Omega)$ of eigenfunctions corresponding to the μ_n 's, that is ψ_n is a non-zero solution of

$$\Delta\psi + \mu\psi = 0 \text{ in } \Omega, \quad \partial_\nu\psi = 0 \text{ on } \Gamma, \quad (31)$$

with $\mu = \mu_n$. The numbers $\widehat{\varphi}(n)$ are the Fourier coefficients of φ corresponding to ψ_n , that is

$$\widehat{\varphi}(n) = \int_{\Omega} \varphi(x) \psi_n(x) dx, \quad n \in \mathbb{N}.$$

Since $\mu_1 = 0$ and $\psi_1 = 1/\sqrt{|\Omega|}$, we can infer that

$$e^{\mu_m t} \left[h(x, t) - \frac{1}{\sqrt{|\Omega|}} \int_{\Omega} \varphi dx \right] \rightarrow \sum_{n=m}^{m+k-1} \widehat{\varphi}(n) \psi_n(x) \text{ as } t \rightarrow \infty, \quad (32)$$

where m is the first integer such that $\widehat{\varphi}(n) \neq 0$ and $m+1, \dots, m+k-1$ are all the integers such that $\mu_m = \mu_{m+1} = \dots = \mu_{m+k-1}$. Thus, similarly to what happens for the case of a grounded conductor, as $t \rightarrow \infty$, a hot spot $x(t)$ of h tends to a maximum point of the function at the right-hand side of (32).

Now, roughly speaking, the conjecture states that, for “most” initial conditions φ , the distance from Γ of any hot and cold spot of h must tend to zero as $t \rightarrow \infty$, and hence it amounts to prove that the right-hand side of (32) attains its maximum and minimum at points in Γ .

It should be noticed now that the quotes around the word *most* are justified by the fact that the conjecture does not hold for all initial conditions. In fact, as shown in [10], if $\Omega = (0, 2\pi) \times (0, 2\pi) \subset \mathbb{R}^2$, the function defined by

$$h(x_1, x_2, t) = -e^{-t}(\cos x_1 + \cos x_2), \quad (x_1, x_2) \in \Omega, \quad t > 0,$$

is a solution of (29) — with $\varphi(x_1, x_2) = -(\cos x_1 + \cos x_2)$ — that attains its maximum at $(-\pi, \pi)$ for any $t > 0$. However, it turns out that in this case $h(x_1, x_2, t) = -e^{-\mu_4 t} \psi_4(x_1, x_2)$. Thus, it is wiser to rephrase the conjecture by asking whether or not the hot and cold spots tend to Γ if the coefficient $\widehat{\varphi}(2)$ of the first non-constant eigenfunction ψ_2 is not zero or, which is the same, whether or not maximum and minimum points of ψ_2 in $\overline{\Omega}$ are attained only on Γ .

In [55], a weaker version of this last statement is proved to hold for domains of the form $D \times (0, a)$, where $D \subset \mathbb{R}^{N-1}$ has a boundary of class $C^{0,1}$. In [55], the conjecture has also been reformulated for convex domains. Indeed, we now know that it is false for fairly general domains: in [20] a planar domain with two holes is constructed, having a simple second eigenvalue and such that the corresponding eigenfunction attains its strict maximum at an interior point of the domain. It turns out that in that example the minimum point is on

the boundary. Nevertheless, in [12] it is given an example of a domain whose second Neumann eigenfunction attains both its maximum and minimum points at interior points. In both examples the conclusion is obtained by probabilistic methods.

Besides in [55], positive results on this conjecture can be found in [9, 10, 11, 27, 50, 69, 73, 83]. In [10], the conjecture is proved for planar convex domains Ω with two orthogonal axis of symmetry and such that

$$\frac{\text{diam}(\Omega)}{\text{width}(\Omega)} > 1.54.$$

This restriction is removed in [50]. In [73], Ω is assumed to have only one axis of symmetry, but ψ_2 is assumed anti-symmetric in that axis. A more general result is contained in [9]: the conjecture holds true for domains of the type

$$\Omega = \{(x_1, x_2) : f_1(x_1) < x_2 < f_2(x_1)\},$$

where f_1 and f_2 have unitary Lipschitz constant. In [27], a modified version is considered: it holds true for general domains, if *vigorous maxima* are considered (see [27] for the definition). If no symmetry is assumed for a convex domain Ω , Y. Miyamoto [69] has verified the conjecture when

$$\frac{\text{diam}(\Omega)^2}{|\Omega|} < 1.378$$

(for a disk, this ratio is about 1.273).

For unbounded domains, the situation changes. For the half-space, Jimbo and Sakaguchi proved in [51] that there is a time T after which the hot spot equals a point on the boundary that depends on φ . In [51], the case of the exterior Ω of a ball $\overline{B_R}$ is also considered for a radially symmetric φ . For a suitably general φ , Ishige [41] has proved that the behavior of the hot spot is governed by the point

$$A_\varphi = \frac{\int_{\Omega} x \left(1 + \frac{R^N}{N-1} |x|^{-N}\right) \varphi(x) dx}{\int_{\Omega} \varphi(x) dx}.$$

If $A_\varphi \in B_R$, then $\mathcal{H}(t)$ tends to the boundary point $RA_\varphi/|A_\varphi|$, while if $A_\varphi \notin B_R$, then $\mathcal{H}(t)$ tends to A_φ itself.

Results concerning the behavior of hot spots for parabolic equations with a rapidly decaying potential can be found in [43, 44].

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Author's address:

Rolando Magnanini
Dipartimento di Matematica "U. Dini"
Università di Firenze
Viale Morgagni, 67/A
50134 Firenze, Italia
E-mail: magnanin@math.unifi.it

Received March 29, 2016
Accepted April 4, 2016