Rend. Istit. Mat. Univ. Trieste
Volume 48 (2016), 5–19
DOI: 10.13137/2464-8728/13149

Exponential decay of solutions to initial boundary value problem for anisotropic visco-elastic systems

Gen Nakamura and Marcos Oliva

Dedicated to Prof. Giovanni Alessandrini on the occasion of his 60th birthday

Abstract. *The paper concerns the asymptotic behaviour of solutions of initial boundary value problem for a general anisotropic viscoelastic system in the form of integrodifferential system of equations with homogeneous mixed boundary condition. We put a usual assumption on the relaxation tensor and assume that the inhomogeneous term of the equation and boundary data are zero. Then, by using the energy method, we show that the solutions decays exponentially with respect to time as it tends to infinity.*

Keywords: anisotropic viscolastic system, decay of solutions.
MS Classification 2010: 73F15, 35B40, 35Q72.

1. Introduction

In this paper we will study the asymptotic behavior of solutions of the initial boundary value problem for general anisotropic viscoelastic integrodifferential system abbreviated by AVIS with homogeneous mixed type boundary condition. The main objective of this paper is to show the exponential decay of solutions with respect to time t as $t \to \infty$ of solutions when the initial data are zero and the relaxation tensor G satisfies a usual asymptotic behavior with respect to time as it tends to infinity. For this usual asymptotic behavior of G, see [1]. In many measurement devices such as a clinical diagnosing modality called the magnetic resonance elastography ([7]) and a rhelogical measurement device called the pendulum type viscoelastic spectroscopy ([8]) which use time harmonic vibrations it is important to have a very short transition time between time harmonic vibrations with different frequencies ω_1 and ω_2 when the frequency of vibration changes from ω_1 to ω_2. This can be ensured if the solutions decay exponential as $t \to \infty$ (see the argument in [4]).

In order to formulate the initial boundary value problem let $\Omega \subset \mathbb{R}^n$, $(2 \leq n \in \mathbb{N})$ be a bounded domain with C^1 smooth boundary $\partial\Omega$. Divide $\partial\Omega$ into $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where $\Gamma_D, \Gamma_N \subset \partial\Omega$ are open and assume that $\Gamma_D \neq \emptyset$, $\Gamma_D \cap \Gamma_N = \emptyset$ and if $n \geq 3$ then their boundaries $\partial\Gamma_D$, $\partial\Gamma_N$ are C^1 smooth if $n \geq 3$.

Consider the following initial boundary value problem

$$\begin{cases} \rho\partial_t^2 u(\cdot,t) = \nabla \cdot \left\{ C(\cdot)\nabla u(\cdot,t) - \displaystyle\int_0^t G(\cdot, t-\tau)\nabla u(\cdot,\tau)\, d\tau \right\} \ (t>0) \\ u = 0 \text{ on } \Gamma_D \times (0,\infty),\ Tu = 0 \text{ on } \Gamma_N \times (0,\infty) \\ u = 0,\ \partial_t u = f \in L^2(\Omega) \text{ on } \Omega \times \{0\}, \end{cases} \tag{1}$$

where $\partial_t = \frac{\partial}{\partial t}$, Tu is the traction given by

$$Tu(\cdot,t) = \left(C(\cdot)\nabla u(\cdot,t) - \int_0^t G(\cdot,t)\nabla u(\cdot,\tau)\, d\tau \right) \nu$$

with the unit outer normal vector ν of $\partial\Omega$. Here $0 < \rho_0 \leq \rho \in L^\infty(\Omega)$ with a positive constant ρ_0, $C = (C_{ijk\ell})$ and $G = (G_{ijk\ell})$ denote the elasticity tensor and relaxation tensor, respectively. Here we note that it is enough to consider the initial condition given above due to the Duhamel principle.

We assume the following assumptions on C and G.

(i) $C \in L^\infty(\Omega)$ and $G = e^{-\kappa t}\hat{G}$ with $\hat{G} = (\hat{G}_{ijk\ell}) \in L^\infty(\Omega)$ and some constant $\kappa > 0$.

(ii) (major symmetry) $C_{ijk\ell} = C_{k\ell ij}, G_{ijk\ell} = G_{k\ell ij}$ a.e. in $\Omega, 1 \leq i,j,k,\ell \leq n$.

(iii) (strong convexity) There exist constants $\alpha_0 > 0$ and $\beta_0 > 0$ such that for any $n \times n$ symmetric matrix $w = (w_{ij})$

$$\alpha_0|w|^2 \leq (Cw) : w \leq \beta_0|w|^2, \quad \alpha_0|w|^2 \leq (\hat{G}w) : w \leq \beta_0|w|^2 \tag{2}$$

where the notation ":" is defined as $(Cw) : w = \sum_{i,j,k,\ell=1}^n C_{ijk\ell}w_{ij}w_{k\ell}$.

(iv) There exists some constants $\mu_0 > 0$, $\nu_0 > 0$ such that for any $u(\cdot,t) \in C^0([0,\infty); H^1(\Omega))$,

$$\begin{aligned} \mu_0 \int_\Omega \left|\nabla u(\cdot,t)\right|^2 dx & \\ \leq \int_\Omega \left\{ \left(C(\cdot) - \int_0^\infty G(\cdot,\tau)\, d\tau \right) \nabla u(\cdot,t) \right\} : \nabla u(\cdot,t)\, dx & \\ \leq \nu_0 \int_\Omega \left|\nabla u(\cdot,t)\right|^2 dx, \qquad & t \geq 0. \end{aligned} \tag{3}$$

REMARK 1.1. The last assumption can be given in the form

$$\mu_0 \int_\Omega |\nabla u|^2\, dx \le \int_\Omega \left\{(C - \kappa^{-1}\hat{G})\nabla u\right\} : \nabla u\, dx \le \nu_0 \int_\Omega |\nabla u|^2\, dx$$

for any $u \in H^1(\Omega)$.

Our main result is as follows.

THEOREM 1.2 (Main result). *The solution $u \in C^3([0,\infty); H^1(\Omega))$ to* (1) *which exists provided that the initial data f satisfies the smoothness condition of order 3 will converge to zero exponentially fast in time t.*

REMARK 1.3. For the definition of the smoothness condition of order 3, see [2]. Also, the existence of $u \in C^3([0,\infty); H^1(\Omega))$ easily follows from Theorem 2.2 therein.

There are several studies on the asymptotic behavior of solutions of AVIS as follows. Some abstract schemes for an integrodifferential equation were developed given in [2, 3] and applied showed that solutions of AVIS satisfying the Dirichlet boundary condition decay to zero as the time tends to infinity. Concerning the decay rate of the solutions, a polynomial order decay was shown in [6] by the energy method introducing an energy norm which is effective to analyze the asymptotic behavior of solutions. The first result on the exponential decay of solution was given in [5]. More precisely the author studied a special isotropic viscoelastic integrodifferential system with exponentially decaying relaxation function and gave the exponential decay of solutions satisfying the Dirichlet boundary condition. Our method is based on the aforementioned energy method given in [6] with careful estimates of constants in energy inequalities.

The rest of this paper is organized as follows. In Section 2 we introduce some notations and give the strategy of proof. Then we provide some basic identities and inequalities given in [6] in Section 3. Since we are concerned about the constants in these identities and inequalities we will give their proofs. Following the arguments in [6], we carefully carry out the strategy in Section 4. At last in Section 5, we will give some conclusion and discussion.

Here after for simplicity we will assume $\rho = 1$.

2. Notations and strategy of proof

2.1. Notations

We will use the following notations

$$E(t,u) := \frac{1}{2}\left[\int_\Omega |u(\cdot,t)|^2 + (G\Box\partial u)(\cdot,t)\,dx \right.$$
$$\left. + \int_\Omega \{(C(\cdot) - \int_0^t G(\cdot,\tau)\,d\tau)\nabla u(\cdot,t)\} : \nabla u(\cdot,t)\,dx\right],$$

$$G\Box\partial u(\cdot,t) := \int_0^t \big\{G(\cdot,t-\tau)\nabla\big(u(\cdot,t)-u(\cdot,\tau)\big)\big\} : \nabla\big(u(\cdot,t)-u(\cdot,\tau)\big)\,d\tau\,,$$

$$K(t,u) := \frac{1}{2}\int_\Omega |\ddot{u}|^2\,dx + \frac{1}{2}\int_\Omega \big(C(\cdot)\nabla\dot{u}(\cdot,t)\big) : \nabla\dot{u}(\cdot,t)\,dx$$
$$-\int_\Omega \big(G(\cdot,0)\nabla u(\cdot,t)\big) : \nabla\dot{u}(\cdot,t)\,dx + \gamma\int_\Omega \big(C(\cdot)\nabla u(\cdot,t)\big) : \nabla\dot{u}(\cdot,t)\,dx$$
$$-\int_\Omega \left(\int_0^t F(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\right) : \nabla\dot{u}(\cdot,t)\,dx\,,$$
$$\text{with } F(\cdot,t) := \gamma G(\cdot,t) + \dot{G}(\cdot,t)\,,$$

$$I(t,u) := \int_\Omega \ddot{u}(\cdot,t)\dot{u}(\cdot,t)\,dx - \frac{1}{2}\int_\Omega \big(G(\cdot,0)\nabla u(\cdot,t)\big) : \nabla u(\cdot,t)\,dx$$
$$-\frac{1}{2}\int_\Omega \left(\int_0^t \dot{G}(\cdot,\tau)\,d\tau\,\nabla u(\cdot,t)\right) : \nabla u(\cdot,t)\,dx + \frac{1}{2}\int_\Omega \dot{G}\Box\partial u(\cdot,t)\,dx\,,$$

$$\mathcal{L}(t,u) := N_1E(t,u) + N_2E(t,\dot{u}) + K(t,u) + (\gamma-c)I(t,u) + c_p\int_\Omega \dot{u}u\,dx,$$

where N_1, N_2, γ, c, c_p are positive constants which will satisfy some condition given later in Subsection 4.4 and $c_0 := \big(\sup_{x\in\Omega}\int_0^\infty |G(x,t)|\,dt\big)^{1/2}$ and c_1 is the Poincaré constant of $u(\cdot,t)$.

2.2. Strategy of the proof

By basically following the argument in [6], we estimate $\frac{d}{dt}\mathcal{L}(t,u)$ from above by a negative constant times $\mathcal{L}(t,u)$ and $\mathcal{L}(t,u)$ from below by a positive constant times the sum $\int_\Omega |\nabla u(\cdot,t)|^2\,dx$ with some other positive terms depending on u.

By adjusting these constants N_1, N_2, γ, c, c_p, we have

$$\frac{d}{dt}\mathcal{L}(t,u) \le -M_1\mathcal{L}(t,u),\ \ t>0, \tag{4}$$

$$\mathcal{L}(t,u) \geq M_2 \int_\Omega \left\{ |\ddot{u}|^2 + |\dot{u}|^2 + |\nabla \dot{u}|^2 + |\nabla u|^2 \right\} dx,\ t > 0$$

for some positive constants M_1, M_2.

3. Basic identities and inequalities

The key to derive estimate (4) is based on some basic identities and inequalities. In deriving these identities and inequalities, we show each step where we need the mixed type boundary condition and how constants of inequalities come in. Henceforth in this paper we assume that $u \in C^3([0,\infty); H^1(\Omega))$ is the solution to (1) with the initial data f satisfying the smoothness condition of order 3.

3.1. Basic identities

We first simply cite the following lemma given as Lemma 2.1. in [6].

LEMMA 3.1. *For any $v \in C^1([0,\infty); H^1(\Omega))$ we have*

$$\begin{aligned}
&\int_\Omega \left(\int_0^t G(\cdot, t-\tau) \nabla v(\cdot,\tau)\, d\tau \right) : \nabla \dot{v}(\cdot,t)\, dx = \\
&\quad -\frac{1}{2} \int_\Omega \left(\frac{d}{dt} G \Box \partial v \right)(\cdot,t)\, dx + \frac{1}{2} \int_\Omega \left(\dot{G} \Box \partial v \right)(\cdot,t)\, dx \\
&\quad +\frac{1}{2} \int_\Omega \frac{d}{dt} \left(\int_0^t G(\cdot,\tau)\, d\tau\, \nabla v(\cdot,t) \right) : \nabla v(\cdot,t)\, dx \\
&\quad - \int_\Omega \big(G(\cdot,t) \nabla v(\cdot,t) \big) : \nabla v(\cdot,t)\, dx \ \ t > 0.
\end{aligned}$$

LEMMA 3.2.

$$\begin{aligned}
\frac{d}{dt} E(t,u) &= \int_\Omega \big(G(\cdot,t) \nabla u(\cdot,t) \big) : \nabla u(\cdot,\tau)\, dx + \frac{1}{2} \int_\Omega \dot{G} \Box \partial u(\cdot,x)\, dx, \\
\frac{d}{dt} E(t,\dot{u}) &= \int_\Omega \big(G(\cdot,t) \nabla \dot{u}(\cdot,t) \big) : \nabla u(\cdot,\tau)\, dx + \frac{1}{2} \int_\Omega \dot{G} \Box \partial \dot{u}(\cdot,x)\, dx \\
&\quad + \int_\Omega \big(G(\cdot,t) \nabla u(\cdot,0) \big) : \nabla \ddot{u}(\cdot,t)\, dx,\ \ t > 0.
\end{aligned}$$

Proof. Let us multiply the viscoelastic equation in (1) by $\dot{u}(\cdot,x)$ to get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \Big\{ \int_\Omega |\dot{u}(\cdot,t)|^2 + \big(C(\cdot) \nabla u(\cdot,t) \big) : \nabla u(\cdot,t)\, dx \Big\} \\
&\qquad = \int_\Omega \big(\int_0^t G(\cdot,t-\tau) \nabla u(\cdot,\tau)\, d\tau \big); \nabla \dot{u}(\cdot,t)\, dx
\end{aligned}$$

Using Lemma 3.1 our first assertion holds. To show the second identity, we take the time derivative of the viscoelastic equation in (1) so that

$$\overset{(3)}{u}(\cdot,t)+\nabla\Big\{-C(\cdot)\nabla\dot{u}(\cdot,t)\ +G(\cdot,0)\nabla u(\cdot,t)\ +\int_0^t\dot{G}(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\Big\}=0,$$

where $\overset{(3)}{u}(\cdot,t)$ denotes the third order derivtive on $u(\cdot,t)$ with respect to t. Integrating by parts, this yields

$$\overset{(3)}{u}(\cdot,t)+\nabla\Big\{-C(\cdot)\nabla\dot{u}(\cdot,t)+\int_0^t G(\cdot,t-\tau)\nabla\dot{u}(\cdot,\tau)\,d\tau\Big\}=-\nabla\big(G(\cdot,t)\nabla u(\cdot,0)\big).$$

Finally multiplying this by $\ddot{u}(\cdot,t)$ and using again Lemma 3.1, we have the second identity. $\square$

Lemma 3.3.

$$\begin{aligned}
&\frac{d}{dt}\big\{K(t,u)+(\gamma-c)I(t,u)\big\}=\\
&-c\int_\Omega|\ddot{u}(\cdot,t)|^2\,dx+c\int_\Omega\big(C(\cdot)\nabla\dot{u}(\cdot)\big):\nabla\dot{u}(\cdot,t)\,dx\\
&-\int_\Omega\big(G(\cdot,0)\nabla\dot{u}(\cdot,t)\big):\nabla\dot{u}(\cdot,t)\,dx-(\gamma-c)\int_\Omega\big(\dot{G}(\cdot,t)\nabla u(\cdot,t)\big):\nabla u(\cdot,t)\,dx\\
&+\frac{1}{2}(\gamma-c)\int_\Omega\ddot{G}\Box\partial u(\cdot,t)\,dx-\int_\Omega\big(F(\cdot,t)\nabla u(\cdot,t)\big):\nabla\dot{u}(\cdot,t)\,dx\\
&+\int_\Omega\Big\{\int_0^t\dot{F}(\cdot,t-\tau)\nabla\big(u(\cdot,t)-u(\cdot,\tau)\big)\,d\tau\Big\}:\nabla\dot{u}(\cdot,t)\,dx,\ \ t>0.
\end{aligned}$$

Proof. First we sum γ the viscoelastic equation and the time derivative of the viscoelastic equation in (1) to obtain

$$\begin{aligned}
&\overset{(3)}{u}(\cdot,t)\ +\gamma\ddot{u}(\cdot,t)+\nabla\Big\{-C(\cdot)\nabla\dot{u}(\cdot,t)\ +G(\cdot,0)\nabla u(\cdot,t)\Big\}\\
&\qquad=-\nabla\left\{\int_0^t F(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\ +\gamma C(\cdot)\nabla u(\cdot,t)\right\}
\end{aligned}\tag{5}$$

Hence multiplying by $\ddot{u}(\cdot,t)$ and integrating in Ω we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega|\ddot{u}(\cdot,t)|^2+(C(\cdot)\nabla\dot{u}(\cdot,t)):\nabla\dot{u}(\cdot,t)\,dx\Big\}\\
&\quad=-\gamma\int_\Omega|\ddot{u}(\cdot,t)|^2\,dx+\int_\Omega(G(\cdot,0)\nabla u(\cdot,t)):\nabla\ddot{u}(\cdot,t)\,dx\\
&\quad\quad-\gamma\int_\Omega(C(\cdot)\nabla u(\cdot,t)):\nabla\ddot{u}(\cdot,t)\,dx\\
&\quad\quad+\int_\Omega\left(\int_0^t F(\cdot,t-\tau)\nabla u(\cdot,\tau)d\tau\right):\nabla\ddot{u}(\cdot,t)\,dx.
\end{aligned}\tag{6}$$

Having in mind (6) and the following identities

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (G(\cdot,0)\nabla u(\cdot,t)) : \nabla \dot u(\cdot,t)\,dx \\
&\qquad = \int_\Omega (G(\cdot,0)\nabla u(\cdot,t)) : \nabla \ddot u(\cdot,t)\,dx + \int_\Omega (G(\cdot,0)\nabla \dot u(\cdot,t)) : \nabla \dot u(\cdot,t)\,dx
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (C(\cdot)\nabla u(\cdot,t)) : \nabla \dot u(\cdot,t)\,dx \\
&\qquad = \gamma\int_\Omega (C(\cdot)\nabla u(\cdot,t)) : \nabla \ddot u(\cdot,t)\,dx + \gamma\int_\Omega (C(\cdot)\nabla \dot u(\cdot,t)) : \nabla \dot u(\cdot,t)\,dx
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(\int_0^t F(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\Big) : \nabla \dot u(\cdot,t)\,dx \\
&\qquad = \int_\Omega \Big(\int_0^t F(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\Big) : \nabla \ddot u(\cdot,t)\,dx \\
&\qquad\quad + \int_\Omega \Big(\int_0^t \dot F(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\Big) : \nabla \dot u(\cdot,t)\,dx \\
&\qquad\quad + \int_\Omega \Big(F(\cdot,0)\nabla u(\cdot,t)\Big) : \nabla \dot u(\cdot,t)\,dx
\end{aligned} \tag{9}
$$

we obtain

$$
\begin{aligned}
\frac{d}{dt}K(t,u) = &-\gamma\int_\Omega |\ddot u(\cdot,t)|^2\,dx - \int_\Omega \big(G(\cdot,0)\nabla \dot u(\cdot,t)\big) : \nabla \ddot u(\cdot,t)\,dx \\
&+ \gamma\int_\Omega \big(C(\cdot)\nabla \dot u(\cdot,t)\big) : \nabla \dot u(\cdot,t)\,dx \\
&- \int_\Omega \big(F(\cdot,0)\nabla u(\cdot,t)\big) : \nabla \dot u(\cdot,t)\,dx \\
&- \int_\Omega \Big(\int_0^t \dot F(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\Big) : \nabla \dot u(\cdot,t)\,dx.
\end{aligned} \tag{10}
$$

Now multiplying (5) with $\gamma = 0$ by $\dot u(\cdot,t)$ and integrating in Ω we have

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \ddot u(\cdot,t)\dot u(\cdot,t)\,dx = &\int_\Omega |\ddot u(\cdot,t)|^2\,dx - \int_\Omega \big(C(\cdot)\nabla \dot u(\cdot,t)\big) : \nabla \dot u(\cdot,t)\,dx \\
&+ \int_\Omega \big(G(\cdot,0)\nabla u(\cdot,t)\big) : \nabla \dot u(\cdot,t)\,dx \\
&+ \int_\Omega \Big(\int_0^t \dot G(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\Big) : \nabla \dot u(\cdot,t)\,dx.
\end{aligned} \tag{11}
$$

Further by the definition of $I(t,u)$, (11) and Lemma 3.1 we have

$$\begin{aligned}\frac{d}{dt}I(t,u) &= \int_\Omega |\ddot{u}(\cdot,t)|^2\,dx - \int_\Omega \big(C(\cdot)\nabla\dot{u}(\cdot,t)\big) : \nabla\dot{u}(\cdot,t)\,dx \\ &\quad - \int_\Omega \big(\dot{G}(\cdot,t)\nabla u(\cdot,t)\big) : \nabla u(\cdot,t)\,dx + \frac{1}{2}\int_\Omega \ddot{G}\Box\partial u\,dx.\end{aligned} \tag{12}$$

Finally putting together (10) and (12) the proof is complete. □

3.2. Basic inequalities

LEMMA 3.4. *For any $u, v \in C^1([0,\infty); H^1(\Omega))$ we have*

$$\begin{aligned}&\left|\int_\Omega \left(\int_0^t G(\cdot,t-\tau)(\nabla u(\cdot,t) - \nabla u(\cdot,\tau))d\tau\right) : \nabla v(\cdot,t)\,dx\right| \\ &\qquad \le c_0 \left(\int_\Omega (G\Box\partial u)(\cdot,t)\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v(\cdot,t)|^2\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Proof. Using Hölder's inequality we have

$$\begin{aligned}&\left|\int_\Omega \left(\int_0^t G(\cdot,t-\tau)(\nabla u(\cdot,t) - \nabla u(\cdot,\tau))d\tau\right) : \nabla v(\cdot,t)\,dx\right| \\ &= \left|\int_\Omega \left(\int_0^t G^{\frac{1}{2}}(\cdot,t-\tau)G^{\frac{1}{2}}(\cdot,t-\tau)(\nabla u(\cdot,t) - \nabla u(\cdot,\tau))d\tau\right) : \nabla v(\cdot,t)\,dx\right| \\ &\le c_0 \int_\Omega \Big((G\Box\partial u)^{\frac{1}{2}}(\cdot,t)\,|\nabla v(\cdot,t)|\,dx\Big)\,dx \\ &\le c_0 \left(\int_\Omega (G\Box\partial u)(\cdot,t)\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v(\cdot,t)|^2\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

□

LEMMA 3.5.

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \dot{u}(\cdot,t)u(\cdot,t)\,dx \\ &\quad = \int_\Omega |\dot{u}(\cdot,t)|^2\,dx - \int_\Omega \left\{\left(C(\cdot) - \int_0^t G(\cdot,\tau)\,d\tau\right)\nabla u(\cdot,t)\right\} : \nabla u(\cdot,t)\,dx \\ &\qquad - \int_\Omega \left(\int_0^t G(\cdot,t-\tau)\nabla(u(\cdot,t) - u(\cdot,\tau))\,d\tau\right) : \nabla u(\cdot,t)\,dx \\ &\quad \le c_1 \int_\Omega |\nabla\dot{u}(\cdot,t)|^2\,dx - \big(\mu_0 + \alpha\kappa^{-1}e^{-\kappa t}\big)\int_\Omega |\nabla u(\cdot,t)|^2\,dx \\ &\qquad + \frac{c_0}{2}\left(\varepsilon\int_\Omega G\Box\partial u(\cdot,t)\,dx + \frac{1}{\varepsilon}\int_\Omega |\nabla u(\cdot,t)|^2\,dx\right), \quad t > 0,\ 0 \le \varepsilon \le 1.\end{aligned}$$

Proof. Using the viscoelastic equation in (1) and integrating by parts we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega \dot u(\cdot,t)u(\cdot,t)\,dx &= \int_\Omega |\dot u(\cdot,t)|^2\,dx + \int_\Omega \ddot u(\cdot,t)u(\cdot,t)\,dx\\ &= \int_\Omega |\dot u(\cdot,t)|^2\,dx - \int_\Omega \{C(\cdot)\nabla u(\cdot,t)\} : \nabla u(\cdot,t)\,dx\\ &\quad + \int_\Omega \left(\int_0^t G(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\right) : \nabla u(\cdot,t)\,dx\end{aligned}$$

Now if we add and subtract $\int_\Omega \left(\int_0^t G(\cdot,\tau)\,d\tau\,\nabla u(\cdot,t)\right) : \nabla u(\cdot,t)dx$ we have the equality in Lemma 3.5, for the inequality we use the Poincaré inequality, (3), Lemma 3.4, and Young's inequality. □

Lemma 3.6.

$$\begin{aligned}&\left|\int_\Omega \left(\int_0^t F(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\right) : \nabla \dot u(\cdot,t)\,dx\right|\\ &\quad\le \frac{1}{2}c_0(\gamma-\kappa)\left\{\varepsilon\int_\Omega G\Box\partial u(\cdot,t)\,dx + \frac{1}{\varepsilon}\int_\Omega |\nabla\dot u(\cdot,t)|^2\,dx\right.\\ &\qquad \left. + c_0\left(\xi\int_\Omega |\nabla u(\cdot,t)|^2\,dx + \frac{1}{\xi}\int_\Omega |\nabla \dot u(\cdot,t)|^2\,dx\right)\right\}\end{aligned}$$

with ε, $\xi > 0$.

Proof. By the definition of $F(\cdot,t)$ and $\dot G(\cdot,t) = -\kappa G(\cdot,t)$ we have

$$\begin{aligned}&\left|\int_\Omega \left(\int_0^t F(\cdot,t-\tau)\nabla u(\cdot,\tau)\,d\tau\right) : \nabla \dot u(\cdot,t)\,dx\right|\\ &\quad\le (\gamma-\kappa)\left|\int_\Omega \left(\int_0^t G(\cdot,t-\tau)\Big(\nabla u(\cdot,t) - \nabla u(\cdot,\tau)\Big)\,d\tau\right) : \nabla\dot u(\cdot,t)\,dx\right|\\ &\qquad + (\gamma-\kappa)\left|\int_\Omega \left(\int_0^t G(\cdot,t-\tau)d\tau\,\nabla u(\cdot,t)\right) : \nabla\dot u(\cdot,t)\,dx\right|\end{aligned}$$

Now using Lemma 3.4 and Young's inequality the proof is complete. □

4. Estimates

4.1. $\frac{d}{dt}\mathcal{L}(t,u) \le -M_1\mathcal{L}(t,u)$, $t \ge 0$

We will bound from above $\frac{d}{dt}\mathcal{L}(t,u)$. From Lemma 3.3 and (1) we obtain:

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{L}(t,u) = -c\int_\Omega |\ddot{u}(\cdot,t)|^2\,dx + c\int_\Omega (C(\cdot)\nabla\dot{u}(\cdot,t)) : \nabla\dot{u}(\cdot,t)\,dx \\
&\quad - \int_\Omega \big(G(\cdot,0)\nabla\dot{u}(\cdot,t)\big) : \nabla\dot{u}(\cdot,t)\,dx + (\gamma - c)\kappa\int_\Omega (G(\cdot,t)\nabla u(\cdot,t)) : \nabla\dot{u}(\cdot,t)\,dx \\
&\quad + \frac{1}{2}(\gamma-c)\kappa^2\int_\Omega G\Box\partial u(\cdot,t)\,dx - (\gamma-\kappa)\int_\Omega (G(\cdot,t)\nabla u(\cdot,t)) : \nabla\dot{u}(\cdot,t)\,dx \\
&\quad - \kappa(\gamma-\kappa)\int_\Omega \left\{\int_0^t G(\cdot,t-\tau)(\nabla u(\cdot,t) - \nabla(\cdot,\tau))d\tau\right\} : \nabla\dot{u}(\cdot,t)\,dx \\
&\quad - N_1\int_\Omega (G(\cdot,t)\nabla u(\cdot,t)) : \nabla u(\cdot,t)\,dx - \frac{1}{2}N_1\kappa\int_\Omega G\Box\partial u(\cdot,t)\,dx \\
&\quad - N_2\int_\Omega (G(\cdot,t)\nabla\dot{u}(\cdot,t)) : \nabla\dot{u}(\cdot,t)\,dx - \frac{N_2}{2}\kappa\int_\Omega G\Box\partial\dot{u}(\cdot,t)\,dx \\
&\quad + c_p\frac{d}{dt}\int_\Omega \dot{u}(\cdot,t)u(\cdot,t)\,dx.
\end{aligned}
$$

Using (2), (3), Lemma 3.4, Lemma 3.5 and Young's inequality we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t,u) \le -c\int_\Omega |\ddot{u}(\cdot,t)|^2\,dx &+ \frac{1}{2}\Big\{\big(2(\gamma-c)\kappa\beta_0 + (\gamma-\kappa)\beta_0 - 2N_1\alpha_0 \\
&\qquad -2c_p\alpha_0\kappa^{-1}\big)\,e^{-\kappa t} + \frac{c_pc_0}{2\eta} - c_p\mu_0\Big\}\int_\Omega |\nabla u(\cdot,t)|^2\,dx \\
&+ \frac{1}{2}\Big\{\big((\gamma-\kappa)\beta_0 - 2N_2\alpha_0\big)e^{-\kappa t} + 2c\beta_0 + 2c_pc_1 \\
&\qquad + \frac{c_0(\gamma-\kappa)\kappa}{\xi} - \alpha_0\Big\}\int_\Omega |\nabla\dot{u}(\cdot,t)|^2\,dx \\
&+ \frac{1}{2}\Big\{(\gamma-c)\kappa^2 + \kappa(\gamma-\kappa)c_0\xi + c_pc_0\eta \\
&\qquad - N_1\kappa\Big\}\int_\Omega G\Box\partial u(\cdot,t)\,dx - \frac{1}{2}N_2\kappa\int_\Omega G\Box\partial\dot{u}(\cdot,t)\,dx.
\end{aligned}
$$

4.2. Estimating $-\mathcal{L}(t,u)$ from below

In this subsection we bound $-\mathcal{L}(t,u)$ from below.

$$\begin{aligned}
-\mathcal{L}(t,u) = &-\frac{1}{2}\int_\Omega |\ddot{u}(\cdot,t)|^2\,dx - \frac{1}{2}\int_\Omega (C(\cdot)\nabla\dot{u}(\cdot,t)) : \nabla\dot{u}(\cdot,t)\,dx \\
&+ \int_\Omega (G(\cdot,0)\nabla u(\cdot,t)) : \nabla\dot{u}(\cdot,t)\,dx - \gamma\int_\Omega (C(\cdot)\nabla u(\cdot,t)) : \nabla\dot{u}(\cdot,t)\,dx \\
&+ (\gamma-\kappa)\int_\Omega \left(\int_0^t G(\cdot,t-\tau)\nabla u(\cdot,\tau)d\tau\right) : \nabla\dot{u}(\cdot,t)\,dx \\
&- (\gamma-c)\int_\Omega \ddot{u}(\cdot,t)\dot{u}(\cdot,t)\,dx + \frac{1}{2}(\gamma-c)\int_\Omega (G(\cdot,0)\nabla u(\cdot,t)) : \nabla u(\cdot,t)\,dx \\
&+ \frac{\gamma-c}{2}\int_\Omega \left\{\left(\int_0^t \dot{G}(\cdot,t-\tau)d\tau\right)\nabla u(\cdot,t)\right\} : \nabla u(\cdot,t)\,dx \\
&- \frac{1}{2}(\gamma-c)\int_\Omega \dot{G}\Box\partial u(\cdot,t)\,dx - \frac{1}{2}N_1\int_\Omega |\dot{u}(\cdot,t)|^2\,dx - \frac{1}{2}N_1\int_\Omega G\Box\partial u(\cdot,t)\,dx \\
&- \frac{1}{2}N_1\int_\Omega \left\{\left(C(\cdot) - \int_0^t G(\cdot,\tau)\,d\tau\right)\nabla u(\cdot,t)\right\} : \nabla u(\cdot,t)\,dx \\
&- \frac{1}{2}N_2\int_\Omega |\ddot{u}(\cdot,t)|^2\,dx - \frac{1}{2}N_2\int_\Omega G\Box\partial\dot{u}\,dx \\
&- N_2\int_\Omega \left\{\left(C(\cdot) - \int_0^t G(\cdot,\tau)\,d\tau\right)\nabla\dot{u}(\cdot,t)\right\} : \nabla\dot{u}(\cdot,t)\,dx \\
&- c_p\int_\Omega \dot{u}(\cdot,t)u(\cdot,t)\,dx.
\end{aligned}$$

Using (2), (3), Lemma 3.6, $G = e^{-\kappa t}\hat{G}$ and Young's inequality, we have:

$$\begin{aligned}
-\mathcal{L}(t,u) \geq &-\frac{1}{2}\big\{1 + N_2 + \gamma - c\big\}\int_\Omega |\ddot{u}(\cdot,t)|^2\,dx \\
&+ \frac{1}{2}\Big\{\big(\alpha_0(\gamma-\kappa) - N_1\beta_0\kappa^{-1}\big)e^{-\kappa t} + N_1\alpha_0\kappa^{-1} - N_1\beta_0 - c_0^2(\gamma-\kappa) \\
&\qquad - c_p - (\gamma+1)\beta_0\Big\}\int_\Omega |\nabla u(\cdot,t)|^2\,dx \\
&+ \frac{1}{2}\Big\{-2N_2\kappa^{-1}\beta_0 e^{-\kappa t} + 2N_2\kappa^{-1}\alpha_0 - 2N_2\beta_0 - \beta_0 - c_1(\gamma-c) \\
&\qquad - c_1N_1 - c_0(\gamma-\kappa)(c_0+1) - c_p - (\gamma+1)\beta_0\Big\}\int_\Omega |\nabla\dot{u}(\cdot,t)|^2\,dx \\
&+ \frac{1}{2}\big\{(\gamma-c)\kappa - N_1 - c_0(\gamma-\kappa)\big\}\int_\Omega G\Box\partial u(\cdot,t)\,dx \\
&- \frac{1}{2}N_2\int_\Omega G\Box\partial\dot{u}(\cdot,t)\,dx.
\end{aligned}$$

It is easy to see that if we take the constants like in Subsection 4.4 given later, all the coefficients in both bounds are less than a negative constant and are bounded. Hence there is a constant $M \geq 0$ such that $\frac{d}{dt}\mathcal{L}(t,u) \leq -M\mathcal{L}(t,u)$ for any $t \geq 0$.

4.3. Estimating $\mathcal{L}(t,u)$ from below

In this section we will bound $\mathcal{L}(t,u)$ from below. For this we use (3) to get the next two inequalities:

$$N_1E(t,u) \geq \frac{1}{2}N_1\int_\Omega |\dot{u}(\cdot,t)|^2\,dx + \frac{1}{2}N_1\int_\Omega G\Box\partial u(\cdot,t)\,dx + N_1\mu_0\int_\Omega |\nabla u(\cdot,t)|^2\,dx,$$

$$N_2E(t,\dot{u}) \geq \frac{1}{2}N_2\int_\Omega |\ddot{u}(\cdot,t)|^2\,dx + \frac{1}{2}N_2\int_\Omega G\Box\partial \dot{u}(\cdot,t)\,dx + N_2\mu_0\int_\Omega |\nabla \dot{u}(\cdot,t)|^2\,dx.$$

We apply (4), Lemma 3.6 with $\varepsilon = \xi = 1, G = e^{-\kappa t}\hat{G}$ and Young inequality with $p = q = 2$ to obtain

$$\begin{aligned}
K(t,u) + (\gamma - c)I(t,u) \geq \frac{1}{2}\Bigg[&\int_\Omega |\ddot{u}(\cdot,t)|^2\,dx + \alpha_0\int_\Omega |\nabla\dot{u}(\cdot,t)|^2\,dx \\
&- \beta_0\left(\int_\Omega |\nabla u(\cdot,t)|^2\,dx \int_\Omega |\nabla\dot{u}(\cdot,t)|^2\,dx\right) \\
&+ \gamma\alpha_0\left(\int_\Omega |\nabla u(\cdot,t)|^2\,dx + \int_\Omega |\nabla\dot{u}(\cdot,t)|^2\,dx\right) \\
&- c_0(\gamma - \kappa)\Bigg\{\int_\Omega G\Box\partial u(\cdot,t)\,dx + \int_\Omega |\nabla\dot{u}(\cdot,t)|^2\,dx \\
&\qquad + c_0\left(\int_\Omega |\nabla u(\cdot,t)|^2\,dx + \int_\Omega |\nabla\dot{u}(\cdot,t)|^2\,dx\right)\Bigg\} \\
&- (\gamma - c)\left\{\int_\Omega |\dot{u}(\cdot,t)|^2\,dx + \int_\Omega |\ddot{u}(\cdot,t)|^2\right\} \\
&- \beta_0(\gamma - c)\int_\Omega |\nabla u(\cdot,t)|^2\,dx \\
&- \kappa(\gamma - c)\int_\Omega G\Box\partial u(\cdot,t)\,dx\Bigg].
\end{aligned}$$

By using Young inequality with $p = q = 2$ and Poincaré inequality, we have

$$c_p\int_\Omega \dot{u}(\cdot,t)u(\cdot,t)\,dx \geq -\frac{1}{2}\left[\int_\Omega |\dot{u}(\cdot,t)|^2\,dx + c_1\int_\Omega |\nabla u(\cdot,t)|^2\,dx\right].$$

Putting together the last four inequalities and the definition of $\mathcal{L}(t,u)$ we obtain:

$$\begin{aligned}\mathcal{L}(t,u) &\geq \frac{1}{2}\Big[1 + N_2 - (\gamma - c)\Big]\int_\Omega |\ddot{u}(\cdot,t)|^2\,dx \\ &+ \frac{1}{2}\Big[2N_1\mu_0 + \gamma\alpha_0 - \beta_0 - c_0^2(\gamma-\kappa) - \beta_0(\gamma-c) - c_p c_1\Big]\int_\Omega |\nabla u(\cdot,t)|^2\,dx \\ &+ \frac{1}{2}\Big[2N_2\mu_0 + \gamma\alpha_0 + \alpha_0 - \beta_0 - c_0^2(\gamma-\kappa) - c_0(\gamma-\kappa)\Big]\int_\Omega |\nabla\dot{u}(\cdot,t)|^2\,dx \\ &+ \frac{1}{2}\Big[N_1 - c_0(\gamma-\kappa) - \kappa(\gamma-c)\Big]\int_\Omega G\Box\partial u(\cdot,t)\,dx \\ &+ \frac{1}{2}N_2\int_\Omega G\Box\partial\dot{u}(\cdot,t)\,dx + \frac{1}{2}\Big[N_1 - (\gamma-c) - c_p\Big]\int_\Omega |\dot{u}(\cdot,t)|^2\,dx.\end{aligned}$$

Now if we take the constants like in Subsection 4.4 we see that exists $M_2 \geq 0$ such that

$$\mathcal{L}(t,u) \geq M_2\int_\Omega \Big\{|\ddot{u}(\cdot,t)|^2 + |\dot{u}(\cdot,t)|^2 + |\nabla\dot{u}(\cdot,t)|^2 + |\nabla u(\cdot,t)|^2\Big\}\,dx,\ t > 0\,.$$

4.4. Conditions for constants

We summarize the conditions for constants N_1, N_2, γ, c, c_p as follows.

$$\begin{aligned}N_1 > \max\Big\{&2\big((\gamma-c)\kappa - c_0(\gamma-\kappa)\big),\, 2\big((\gamma-c)\kappa + c_0(\gamma-\kappa)\xi + c_0\,c_p\kappa^{-1}\eta\big), \\ &(\beta_0 - \kappa^{-1}\alpha_0)^{-1}\big((\alpha_0 - c_0^2)(\gamma-c) - c_0^2(\gamma-\kappa) - c_p\,c_1 - (\gamma+1)\beta_0\big), \\ &\frac{1}{2\mu_0}\big(\beta_0 - \gamma\alpha_0 + c_0^2(\gamma-\kappa) + \beta_0(\gamma-c) + c_p\,c_1\big), \\ &c_0(\gamma-\kappa) + \kappa(\gamma-c), (\gamma-c) + c_p\Big\}, \\ N_2 > \max\Big\{&(\gamma-c) - 1,\, \frac{1}{2\mu_0}\big(-\alpha_0 + \beta_0 - \gamma\alpha_0 + c_0(c_0+1)(\gamma-\kappa)\big)\Big\}, \\ \xi > 6&c_0(\gamma-\kappa)\kappa\alpha_0^{-1},\ c < \frac{\alpha_0}{6\beta_0},\ c_p < \frac{\alpha_0}{6c_1},\ \eta > \frac{c_0}{N_0}\,,\end{aligned}$$

with $c_0 := \big(\sup_{x\in\Omega}\int_0^\infty |G(x,t)|\,dt\big)^{1/2}$ and the Poincaré constant c_1 provided that $\beta_0 > \kappa^{-1}\alpha_0$.

It is easy to see that there are constants N_1, N_2, γ, c, c_p which satisfy these conditions. Hence, from Subsections 4.1 and 4.3, we have obtained the exponential decay of solution to (1).

5. Conclusion and discussion

Conclusion
We have shown an exponential decay of solutions of (1) by applying the energy method given in [6] and carefully concerning the constants which appear in this method.

Discussion
We will give the following list for some discussion on our result.

1. The case with density can be handled in a similar way.
2. The case with general inhomogeneous data can be handled using the Duhamel principle.
3. How the exponential decay rate depends on the assumptions and coefficients is not clear.
4. As a future work, we would like to extend this work to a more general relaxation tensor.

6. Acknowledgements

The first author was partially supported by Korea Research Foundation (49771-01). The second author was supported by ERC Grant "Geometric function theory, inverse problems and fluid dynamics" and this work is part of his thesis work.

References

[1] R.M. Christensen, *Theory of viscoelasticity*, second ed., Academic Press, New York, 1982.
[2] C. M. Dafermos, *An abstract Volterra equation with applications to linear viscoelasticity*, J. Differential Equations **7** (1970), 554–569.
[3] C. M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37** (1970), 297–308.
[4] N. Higashimori, *Identification of viscoelastic properties by magnetic resonance elastography*, J. Phys.: Conf. Ser. **73** (2007).
[5] J. E. Muñoz Rivera, *Asymptotic behaviour in linear viscoelasticity*, Quart. Appl. Math. **52** (1994), no. 4, 629–648.
[6] J. E. Muñoz Rivera and E. Cabanillas Lapa, *Decay rates of solutions of an anisotropic inhomogeneous n-dimensional viscoelastic equation with polynomially decaying kernels*, Comm. Math. Phys. **177** (1996), no. 3, 583–602.

[7] R. Muthupillai, D.J. Lomas, P. J. Rossman, J. F. Greenlead, A. Manduca, and R. L. Ehman, *Magnetic resonance elastography by direct visualization of propagating acoustic strain waves*, Science **269** (1995), 1854–1857.

[8] Y.-C. Wang, C.-C. Ko, and Siau L.-M., *Accurate determination of torsional and pure bending moment for viscoelastic measurements*, Int. J. Mod. Phys. Conference Series **24** (2013).

Authors' addresses:

Gen Nakamura
Department of Mathematics,
Inha University, Incheon, Republic of Korea
E-mail: `nakamuragenn@gmail.com`

Marcos Oliva
Department of Mathematics
Universidad Autónoma de Madrid, Madrid, Spain
E-mail: `marcos.delaoliva@uam.es`

Received August 7, 2015
Accepted November 27, 2015