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Discrete inequalities of Jensen type for λ-convex functions on linear spaces

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Abstract. *Some discrete inequalities of Jensen type for λ-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.*

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1. Introduction

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1.1 ([38]). *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1}$$

Some further properties of this class of functions can be found in [28, 29, 31, 44, 47, 48]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0, \infty)$ where C is a convex subset of the real or complex linear space X and the inequality (1) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 1.2 ([31]). *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contains all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx+(1-t)y) \leq \max\{f(x), f(y)\} \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [31, 45] while for quasi convex functions, the reader can consult [30].

If $f : C \subseteq X \to [0,\infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (2) (or (3)) holds true for $x, y \in C$ and $t \in [0,1]$.

Definition 1.3 ([7]). *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx+(1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1, 2, 7, 8, 26, 27, 39, 41, 50].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p, p \geq 1$ is convex on X. Utilising the elementary inequality $(a+b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0,1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx+(1-t)y) &= \|tx+(1-t)y\|^s \leq (t\|x\| + (1-t)\|y\|)^s \\ &\leq (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0,1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 1.4 ([53]). *Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx+(1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [53, 6, 42, 51, 49, 52].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 1.5. *We say that the function* $f : C \subseteq X \to [0,\infty)$ *is of* s*-Godunova-Levin type, with* $s \in [0,1]$, *if*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{5}$$

for all $t \in (0,1)$ *and* $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[37], [40]-[42] and [45]-[52].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \geq h(t)\,h(s) \text{ for any } t, s \in J. \tag{6}$$

If the inequality (6) is reversed, then h is said to be *submultiplicative*. If the equality holds in (6) then h is said to be a multiplicative function on J.

In [53] it has been noted that if $h : [0,\infty) \to [0,\infty)$ with $h(t) = (x+c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \geq 1$, then for $p \in (0,1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative. We observe that, if h, g are nonnegative and supermultiplicative, the same is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative. The case of h-convex function with h supermultiplicative is of interest due to several Jensen type inequalities one can derive.

The following results were obtained in [53] for functions of a real variable. However, with similar proofs they can be extended to h-convex function defined on convex subsets in linear spaces.

Theorem 1.6. *Let* $h : J \to [0,\infty)$ *be a supermultiplicative function on J. If the function* $f : C \subseteq X \to [0,\infty)$ *is* h*-convex on the convex subset* C *of the linear space* X, *then for any* $w_i \geq 0$, $i \in \{1,...,n\}$, $n \geq 2$ *with* $W_n := \sum_{i=1}^n w_i > 0$ *we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq \sum_{i=1}^n h\left(\frac{w_i}{W_n}\right) f(x_i). \tag{7}$$

In particular, we have the unweighted inequality

$$f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \leq h\left(\frac{1}{n}\right)\sum_{i=1}^n f(x_i). \tag{8}$$

COROLLARY 1.7 ([27]). *If the function $f : C \subseteq X \to [0,\infty)$ is Breckner s-convex on the convex subset C of the linear space X with $s \in (0,1)$, then for any $x_i \in C$, $w_i \geq 0$, $i \in \{1,...,n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq \frac{1}{W_n^s}\sum_{i=1}^n w_i^s f(x_i). \tag{9}$$

If $(X, \|\cdot\|)$ is a normed linear space, then for $s \in (0,1)$, $x_i \in X$, $w_i \geq 0$, $i \in \{1,...,n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have the norm inequality

$$\left\|\sum_{i=1}^n w_i x_i\right\|^s \leq \sum_{i=1}^n w_i^s \|x_i\|^s. \tag{10}$$

COROLLARY 1.8. *If the function $f : C \subseteq X \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, on the convex subset C of the linear space X, then for any $x_i \in C$, $w_i > 0$, $i \in \{1,...,n\}$, $n \geq 2$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq W_n^s \sum_{i=1}^n \frac{1}{w_i^s} f(x_i). \tag{11}$$

This result generalizes the Jensen type inequality obtained in [44] for $s = 1$.

Let K be a finite non-empty set of positive integers. We can define the index set function, see also [53],

$$J(K) := \sum_{i\in K} h(w_i) f(x_i) - h(W_K) f\left(\frac{1}{W_K}\sum_{i\in K} w_i x_i\right), \tag{12}$$

where $W_K := \sum_{i\in K} w_i > 0$, $x_i \in C$, $i \in K$.

We notice that if $h : [0,\infty) \to [0,\infty)$ is a supermultiplicative function on $[0,\infty)$ and the function $f : C \subseteq X \to [0,\infty)$ is h-convex on the convex subset C of the linear space X, then

$$J(K) \geq h(W_K)\left[\sum_{i\in K} h\left(\frac{w_i}{W_K}\right) f(x_i) - f\left(\frac{1}{W_K}\sum_{i\in K} w_i x_i\right)\right] \geq 0. \tag{13}$$

THEOREM 1.9. *Assume that $h : [0,\infty) \to [0,\infty)$ is a supermultiplicative function on $[0,\infty)$ and the function $f : C \subseteq X \to [0,\infty)$ is h-convex on the convex subset C of the linear space X. Let M and K be finite non-empty sets of positive integers, $w_i > 0$, $x_i \in C$, $i \in K \cup M$. Then*

$$J(K \cup M) \geq J(K) + J(M) \geq 0, \tag{14}$$

i.e., J is a superadditive index set functional.

This results was proved in an equivalent form in [53] for functions of a real variable. The proof is similar for functions defined on convex sets in linear spaces.

Corollary 1.10. *With the assumptions of Theorem 1.9 and if we note* $M_k := \{1, ..., k\}$, *then*

$$J(M_n) \geq J(M_{n-1}) \geq ... \geq J(M_2) \geq 0 \tag{15}$$

and

$$\begin{aligned} &J(M_n) \\ &\geq \max_{1\leq i<j\leq n}\left\{h(w_i)f(x_i) + h(w_j)f(x_j) - h(w_i+w_j)f\left(\frac{w_ix_i+w_jx_j}{w_i+w_j}\right)\right\} \\ &\geq 0. \end{aligned} \tag{16}$$

If we consider the functional

$$J_s(K) := \sum_{i\in K} w_i^s \|x_i\|^s - \left\|\sum_{i\in K} w_ix_i\right\|^s$$

for $s \in (0,1)$, then we have the norm inequalities

$$\begin{aligned} \sum_{i=1}^n w_i^s \|x_i\|^s - \left\|\sum_{i=1}^n w_ix_i\right\|^s &\geq \sum_{i=1}^{n-1} w_i^s \|x_i\|^s - \left\|\sum_{i=1}^{n-1} w_ix_i\right\|^s \\ &\geq ... \geq \sum_{i=1}^{2} w_i^s \|x_i\|^s - \left\|\sum_{i=1}^{2} w_ix_i\right\|^s \geq 0 \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\sum_{i=1}^n w_i^s \|x_i\|^s - \left\|\sum_{i=1}^n w_ix_i\right\|^s \\ &\geq \max_{1\leq i<j\leq n}\left\{w_i^s\|x_i\|^s + w_j^s\|x_j\|^s - \|w_ix_i+w_jx_j\|^s\right\} \geq 0 \end{aligned} \tag{18}$$

where $w_i \geq 0$, $x_i \in X$, $i \in \{1, ..., n\}$, $n \geq 2$.

2. λ-convex functions

We start with the following definition (see also [24]):

Definition 2.1. *Let* $\lambda : [0,\infty) \to [0,\infty)$ *be a function with the property that* $\lambda(t) > 0$ *for all* $t > 0$. *A mapping* $f : C \to \mathbb{R}$ *defined on convex subset* C *of a linear space* X *is called* λ*-convex on* C *if*

$$f\left(\frac{\alpha x + \beta y}{\alpha+\beta}\right) \leq \frac{\lambda(\alpha)f(x) + \lambda(\beta)f(y)}{\lambda(\alpha+\beta)} \tag{19}$$

for all $\alpha, \beta \geq 0$ *with* $\alpha + \beta > 0$ *and* $x, y \in C$.

We observe that if $f : C \to \mathbb{R}$ is λ-convex on C, then f is h-convex on C with $h(t) = \frac{\lambda(t)}{\lambda(1)}$, $t \in [0, 1]$. If $f : C \to [0, \infty)$ is h-convex function with h supermultiplicative on $[0, \infty)$, then f is λ-convex with $\lambda = h$.

Indeed, if $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$ then

$$\begin{aligned} f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) &\leq h\left(\frac{\alpha}{\alpha + \beta}\right) f(x) + h\left(\frac{\beta}{\alpha + \beta}\right) f(y) \\ &\leq \frac{h(\alpha) f(x) + h(\beta) f(y)}{h(\alpha + \beta)}. \end{aligned}$$

The following proposition contain some properties of λ-convex functions [24].

Proposition 2.2. *Let* $f : C \to \mathbb{R}$ *be a* λ*-convex function on* C.

(i) If $\lambda(0) > 0$, *then we have* $f(x) \geq 0$ *for all* $x \in C$;

(ii) If there exists $x_0 \in C$ *so that* $f(x_0) > 0$, *then*

$$\lambda(\alpha + \beta) \leq \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, *i.e. the mapping* λ *is subadditive on* $(0, \infty)$.

(iii) If there exists $x_0, y_0 \in C$ *with* $f(x_0) > 0$ *and* $f(y_0) < 0$, *then*

$$\lambda(\alpha + \beta) = \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, *i.e. the mapping* λ *is additive on* $(0, \infty)$.

We have the following result providing many examples of subadditive functions $\lambda : [0, \infty) \to [0, \infty)$.

Theorem 2.3 ([24]). *Let* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *a power series with nonnegative coefficients* $a_n \geq 0$ *for all* $n \in \mathbb{N}$ *and convergent on the open disk* $D(0, R)$ *with* $R > 0$ *or* $R = \infty$. *If* $r \in (0, R)$ *then the function* $\lambda_r : [0, \infty) \to [0, \infty)$ *given by*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r \exp(-t))}\right] \tag{20}$$

is nonnegative, increasing and subadditive on $[0, \infty)$.

We have the following fundamental examples of power series with positive coefficients:

$$h(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \;\; z \in D(0,1) \tag{21}$$

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \exp(z) \qquad z \in \mathbb{C},$$

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \;\; z \in \mathbb{C};$$

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \;\; z \in \mathbb{C};$$

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1-z}, \;\; z \in D(0,1).$$

Other important examples of functions as power series representations with positive coefficients are:

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \frac{1}{2} \ln \left(\frac{1+z}{1-z} \right), \qquad z \in D(0,1); \tag{22}$$

$$h(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!} z^{2n+1} = \sin^{-1}(z), \qquad z \in D(0,1);$$

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \tanh^{-1}(z), \qquad z \in D(0,1);$$

$$h(z) =_2 F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0,$$

$$z \in D(0,1);$$

where Γ is *Gamma function.*

REMARK 2.4. *Now, if we take* $h(z) = \frac{1}{1-z}$, $z \in D(0,1)$, *then*

$$\lambda_r(t) = \ln \left[\frac{1 - r\exp(-t)}{1-r} \right] \tag{23}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ *for any* $r \in (0,1)$.

If we take $h(z) = \exp(z)$, $z \in \mathbb{C}$ *then*

$$\lambda_r(t) = r\left[1 - \exp(-t)\right] \tag{24}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ *for any* $r > 0$.

Corollary 2.5 ([24]). *Let $h(z)=\sum_{n=0}^{\infty}a_n z^n$ be a power series with nonnegative coefficients $a_n\geq 0$ for all $n\in\mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$ and $r\in(0,R)$. For a mapping $f:C\to\mathbb{R}$ defined on convex subset C of a linear space X, the following statements are equivalent:*

(i) The function f is λ_r-convex with $\lambda_r:[0,\infty)\to[0,\infty)$,

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right];$$

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}\leq\left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)}\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)}\tag{25}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

(iii) We have the inequality

$$\frac{[h(r\exp(-\alpha))]^{f(x)}[h(r\exp(-\beta))]^{f(y)}}{[h(r\exp(-\alpha-\beta))]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}}\leq[h(r)]^{f(x)+f(y)-f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}\tag{26}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

Remark 2.6. *We observe that, in the case when*

$$\lambda_r(t)=r[1-\exp(-t)],\ t\geq 0,$$

then the function f is λ_r-convex on convex subset C of a linear space X iff

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{[1-\exp(-\alpha)]f(x)+[1-\exp(-\beta)]f(y)}{1-\exp(-\alpha-\beta)}\tag{27}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

We observe that this definition is independent of $r>0$.

The inequality (27) is equivalent with

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{\exp(\beta)[\exp(\alpha)-1]f(x)+\exp(\alpha)[\exp(\beta)-1]f(y)}{\exp(\alpha+\beta)-1}\tag{28}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

We can give now more examples of subadditive functions that can be used to define λ-convex mappings on linear spaces.

Let $I=(0,\infty)$ or $[0,\infty)$. A function $h:I\to\mathbb{R}$ is called *superadditive (subadditive)* on I if

(iii) $h(t+s)\geq(\leq)\,h(t)+h(s)$ for any $t,s\in I$

and *nonnegative (strictly positive)* on I if, obviously, it satisfies

(iv) $h(t) \geq (>) \, 0$ for each $t \in I$.

The following result holds:

THEOREM 2.7. *If $h : I \to [0, \infty)$ is a superadditive (subadditive) function on I and $p \geq 1$ $(0 < p < 1)$ then the function*

$$\Psi_p : I \to [0, \infty), \Psi_p(t) = t^{1-\frac{1}{p}} h(t) \tag{29}$$

is superadditive (subadditive) on I.

Proof. First of all we observe that the following elementary inequality holds:

$$(\alpha + \beta)^p \geq (\leq) \, \alpha^p + \beta^p \tag{30}$$

for any $\alpha, \beta \geq 0$ and $p \geq 1$ $(0 < p < 1)$.

Indeed, if we consider the function $f_p : [0, \infty) \to \mathbb{R}$, $f_p(t) = (t+1)^p - t^p$ we have $f_p'(t) = p\left[(t+1)^{p-1} - t^{p-1}\right]$. Observe that for $p > 1$ and $t > 0$ we have that $f_p'(t) > 0$ showing that f_p is strictly increasing on the interval $[0, \infty)$. Now for $t = \frac{\alpha}{\beta}$ $(\beta > 0, \alpha \geq 0)$ we have $f_p(t) > f_p(0)$ giving that $\left(\frac{\alpha}{\beta} + 1\right)^p - \left(\frac{\alpha}{\beta}\right)^p > 1$, i.e., the desired inequality (30).

For $p \in (0, 1)$ we have that f_p is strictly decreasing on $[0, \infty)$ which proves the second case in (30).

Now, if h is superadditive (subadditive) and $p \geq 1$ $(0 < p < 1)$ then we have by (30) that

$$h^p(t+s) \geq (\leq) \, [h(t) + h(s)]^p \geq (\leq) \, h^p(t) + h^p(s) \tag{31}$$

for all $t, s \in I$. Utilising (31) we have for any $t, s \in I$ that

$$\begin{aligned} \frac{h^p(t+s)}{t+s} &\geq (\leq) \, \frac{h^p(t) + h^p(s)}{t+s} = \frac{t \cdot \frac{h^p(t)}{t} + s \cdot \frac{h^p(s)}{s}}{t+s} \\ &= \frac{t \cdot \left[\frac{h(t)}{t^{1/p}}\right]^p + s \cdot \left[\frac{h(s)}{s^{1/p}}\right]^p}{t+s} =: I. \end{aligned} \tag{32}$$

Since for $p \geq 1$ $(0 < p < 1)$ the power function $g(t) = t^p$ is convex (concave), then

$$I \geq (\leq) \left[\frac{t \cdot \frac{h(t)}{t^{1/p}} + s \cdot \frac{h(s)}{s^{1/p}}}{t+s}\right]^p = \left[\frac{h(t)\, t^{1-1/p} + h(s)\, s^{1-1/p}}{t+s}\right]^p \tag{33}$$

for any $t, s \in I$.

By combining (32) with (23) we get

$$\frac{h^p(t+s)}{t+s} \geq (\leq) \left[\frac{h(t)\,t^{1-1/p} + h(s)\,s^{1-1/p}}{t+s}\right]^p,$$

which is equivalent with

$$\frac{h(t+s)}{(t+s)^{1/p}} \geq (\leq) \frac{h(t)\,t^{1-1/p} + h(s)\,s^{1-1/p}}{t+s}$$

i.e., by multiplying with $t+s$,

$$\Psi_p(t+s) \geq (\leq) \Psi_p(t) + \Psi_p(s)$$

for any $t, s \in I$ and the proof is complete. □

Corollary 2.8. *If $h : I \to [0, \infty)$ is a superadditive (subadditive) function on I and $p, q \geq 1$ $(0 < p, q < 1)$ then the two parameter function*

$$\Psi_{p,q} : I \to [0, \infty), \Psi_{p,q}(t) = t^{q\left(1-\frac{1}{p}\right)} h^q(t) \tag{34}$$

is superadditive (subadditive) on I.

Proof. Observe that $\Psi_{p,q}(t) = [\Psi_p(t)]^q$ for $t \in I$. Therefore, by Theorem 2.7 and the inequality (30) for $q \geq 1$ $(0 < q < 1)$ we have that

$$\begin{aligned}\Psi_{p,q}(t+s) &= [\Psi_p(t+s)]^q \geq (\leq) [\Psi_p(t) + \Psi_p(s)]^q \\ &\geq (\leq) [\Psi_p(t)]^q + [\Psi_p(s)]^q = \Psi_{p,q}(t) + \Psi_{p,q}(s)\end{aligned}$$

for any $t, s \in I$ and the statement is proved. □

Remark 2.9. *If we consider the function $\psi_p(t) := t^{p-1} h^p(t)$ then for $p \geq 1$ $(0 < p < 1)$ and $h : I \to [0, \infty)$ a superadditive (subadditive) function on I, the function ψ_p is also superadditive (subadditive) on I.*

The following result also holds:

Theorem 2.10. *If $h : I \to (0, \infty)$ is a superadditive function on I and $0 < m < 1$, then the function*

$$\Phi_p : I \to [0, \infty), \Phi_p(t) = \frac{t^{1-\frac{1}{m}}}{h(t)} \tag{35}$$

is subadditive on I.

Proof. Let $m := -p \in [-1, 0)$. For $m < 0$ we have the following inequality

$$(\alpha + \beta)^m \leq \alpha^m + \beta^m \tag{36}$$

for any $\alpha, \beta > 0$. Indeed, by the convexity of the function $f_s(t) = t^m$ on $(0, \infty)$ with $m < 0$ we have that

$$(\alpha + \beta)^m \leq 2^{m-1} (\alpha^m + \beta^m)$$

for any $\alpha, \beta > 0$ and since, obviously, $2^{m-1} (\alpha^m + \beta^m) \leq \alpha^m + \beta^m$, then (36) holds true.

Taking into account that h is superadditive, then by (36) we have

$$h^m (t + s) \leq [h(t) + h(s)]^m \leq h^m (t) + h^m (s) \tag{37}$$

for any $t, s \in I$. By (36) we have that

$$\begin{aligned} \frac{h^m (t+s)}{t+s} &\leq \frac{h^m (t) + h^m (s)}{t+s} \\ &= \frac{t \cdot \left[\frac{h(t)}{t^{1/m}}\right]^m + s \cdot \left[\frac{h(s)}{s^{1/m}}\right]^m}{t+s} \\ &= \frac{t \cdot \left[\frac{t^{1/m}}{h(t)}\right]^{-m} + s \cdot \left[\frac{s^{1/m}}{h(s)}\right]^{-m}}{t+s} =: J. \end{aligned} \tag{38}$$

By the concavity of the function $g(t) = t^{-m}$ with $m \in [-1, 0)$ we also have

$$J \leq \left[\frac{t \cdot \frac{t^{1/m}}{h(t)} + s \cdot \frac{s^{1/m}}{h(s)}}{t+s}\right]^{-m}. \tag{39}$$

Making use of (38) and (39) we get

$$\frac{h^m (t+s)}{t+s} \leq \left[\frac{t \cdot \frac{t^{1/m}}{h(t)} + s \cdot \frac{s^{1/m}}{h(s)}}{t+s}\right]^{-m}$$

for any $t, s \in I$, which is equivalent to

$$\frac{h^{-1} (t+s)}{(t+s)^{-1/m}} \leq \frac{\frac{t^{1+1/m}}{h(t)} + \frac{s^{1+1/m}}{h(s)}}{t+s}$$

and, with

$$\frac{(t+s)^{1+1/m}}{h(t+s)} \leq \frac{t^{1+1/m}}{h(t)} + \frac{s^{1+1/m}}{h(s)}$$

for any $t, s \in I$.

This completes the proof. □

The following result may be stated as well:

COROLLARY 2.11. *If $h : I \to [0, \infty)$ is a superadditive function on I and $0 < p, q < 1$ then the two parameter function*

$$\Phi_{p,q} : I \to [0, \infty), \Phi_{p,q}(t) = \frac{t^{q\left(1-\frac{1}{p}\right)}}{h^q(t)} \tag{40}$$

is subadditive on I.

Proof. Observe that $\Phi_{p,q}(t) = [\Phi_p(t)]^q$ for $t \in I$. Therefore, by Theorem 2.10 and the inequality (30) for $0 < q < 1$ we have that

$$\begin{aligned}\Phi_{p,q}(t+s) &= [\Phi_p(t+s)]^q \le [\Phi_p(t) + \Phi_p(s)]^q \\ &\le [\Phi_p(t)]^q + [\Phi_p(s)]^q = \Phi_{p,q}(t) + \Phi_{p,q}(s)\end{aligned}$$

for any $t, s \in I$ and the statement is proved. □

REMARK 2.12. *If we consider the function $\varphi_p(t) := \frac{t^{p-1}}{h^p(t)}$ then for $0 < p < 1$ and $h : I \to [0, \infty)$ a superadditive function on I, the function ψ_p is subadditive on I.*

3. Jensen's type inequalities

The following inequality of Jensen's type holds:

THEOREM 3.1. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$ and a mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X. The following statements are equivalent:*

(i) f is λ-convex on C;

(ii) For all $x_i \in C$ and $p_i \ge 0$ with $i \in \{1, ..., n\}$, $n \ge 2$ so that $P_n > 0$ we have the inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i) f(x_i). \tag{41}$$

Proof. "$(ii) \Rightarrow (i)$". Follows for $n = 2$.

"$(i) \Rightarrow (ii)$". For $n = 2$ the inequality (30) follows by the Definition 2.1.

Assume that the inequality (41) is true for $2, ..., n-1$ ($n \ge 3$) and let prove it for n.

Let $p_i \ge 0$ with $i \in \{1, ..., n\}$, $n \ge 3$ so that $P_n > 0$. If $P_{n-1} = 0$, then $p_1 = ... = p_{n-1} = 0$ and $p_n > 0$ and the inequality (41) becomes

$$f(x_n) \le \frac{\lambda(0)(f(x_1) + ... + f(x_{n-1})) + \lambda(p_n) f(x_n)}{\lambda(p_n)},$$

which is equivalent to

$$\lambda(0)(f(x_1)+...+f(x_{n-1}))\geq 0. \tag{42}$$

Since f is λ-convex on C then for $\beta>0$ and $x\in C$ we have

$$f\left(\frac{0x+\beta y}{0+\beta}\right)\leq\frac{\lambda(0)f(x)+\lambda(\beta)f(y)}{\lambda(\beta)}$$

from where we get

$$\frac{\lambda(0)f(x)}{\lambda(\beta)}\geq 0$$

and since $\lambda(\beta)>0$ we get $\lambda(0)f(x)\geq 0$. This implies that the inequality (42) is true for any $x_1,...,x_{n-1}\in C$.

Now, let assume that $P_{n-1}>0$. Then we have

$$\begin{aligned} f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right) &= f\left(\frac{P_{n-1}\cdot\frac{1}{P_{n-1}}\sum_{i=1}^{n-1}p_ix_i+p_nx_n}{P_{n-1}+p_n}\right)\\ &\leq\frac{\lambda(P_{n-1})f\left(\frac{1}{P_{n-1}}\sum_{i=1}^{n-1}p_ix_i\right)+\lambda(p_n)f(x_n)}{\lambda(P_n)}. \end{aligned}$$

By the induction hypothesis we have

$$f\left(\frac{1}{P_{n-1}}\sum_{i=1}^{n-1}p_ix_i\right)\leq\frac{1}{\lambda(P_{n-1})}\sum_{i=1}^{n-1}\lambda(p_i)f(x_i)$$

and thus, by the above inequality, we can state that

$$\begin{aligned} f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right) &\leq\frac{\lambda(P_{n-1})\frac{1}{\lambda(P_{n-1})}\sum_{i=1}^{n-1}\lambda(p_i)f(x_i)+\lambda(p_n)f(x_n)}{\lambda(P_n)}\\ &=\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f(x_i), \end{aligned}$$

and the theorem is thus proved. □

COROLLARY 3.2. *Let $f:C\to\mathbb{R}$ be a λ-convex function on C and $\alpha_i\in[0,1]$, $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}\alpha_i=1$. Then for any $x_i\in C$ with $i\in\{1,...,n\}$ we have the inequality*

$$f\left(\sum_{i=1}^{n}\alpha_ix_i\right)\leq\frac{1}{\lambda(1)}\sum_{i=1}^{n}\lambda(\alpha_i)f(x_i). \tag{43}$$

In particular, we have

$$f\left(\frac{x_1+...+x_n}{n}\right)\le c(n)\,\frac{f(x_1)+...+f(x_n)}{n} \tag{44}$$

where

$$c(n):=\frac{n\lambda\left(\frac{1}{n}\right)}{\lambda(1)},\ n\ge 2.$$

We have the following version of Jensen's inequality:

Corollary 3.3. *Let $f:C\to\mathbb{R}$ be a λ-convex function on C and $x_i\in C$ and $p_i\ge 0$ with $i\in\{1,...,n\}$, $n\ge 2$ so that $P_n>0$. Then we have the inequality*

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\le\frac{1}{\lambda(1)}\sum_{i=1}^n\lambda\left(\frac{p_i}{P_n}\right)f(x_i). \tag{45}$$

The proof follows by (43) for $\alpha_i=\frac{p_i}{P_n}$, $i\in\{1,...,n\}$.

Corollary 3.4. *Let $h(z)=\sum_{n=0}^\infty a_n z^n$ a power series with nonnegative coefficients $a_n\ge 0$ for all $n\in\mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. For a mapping $f:C\to\mathbb{R}$ defined on convex subset C of a linear space X, the following statements are equivalent:*

(i) The function f is λ_r-convex with $\lambda_r:[0,\infty)\to[0,\infty)$

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right]$$

on C;

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-P_n))}\right]^{f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)}\le\prod_{i=1}^n\left[\frac{h(r)}{h(r\exp(-p_i))}\right]^{f(x_i)} \tag{46}$$

for any $x_i\in C$ and $p_i\ge 0$ with $i\in\{1,...,n\}$, $n\ge 2$ so that $P_n>0$.

Now, let define the mapping:

$$J(I,p,x,f):=\sum_{i\in I}\lambda(p_i)f(x_i)-\lambda(P_I)f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right),$$

where $p:=(p_i)_{i\in\mathbb{N}}\ge 0$, $I\in\mathcal{F}(\mathbb{N}):=\{I\subset\mathbb{N}|\ I\text{ is finite}\}$, $x:=(x_i)_{i\in\mathbb{N}}\subset C$ and $P_I:=\sum_{i\in I}p_i>0$.

Theorem 3.5. *Assume that $f : C \to \mathbb{R}$ is a λ-convex function on C and p, x are as above. Then*

(i) For all $I, K \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}$ with $I \cap K = \emptyset$ we have the inequality

$$J(I \cup K, p, x, f) \geq J(I, p, x, f) + J(K, p, x, f) \geq 0, \tag{47}$$

i.e. the mapping $J(\cdot, p, x, f)$ is superadditive as an index set map on $\mathcal{F}(\mathbb{N})$;

(ii) For all $I, K \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}$ with $K \subsetneq I$ one has the inequality

$$J(I, p, x, f) \geq J(K, p, x, f) \geq 0, \tag{48}$$

i.e. the mapping $J(\cdot, p, x, f)$ is monotonic nondecreasing as an index set map on $\mathcal{F}(\mathbb{N})$.

Proof. (i) Let $I, K \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}$ with $I \cap K = \emptyset$, then

$$\begin{aligned}
&J(I \cup K, p, x, f) \\
&= \sum_{i \in I} \lambda(p_i) f(x_i) + \sum_{j \in K} \lambda(p_j) f(x_j) \\
&- \lambda(P_I + P_K) f\left[\frac{1}{P_I + P_K}\left(\sum_{i \in I} p_i x_i + \sum_{j \in K} p_j x_j\right)\right] \\
&= \sum_{i \in I} \lambda(p_i) f(x_i) + \sum_{j \in K} \lambda(p_j) f(x_j) \\
&- \lambda(P_I + P_K) f\left[\frac{P_I}{P_I + P_K}\left(\frac{\sum_{i \in I} p_i x_i}{P_I}\right) + \frac{P_K}{P_I + P_K}\left(\frac{\sum_{j \in K} p_j x_j}{P_K}\right)\right].
\end{aligned}$$

As f is λ-convex function on C, then

$$\begin{aligned}
&f\left[\frac{P_I}{P_I + P_K}\left(\frac{\sum_{i \in I} p_i x_i}{P_I}\right) + \frac{P_K}{P_I + P_K}\left(\frac{\sum_{j \in K} p_j x_j}{P_K}\right)\right] \\
&\leq \frac{\lambda(P_I) f\left(\frac{\sum_{i \in I} p_i x_i}{P_I}\right) + \lambda(P_K) f\left(\frac{\sum_{j \in K} p_j x_j}{P_K}\right)}{\lambda(P_I + P_K)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
J(I \cup K, p, x, f) &\geq \sum_{i \in I} \lambda(p_i) f(x_i) + \sum_{j \in K} \lambda(p_j) f(x_j) \\
&- \lambda(P_I) f\left(\frac{\sum_{i \in I} p_i x_i}{P_I}\right) - \lambda(P_K) f\left(\frac{\sum_{j \in K} p_j x_j}{P_K}\right) \\
&= J(I, p, x, f) + J(K, p, x, f)
\end{aligned}$$

and the inequality (47) is proved.

(ii) By the use of the inequality (47) we have

$$\begin{aligned} J(I,p,x,f) &= J(K\cup(I\setminus K),p,x,f) \geq J(K,p,x,f)+J(I\setminus K,p,x,f) \\ &\geq J(K,p,x,f) \end{aligned}$$

since $J(I\setminus K,p,x,f)\geq 0$, and the inequality (48) is proved. □

With the above assumptions, and if $p:=(p_i)_{i\in\mathbb{N}}>0$ we can consider the sequence

$$J_n(p,x,f):=\sum_{i=1}^{n}\lambda(p_i)f(x_i)-\lambda(P_n)f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right),\ n\geq 2.$$

Corollary 3.6. *Assume that $f:C\to\mathbb{R}$ is a λ-convex function on C, then*

$$J_n(p,x,f)\geq J_{n-1}(p,x,f)\geq ...\geq J_2(p,x,f)\geq 0 \tag{49}$$

and we have the inequality

$$\begin{aligned} &J_n(p,x,f) \\ &\geq \max_{1\leq i<j\leq n}\left\{\lambda(p_i)f(x_i)+\lambda(p_j)f(x_j)-\lambda(p_i+p_j)f\left(\frac{p_ix_i+p_jx_j}{p_i+p_j}\right)\right\} \\ &\geq 0 \end{aligned} \tag{50}$$

for all $n\geq 2$.

For a function f that is λ_r-convex on C with $\lambda_r:[0,\infty)\to[0,\infty)$ and

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right],$$

we can consider the functional

$$Q(I,p,x,f):=\frac{\prod_{i\in I}\left[\frac{h(r)}{h(r\exp(-p_i))}\right]^{f(x_i)}}{\left[\frac{h(r)}{h(r\exp(-P_I))}\right]^{f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)}},$$

where $p:=(p_i)_{i\in\mathbb{N}}\geq 0$, $I\in\mathcal{F}(\mathbb{N}):=\{I\subset\mathbb{N}|\ I\text{ is finite}\}$, $x:=(x_i)_{i\in\mathbb{N}}\subset C$ and $P_I:=\sum_{i\in I}p_i>0$.

Corollary 3.7. *Assume that $f:C\to\mathbb{R}$ is a λ_r-convex function on C and p, x are as above. Then* *(i) For all $I, K \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}$ with $I \cap K = \emptyset$ we have the inequality*

$$Q(I \cup K, p, x, f) \geq Q(I, p, x, f)\, Q(K, p, x, f), \tag{51}$$

i.e. the mapping $Q(\cdot, p, x, f)$ is supermultiplicative as an index set map on $\mathcal{F}(\mathbb{N})$;

(ii) For all $I, K \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}$ with $K \subsetneq I$ one has the inequality

$$Q(I, p, x, f) \geq Q(K, p, x, f) \geq 1. \tag{52}$$

The proof follows by Theorem 3.5 on observing that

$$\ln Q(I, p, x, f) = J(I, p, x, f)$$

for $\lambda = \lambda_r$. In particular, if we consider the sequence

$$Q_n(p, x, f) := \frac{\prod_{i=1}^{n} \left[\frac{h(r)}{h(r\exp(-p_i))}\right]^{f(x_i)}}{\left[\frac{h(r)}{h(r\exp(-P_n))}\right]^{f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)}},\ n \geq 2$$

then by Corollary 3.6 we have that

$$Q_n(p, x, f) \geq Q_{n-1}(p, x, f) \geq \ldots \geq Q_2(p, x, f) \geq 1 \tag{53}$$

and

$$Q_n(p, x, f) \geq \max_{1 \leq i < j \leq n} \left\{ \frac{\left[\frac{h(r)}{h(r\exp(-p_i))}\right]^{f(x_i)} \left[\frac{h(r)}{h(r\exp(-p_j))}\right]^{f(x_j)}}{\left[\frac{h(r)}{h(r\exp(-p_i - p_j))}\right]^{f\left(\frac{1}{p_i+p_j}(p_i x_i + p_j x_j)\right)}} \right\} \geq 1. \tag{54}$$

Remark 3.8. *If the function $f : C \to \mathbb{R}$ is a λ-convex function on C with*

$$\lambda_r(t) = 1 - \exp(-t),\ t \geq 0,$$

then for any $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$ we have the Jensen's type inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{1 - \exp(-P_n)} \sum_{i=1}^{n} \left[1 - \exp(-p_i)\right] f(x_i). \tag{55}$$

If $\alpha_i \in [0, 1]$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} \alpha_i = 1$, then for any $x_i \in C$ with $i \in \{1, ..., n\}$ we also have the inequality

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \frac{e}{e - 1} \sum_{i=1}^{n} \left[1 - \exp(-\alpha_i)\right] f(x_i). \tag{56}$$

Finally, if $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$, then for any $x_i \in C$ with $i \in \{1, ..., n\}$ we have the inequality:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{e}{e-1}\sum_{i=1}^{n}\left[1-\exp\left(-\frac{p_i}{P_n}\right)\right] f(x_i). \tag{57}$$

4. Inequalities for double sums

We have the following result:

THEOREM 4.1. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C and $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$. For $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ we have the inequalities*

$$\begin{aligned}
&\left[\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)} + \frac{\lambda(\beta)}{\lambda(\alpha+\beta)}\right]\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i) f(x_i) \frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i) \qquad (58)\\
&\geq \frac{1}{\lambda^2(P_n)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_i)\lambda(p_j) f\left(\frac{\alpha x_i + \beta x_j}{\alpha+\beta}\right)\\
&\geq \frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i) f\left(\frac{\alpha x_i + \beta \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j}{\alpha+\beta}\right) \geq f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right).
\end{aligned}$$

Proof. From the λ-convexity of the function f on C we have

$$\frac{\lambda(\alpha) f(x_i) + \lambda(\beta) f(x_j)}{\lambda(\alpha+\beta)} \geq f\left(\frac{\alpha x_i + \beta x_j}{\alpha+\beta}\right) \tag{59}$$

for any $i, j \in \{1, ..., n\}$. If we multiply (59) by

$$\frac{\lambda(p_i)\lambda(p_j)}{\lambda^2(P_n)} \geq 0, \; i, j \in \{1, ..., n\}$$

and sum over i and j from 1 to n we get

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)} f(x_i) + \frac{\lambda(\beta)}{\lambda(\alpha+\beta)} f(x_j)\right]\frac{\lambda(p_i)\lambda(p_j)}{\lambda^2(P_n)} \qquad (60)\\
&\geq \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda(p_i)\lambda(p_j)}{\lambda^2(P_n)} f\left(\frac{\alpha x_i + \beta x_j}{\alpha+\beta}\right).
\end{aligned}$$

Since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}f(x_i)+\frac{\lambda(\beta)}{\lambda(\alpha+\beta)}f(x_j)\right]\frac{\lambda(p_i)\lambda(p_j)}{\lambda^2(P_n)}$$

$$=\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda(p_i)\lambda(p_j)}{\lambda^2(P_n)}f(x_i)+\frac{\lambda(\beta)}{\lambda(\alpha+\beta)}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda(p_i)\lambda(p_j)}{\lambda^2(P_n)}f(x_j)$$

$$=\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}\frac{1}{\lambda^2(P_n)}\sum_{i=1}^{n}\lambda(p_i)f(x_i)\sum_{j=1}^{n}\lambda(p_j)$$

$$+\frac{\lambda(\beta)}{\lambda(\alpha+\beta)}\frac{1}{\lambda^2(P_n)}\sum_{j=1}^{n}\lambda(p_j)f(x_j)\sum_{i=1}^{n}\lambda(p_i)$$

$$=\left[\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}+\frac{\lambda(\beta)}{\lambda(\alpha+\beta)}\right]\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f(x_i)\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i),$$

then by (60) we get the first inequality in (58).

By the Jensen inequality we have the inequality

$$\frac{1}{\lambda(P_n)}\sum_{j=1}^{n}\lambda(p_j)f\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right)\geq f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_j\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right)\right)$$
$$=f\left(\frac{\alpha x_i+\beta\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{\alpha+\beta}\right)$$

for all $i\in\{1,...,n\}$.

If we multiply this inequality by $\frac{\lambda(p_i)}{\lambda(P_n)}$ and sum over i from 1 to n we get

$$\frac{1}{\lambda^2(P_n)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_i)\lambda(p_j)f\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right)$$
$$\geq\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f\left(\frac{\alpha x_i+\beta\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{\alpha+\beta}\right)$$

and the second inequality in (58) is proved.

If we apply Jensen inequality again we get

$$\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f\left(\frac{\alpha x_i+\beta\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{\alpha+\beta}\right)$$
$$\geq f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\left(\frac{\alpha x_i+\beta\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{\alpha+\beta}\right)\right)=f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)$$

and the last part of (58) is proved. □

Corollary 4.2. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C and $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$. We have the inequalities*

$$\inf_{\alpha>0}\left(\frac{2\lambda(\alpha)}{\lambda(2\alpha)}\right)\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f(x_i)\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i) \tag{61}$$
$$\geq \frac{1}{\lambda^2(P_n)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_i)\lambda(p_j)f\left(\frac{x_i+x_j}{2}\right)$$
$$\geq \frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f\left(\frac{x_i+\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{2}\right)\geq f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right).$$

We have the following result as well:

Theorem 4.3. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C and $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$. For $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ we have the inequalities*

$$\left[\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}+\frac{\lambda(\beta)}{\lambda(\alpha+\beta)}\right]\frac{1}{\lambda(P_n^2)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_ip_j)f(x_i) \tag{62}$$
$$\geq \frac{1}{\lambda(P_n^2)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_ip_j)f\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right)\geq f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right).$$

Proof. From the λ-convexity of the function f on C we have

$$\frac{\lambda(\alpha)f(x_i)+\lambda(\beta)f(x_j)}{\lambda(\alpha+\beta)}\geq f\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right) \tag{63}$$

for any $i, j \in \{1, ..., n\}$. If we multiply (63) by

$$\frac{\lambda(p_ip_j)}{\lambda(P_n^2)}\geq 0,\ i,j\in\{1,...,n\}$$

and sum over i and j from 1 to n we get

$$\frac{1}{\lambda(P_n^2)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_ip_j)\left[\frac{\lambda(\alpha)f(x_i)+\lambda(\beta)f(x_j)}{\lambda(\alpha+\beta)}\right] \tag{64}$$
$$\geq \frac{1}{\lambda(P_n^2)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_ip_j)f\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right).$$

We have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda\left(p_ip_j\right)\left[\frac{\lambda\left(\alpha\right)f\left(x_i\right)+\lambda\left(\beta\right)f\left(x_j\right)}{\lambda\left(\alpha+\beta\right)}\right]$$
$$=\frac{\lambda\left(\alpha\right)}{\lambda\left(\alpha+\beta\right)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda\left(p_ip_j\right)f\left(x_i\right)+\frac{\lambda\left(\beta\right)}{\lambda\left(\alpha+\beta\right)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda\left(p_ip_j\right)f\left(x_j\right)$$

and since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda\left(p_ip_j\right)f\left(x_i\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda\left(p_ip_j\right)f\left(x_j\right)$$

then we get from (64) the first inequality in (62).

By Jensen's inequality we have

$$\frac{1}{\lambda\left(\sum_{i=1}^{n}\sum_{j=1}^{n}p_ip_j\right)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda\left(p_ip_j\right)f\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right)$$
$$\geq f\left(\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}p_ip_j}\sum_{i=1}^{n}\sum_{j=1}^{n}p_ip_j\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right)\right)$$
$$=f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)$$

and the last part of (62) is thus proved. □

Corollary 4.4. *Let $f:C\to\mathbb{R}$ be a λ-convex function on C and $x_i\in C$ and $p_i\geq 0$ with $i\in\{1,...,n\}$, $n\geq 2$ so that $P_n>0$. We have the inequalities*

$$\inf_{\alpha>0}\left(\frac{2\lambda\left(\alpha\right)}{\lambda\left(2\alpha\right)}\right)\frac{1}{\lambda\left(P_n^2\right)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda\left(p_ip_j\right)f\left(x_i\right) \tag{65}$$
$$\geq\frac{1}{\lambda\left(P_n^2\right)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda\left(p_ip_j\right)f\left(\frac{x_i+x_j}{2}\right)\geq f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right).$$

It is known that if $(X,\|\cdot\|)$ is a normed linear space, then the function $f\left(x\right)=\|x\|^s$, $s\in(0,1)$ is Breckner s-convex on X.

If $x_i\in X$ and $p_i\geq 0$ with $i\in\{1,...,n\}$, $n\geq 2$ so that $P_n>0$, then from (61) we have

$$
\begin{aligned}
&2^{1-s}\frac{1}{P_n^s}\sum_{i=1}^{n}p_i^s\frac{1}{P_n^s}\sum_{i=1}^{n}p_i^s\left\|x_i\right\|^s \qquad (66)\\
&\geq \frac{1}{P_n^{2s}}\sum_{i=1}^{n}\sum_{j=1}^{n}p_i^s p_j^s\left\|\frac{x_i+x_j}{2}\right\|^s\\
&\geq \frac{1}{P_n^s}\sum_{i=1}^{n}p_i^s\left\|\frac{x_i+\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{2}\right\|^s\geq\left\|\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right\|^s.
\end{aligned}
$$

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