Rend. Istit. Mat. Univ. Trieste
Volume 47 (2015), 187–202
DOI: 10.13137/0049-4704/11224

On the lifting problem in positive characteristic

Paola Bonacini

Abstract. *Given $\mathbb{P}^n_k$, with k algebraically closed field of characteristic $p > 0$, and $X \subset \mathbb{P}^n_k$ integral variety of codimension 2 and degree d, let $Y = X \cap H$ be the general hyperplane section of X. In this paper we study the problem of lifting, i.e. extending, a hypersurface in H of degree s containing Y to a hypersurface of same degree s in $\mathbb{P}^n$ containing X. For $n = 3$ and $n = 4$, in the case in which this extension does not exist we get upper bounds for d depending on s and p.*

Keywords: lifting problem, sporadic zero, surface.
MS Classification 2010: 14H50,14M07,14J99.

1. Introduction

Let $S \subset \mathbb{P}^n_k$, with k algebraically closed field of characteristic 0, be an integral variety of codimension 2 and degree d. Let X be the general hyperplane section of S and suppose that X is contained in a hypersurface of $\mathbb{P}^{n-1}$ of degree s. The lifting problem is the problem of finding some bound $d > f(s, n)$ in such a way that the hypersurface of $\mathbb{P}^{n-1}$ of degree s containing X can be lifted to a hypersurface in $\mathbb{P}^n$ of degree s containing S.

This problem has been first studied for curves in $\mathbb{P}^3$ by Laudal in his Lemma in [14, Corollary p. 147] and then the bound was refined, using two different methods by Gruson and Peskine [8] and by Strano in [25, Corollario 2] to $d > s^2 + 1$. Moreover, this bound is sharp, as we see in the examples in [8], [9] and [26, Proposition 1].

Later on, the problem has been studied for some particular values of n and for $n \geq 3$.

The case $n = 4$ has been studied and solved by Mezzetti and Raspanti in [19] and in [18], showing that the sharp bound is $d > s^2 - s + 1$, and in [17] Mezzetti classifies the border case $d = s^2 - s + 2$.

The case $n = 5$ has been solved with new methods by Mezzetti in [18], with $d > s^2 - 2s + 4$ as sharp bound. Moreover, in this paper she suggests the conjecture that the sharp bound could be:

$$d > s^2 - (n-3)s + \binom{n-2}{2} + 1.$$

Roggero in [20] proves Mezzetti's conjecture in the case $n = 6$, while in the general case (see [23]) the proof needs some additional assumptions on the general plane section of S. In this way Roggero improves a result by Tortora [27], who proved the conjecture under one more additional technical hypothesis than in Roggero's proof.

For the general case, Chiantini and Ciliberto prove in [6] the bound $d > s^2 + (2n-3)s$ for any $n \geq 3$, that is improved at first by Valenzano in [28] and taken to $d > s^2 - 2s + 2$, with $n \geq 5$, and then by Roggero in [22] and taken to $d > s^2 - 3s + 7$, for $n \geq 6$. If $n \geq 7$, Roggero in [21] slightly improves the bound to $d > s^2 - 3s + 6$, which is quite far from the bound $d > s^2 - 4s + 11$ of the conjecture for $n = 7$.

Another approach has been tried by Tortora in [27] and Roggero [23], generalizing the lifting problem to the problem of bounding the degree in such a way that the map $H^0(\mathscr{I}_S(s)) \to H^0(\mathscr{I}_X(s))$ is surjective. In this case, instead of bounding the degree of S by a function $f(s,n)$ they try to bound the degree by a function $f(s,n,a)$, where $a = h^0(\mathscr{I}_S(s))$.

In this paper we show the approach to the problem in the case that the base field has positive characteristic p and that either $n = 3$ (see [3]) or $n = 4$ (see [4]).

2. Lifting problem for curves in $\mathbb{P}^3$: Gruson's and Peskine's proof

In [8] Gruson and Peskine, under the hypothesis that the base field k has characteristic 0, prove the following:

THEOREM 2.1. *Let $C \subset \mathbb{P}^3$ be an integral curve of degree d such that its generic plane section X is contained in a plane curve of degree s. Suppose that $d > s^2 + 1$. Then C is contained in a surface of degree s.*

The idea of the proof is the following. We suppose that C is not contained in any surface of degree s and we prove that $d \leq s^2 + 1$.

Consider the bi-projective space $\check{\mathbb{P}}^3 \times \mathbb{P}^3$. Let $k[\underline{t}]$ and $k[\underline{x}]$ the coordinate rings of $\check{\mathbb{P}}^3$ and $\mathbb{P}^3$, respectively, and consider the incidence variety $M \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$, which is a hypersurface of equation $\sum t_i x_i = 0$. Consider also the projection $p\colon M \to \mathbb{P}^3$.

Take a minimal s such that there exists a plane curve containing the generic plane section X of C. It is not difficult to see that this plane curve Γ determines an integral hypersurface $S \subset M$ such that $S \supset p^{-1}(C)$. Moreover, since we suppose that $h^0 \mathscr{I}_C(s) = 0$, we see that the projection $p_S \colon S \to \mathbb{P}^3$ is dominant and is, of course, generically smooth (because char $k = 0$ and here we see the importance of the characteristic of the base field).

Using the fact that p_S is generically smooth we get the key to the inequality that we want to prove, which is the following exact sequence:

$$0 \to \mathscr{N} \to \Omega_H(1) \to \mathscr{I}_\Delta(s) \to 0,$$

where H is the generic plane, $\mathscr{I}_\Delta \subset \mathscr{O}_\Gamma$ is the ideal sheaf of a zero-dimensional scheme containing $X = C \cap H$ and $\mathscr{N}$ is a locally free sheaf. More precisely, the key is:

$$c_2(\mathscr{N}(1)) = s^2 + 1 - \deg \Delta.$$

The statement follows by the following:

Lemma 2.2 ([8]). *Let $\mathscr{M}$ be a locally free sheaf of rank 2 on $\mathbb{P}^2$ such that $h^0 \mathscr{M}(-1) = 0$. Then $c_2(\mathscr{M}) \geq 0$.*

Indeed, since $\mathscr{N}$ is a locally free sheaf of rank two in H such that $h^0 \mathscr{N} = 0$, we get $c_2(\mathscr{N}(1)) \geq 0$. This means that $\deg C = \deg X \leq \deg \Delta \leq s^2 + 1$, so that the theorem is proved.

Now we need to remark that the following result holds:

Theorem 2.3. *Let $C \subset \mathbb{P}^3$ be a curve (not necessarily reduced or irreducible) of degree d such that its generic plane section is contained in an* integral *plane curve Γ of degree s. If $d > s^2 + 1$, then C is contained in a surface of degree s.*

Indeed, in this case, the proof is the same as the previous one. The only difference is at the beginning, because the surface $S \subset M$ that we get is integral since Γ is integral. From this point on the proof follows in the same way.

The bound given in Theorem 2.1 is sharp. Before giving the example, let us recall the following definition.

Definition 2.4. *A rank 2 vector bundle $\mathscr{E}_0$ on $\mathbb{P}^3$ is said to be a* null-correlation bundle *if there exists an exact sequence $0 \to \mathscr{O}_{\mathbb{P}^3} \xrightarrow{\tau} \Omega_{\mathbb{P}^3}(2) \to \mathscr{E}_0(1) \to 0$ where τ is a nowhere vanishing section of $\Omega_{\mathbb{P}^3}(2)$.*

Remark 2.5. *It is possible to prove (see [1], [29] and [10, Example 8.4.1]) that $\mathscr{E}$ is a stable rank 2 vector bundle on $\mathbb{P}^3$ with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = 1$ if and only if $\mathscr{E}$ is isomorphic to a null-correlation bundle.*

Example 2.6 ([8]). Let $\sigma \in H^0 \mathscr{E}_0(s)$ be a global section whose zero locus is curve C. Then we have an exact sequence $0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{E}_0(s) \to \mathscr{I}_C(2s) \to 0$. Then $h^0 \mathscr{I}_C(s) = h^0 \mathscr{E}_0 = 0$ and $\deg C = c_2(\mathscr{E}_0(s)) = s^2 + 1$. By [10, Proposition 1.4] for a sufficiently general $\sigma \in H^0 \mathscr{E}_0(s)$ C is a smooth connected curve. Since

$h^0\mathscr{E}_0|_H = 1$, we see that $h^0\mathscr{I}_{C\cap H|H}(s) = 1$. This means that C is an integral smooth curve of degree s^2+1, which is not contained in any surface of degree s, such that the minimal curve containing its generic plane section is an integral curve of degree s.

3. Lifting problem for curves in $\mathbb{P}^3$: characteristic p case

In this section we will show what happens for the lifting problem of curves in $\mathbb{P}^3_k$ in the case that $\operatorname{char} k = 0$. First, we need to recall the definition of absolute and relative Frobenius morphism:

Definition 3.1. *The absolute Frobenius morphism of a scheme X of characteristic $p > 0$ is $F_X\colon X \to X$, where F_X is the identity as a map of topological spaces and on each U open set $F_X^{\#}\colon \mathscr{O}_X(U) \to \mathscr{O}_X(U)$ is given by $f \mapsto f^p$ for each $f \in \mathscr{O}_X(U)$. Given $X \to S$ for some scheme S and $X^{p/S} = X \times_{S,F_S} S$, the absolute Frobenius morphisms on X and S induce a morphism $F_{X/S}\colon X \to X^{p/S}$, called the Frobenius morphism of X relative to S.*

Given $\mathbb{P}^n$ for some $n \in \mathbb{N}$, let us consider the bi-projective space $\check{\mathbb{P}}^n \times \mathbb{P}^n$ and let $r \in \mathbb{N}$ be a non negative integer. Let $k[\underline{t}]$ and $k[\underline{x}]$ be the coordinate rings of $\check{\mathbb{P}}^n$ and $\mathbb{P}^n$, respectively. Let $M_r \subset \check{\mathbb{P}}^n \times \mathbb{P}^n$ be the hypersurface of equation $h_r := \sum_{i=0}^n t_i {x_i}^{p^r} = 0$. Note that in the case $r = 0$ M_r is the usual incidence variety M of equation $\sum t_i x_i = 0$. If $r \geq 1$, M_r is determined by the following fibred product:

$$\begin{tikzcd}
M \arrow[dr, dashed, "F_{M_r}"] \arrow[drr, bend left, "(F_M)^r"] \arrow[ddr, bend right, "p"'] & & \\
& M_r \arrow[r, "\pi"] \arrow[d, "p_{M_r}"'] & M \arrow[d, "p"] \\
& \mathbb{P}^n \arrow[r, "F^r"'] & \mathbb{P}^n
\end{tikzcd}$$

where $F\colon \mathbb{P}^n \to \mathbb{P}^n$ is the absolute Frobenius.

In positive characteristic the lifting problem for curves in $\mathbb{P}^3$ has been solved in [3] getting the following result:

Theorem 3.2. *Let $C \subset \mathbb{P}^3$ be a non degenerate reduced curve of degree d in characteristic $p > 0$. Suppose that the generic plane section X is contained in an integral plane curve of degree s. Then C is contained in a surface of degree s, if one of the following conditions is satisfied:*

1. *C is connected, $p \geq s$ and $d > s^2 + 1$;*

2. *C is connected, $p < s$ and $d > s^2 + p^{2n}$, with $p^n < s \leq p^{n+1}$; in particular this holds if $d > 2s^2 - 2s + 1$;*

3. *$p > s$ and $d > s^2 + 1$;*

4. *$p \leq s$ and $d > s^2 + p^{2n}$, with $p^n \leq s < p^{n+1}$. In particular this holds if $d > 2s^2$.*

The idea of the proof is to follow Gruson's and Peskine's Theorem, considering, however, that some differences occur due to the positive characteristic.

The beginning of the proof is the same as the one for the characteristic 0 case. Indeed, by taking a minimal s such that there exists a plane curve containing the generic plane section X of C, we see that this plane curve Γ determines an integral hypersurface $S \subset M$ such that $S \supset p^{-1}(C)$.

The first difficulty given by the characteristic is that the projection $p_S \colon S \to \mathbb{P}^3$ is dominant, but it may not be generically smooth. In order to solve this problem we use the following result:

Theorem 3.3 ([3]). *Let $V \subset \check{\mathbb{P}}^n \times \mathbb{P}^n$ be an integral hypersurface in M such that the projection $\pi \colon V \to \mathbb{P}^n$ is dominant and not generically smooth. Then there exist $r \geq 1$ and $V_r \subset M_r$ integral hypersurface such that π can be factored in the following way:*

$$\begin{tikzcd} V \arrow[rr, "\pi"] \arrow[dr, "F_r"'] & & \mathbb{P}^n \\ & V_r \arrow[ur, "\pi_r"'] & \end{tikzcd}$$

where the projection π_r is dominant and generically smooth and F_r is induced by the commutative diagram:

$$\begin{tikzcd} V \arrow[r, "F_r"] \arrow[d, hook, "j"'] & V_r \arrow[d, hook, "i"] \\ M \arrow[r, "F_{M_r}"'] & M_r \end{tikzcd} \tag{1}$$

So, we factor p_S through a generically smooth morphism $p_{S_r} \colon S_r \to \mathbb{P}^3$, where S_r is an integral hypersurface in $S_r \subset M_r = V(\sum t_i x_i^{p^r}) \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ and, given $p_r \colon M_r \to \mathbb{P}^3$, we also have that $p_r^{-1}(C) \subset S_r$.

Now, it is possible to continue as in Gruson's and Peskine's proof. However, when we use the generically smooth morphism p_{S_r}, we get an exact sequence:

$$0 \to \mathscr{N} \to F^{r\star}\Omega_H(p^r) \to \mathscr{I}_\Delta(s) \to 0,$$

where F is the Frobenius morphism, $\mathscr{N}$ is a rank two vector bundle, and $\mathscr{I}_\Delta \subset \mathscr{O}_\Gamma$ is the ideal sheaf of a 0-dimensional scheme containing X. In this case, we see that $c_2(\mathscr{N}(p^r)) = s^2 + p^{2r} - \deg\Delta$. Since $h^0\mathscr{N}(p^r - 1) = 0$, by Lemma 2.2 we see that $c_2(\mathscr{N}(p^r)) \geq 0$, so that $d \leq \deg\Delta \leq s^2 + p^{2r}$.

Now, as a last step of the proof we need to get a bound on p^r. This is the point where the hypothesis that the curve C is reduced is needed. Indeed, given a generic plane $H = V(l)$, where l is a linear form in the $\{x_i\}$, we can consider the non reduced surface H_r in $\mathbb{P}^3$ given by $l^{p^r} = 0$. Let X_r be the intersection of C with H_r. Then, we get the exact sequence:

$$H^0\left(\mathscr{I}_C(s)\right) \to H^0\left(\mathscr{I}_{\Gamma_r}(s)\right) \to H^1\left(\mathscr{I}_C(s-p^r)\right) \overset{\varphi_H}{\to} H^1\left(\mathscr{I}_C(s)\right),$$

determined by:

$$0 \to \mathscr{I}_C(-p^r) \overset{\varphi_H}{\to} \mathscr{I}_C \to i_\star\mathscr{I}_{\Gamma_r} \to 0.$$

The hypersurface $S_r \subset M_r$ determines a nonzero element $\alpha \in H^1\mathscr{I}_C(s-p^r)$ such that $\alpha \cdot l^{p^r} = 0$. In particular, we see that $h^1\mathscr{I}_C(s-p^r) \neq 0$, which gives us the statement, because C is reduced.

Now, by generalizing the example given in Example 2.6, we show that for any p there exist smooth integral curves of degree $d = s^2 + p^{2n}$, being $s > p$ and n such that $p^n < s \leq p^{n+1}$, that are not contained in any surface of degree s and that have the generic plane section contained in an integral plane curve of degree s.

EXAMPLE 3.4 ([3]). Let $\mathscr{E}_0$ be a null-correlation bundle. Let $n, s \in \mathbb{N}$ be positive integers and let $F\colon \mathbb{P}^3 \to \mathbb{P}^3$ be the absolute Frobenius on $\mathbb{P}^3$. Let us consider the sheaf $\mathscr{E}(s) = F^{n\star}(\mathscr{E}_0) \otimes \mathscr{O}_{\mathbb{P}^3}(s)$. Since $c_1(F^{n\star}(\mathscr{E}_0)) = 0$ and $c_2(F^{n\star}(\mathscr{E}_0)) = p^{2n}$, we see that $c_1(\mathscr{E}(s)) = 2s$ and $c_2(\mathscr{E}(s)) = p^{2n} + s^2$.

Let $\sigma \in H^0(\mathscr{E}(s))$ be a global section such that the zero locus of σ is a curve C. Then we get the exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{E}(s) \to \mathscr{I}_C(2s) \to 0 \tag{2}$$

so that $h^0\left(\mathscr{I}_C(s)\right) = h^0(\mathscr{E})$ and $\deg C = c_2(\mathscr{E}(s)) = p^{2n} + s^2$. Let H be a plane transversal to C and $\Gamma = C \cap H$. Restricting to H the exact sequence (2) we have:

$$0 \to \mathscr{O}_H \to \mathscr{E}(s)|_H \to \mathscr{I}_X(2s) \to 0.$$

By [7, Theorem 3.2] $\mathscr{E}$ is stable and we can choose H sufficiently general in such a way that $\mathscr{E}|_H$ is semi-stable, but not stable. Since $\mathscr{E}$ is stable and $c_1(\mathscr{E}) = 0$, then by [11, Lemma 3.1] $h^0(\mathscr{E}) = 0$, which implies that $h^0(\mathscr{I}_C(s)) = 0$. So C is not contained in any surface of degree s.

Let X be the generic plane section of C. It is possible to see that $h^0\mathscr{I}_X(s) = h^0(\mathscr{E}|_H) = 1$. So there is a unique plane curve of degree s containing X, which means that this plane curve of degree s is the minimal plane curve containing X.

It is also possible to see that $h^0(\mathscr{E}(s)) \neq 0$ if and only if $s \geq p^n$ and that every general nonzero global section of $\mathscr{E}(s)$, for $s \geq p^n$, has as zero locus a curve in $\mathbb{P}^3$. By [10, Proposition 1.4] we see that for $s > p^n$ the zero locus of a generic global section of $\mathscr{E}(s)$ is connected and smooth.

In this way we construct, for any p, n, s, with $s > p^n$, examples of irreducible and smooth curves $C \subset \mathbb{P}^3$ of degree $p^{2n} + s^2$ not contained in any surface of degree s such that the minimal curve containing its generic plane section has degree s. In this situation the minimal curve of degree s containing the generic plane section of C is integral by [2, Theorem 4.1].

In particular, we see that the bound in Theorem 3.2 for connected curves is sharp. Moreover, taking $s = p^n + 1$, we see that there exist connected and reduced curves (in particular nonsingular) of degree $d = 2s^2 - 2s + 1$, not lying on any surface of degree s, whose generic plane section is contained in an integral plane curve of degree s.

4. Lifting problem in $\mathbb{P}^4$

In [18] the lifting problem has been solved in characteristic 0 for surfaces in $\mathbb{P}^4$:

THEOREM 4.1. *Let S be an integral non degenerate surface of degree d in $\mathbb{P}^4$. Let s be the minimal degree of a degenerate surface containing a general hyperplane section of S. If $d > s^2 - s + 2$, then S is contained in a hypersurface of degree s.*

The following example shows that the bound in Theorem 4.1 is sharp.

EXAMPLE 4.2. Let $s \in \mathbb{N}$ such that $s \geq 2$ and let us consider $\mathscr{E} = \mathscr{O}_{\mathbb{P}^4}(1-2s) \oplus \mathscr{O}_{\mathbb{P}^4}(-1-s)^{\oplus 2}$. A general momorphism $\varphi \in \operatorname{Hom}(\mathscr{E}, \Omega_{\mathbb{P}^4}(1-s))$ determines a smooth integral surface $X \subset \mathbb{P}^4$ such that:

$$0 \to \mathscr{O}_{\mathbb{P}^4}(1-2s) \oplus \mathscr{O}_{\mathbb{P}^4}^{\oplus 2}(-1-s) \to \Omega_{\mathbb{P}^4}(1-s) \to \mathscr{I}_X \to 0.$$

Moreover, $h^0\mathscr{I}_X(s) = 0$, while a general hyperplane section of X is contained in one surface of degree s, and by a computation with Chern classes we see that $\deg X = s^2 - s + 2$.

The technique used in Mezzetti's proof is based on focal linear systems and is strictly related to the characteristic 0. In this section we will analyze what happens in the case that the base field has positive characteristic, continuing to follow Gruson's and Peskine's idea. We will also see that this technique provides another proof of Mezzetti's result.

Let $X \subset \mathbb{P}^4$ be an integral surface of degree d. Let $Y = X \cap H$ be the generic hyperplane section of X and let $Z = Y \cap K$ be the generic plane section of Y. Let $\mathscr{I}_X$ be the ideal sheaf of X in $\mathscr{O}_{\mathbb{P}^4}$, $\mathscr{I}_Y$ the ideal sheaf of Y in $\mathscr{O}_H$, with $H \cong \mathbb{P}^3$, and $\mathscr{I}_Z$ the ideal sheaf of Z in $\mathscr{O}_K$, with $K \cong \mathbb{P}^2$. Let us consider for any $s \in \mathbb{N}$ the following maps:

$$\pi_s \colon H^0\mathscr{I}_X(s) \to H^0\mathscr{I}_Y(s) \quad \text{and} \quad \phi_s \colon H^1\mathscr{I}_X(s-1) \to H^1\mathscr{I}_X(s)$$

obtained by the cohomology associated to the exact sequence:

$$0 \to \mathscr{I}_X(s-1) \to \mathscr{I}_X(s) \to \mathscr{I}_Y(s) \to 0.$$

A *sporadic zero* of degree s is an element $\alpha \in \operatorname{coker}(\pi_s) = \ker(\phi_s)$.

Definition 4.3. *The* order *of a sporadic zero α is the maximum integer m such that $\alpha = \beta \cdot H^m$, for some $\beta \in H^1\mathscr{I}_X(s-m-1)$, i.e. such that α is in the image of the map $H^1\mathscr{I}_X(s-m-1) \to \mathscr{I}_X(s-1)$ induced by the injective morphism $\mathscr{I}_X(s-m-1) \to \mathscr{I}_X(s-1)$ defined by the multiplication for H^m.*

Theorem 4.4. *Let α be a sporadic zero of degree s and order m and let $p < s$. Let p^n be such that $p^n \leq m+1$ and $p^{n+1} > m+1$. Suppose that $h^0\mathscr{I}_X(s) = 0$. Then:*

1. *if $s \geq 2m+3$, we have $d \leq s^2 - s + p^n + 1$;*
2. *if $s \leq 2m+2$, we have $d \leq s^2$.*

A sporadic zero of degree s corresponds to a surface containing the generic hyperplane section Y that cannot be lifted to a hypersurface of the same degree containing X. We need to introduce the concept of sporadic zero, because, as in the case of curves, we will have some power of p, for which we need some bound. That bound is provided by the sporadic zero. The first step of the proof lies in the following result:

Proposition 4.5 ([4]). *Let α be a sporadic zero of degree s and let $h^0\mathscr{I}_X(s) = 0$. Then one of the following conditions holds:*

1. $\deg X \leq s^2 - s + 1$;
2. *$h^0\mathscr{I}_Y(s) = 1$ and $h^0\mathscr{I}_Z(s) = 2$.*

This implies that we can suppose that $h^0\mathscr{I}_Y(s) = 1$ and $h^0\mathscr{I}_Z(s) = 2$. In particular, if $s \leq 2m+2$, we get the conclusion. So we suppose that $s \geq 2m+3$ and we also see that the surface R of degree s containing Y that can not be lifted to a hypersurface of degree s containing X is integral. Let $I_R = (f)$ in H be the ideal of R.

It is possible to prove (see [4, Lemma 4.1]) that there exist $r \in \mathbb{N}$ with $p^r \leq m+1$ and $f_i \in H^0\mathscr{O}_H(s)$ for $i = 0, \dots, 4$ such that the subscheme of H associate to the ideal $(f, {x_i}^{p^r} f_j - {x_j}^{p^r} f_i|_H, i, j = 0, \dots, 4)$ is a 1-dimensional scheme E, which can have isolated or embedded 0-dimensional schemes, such that $Y \subset E \subset R$. Moreover, there exists a reflexive sheaf $\mathscr{N}$ of rank 3 such that we have the exact sequence:

$$0 \to \mathscr{N} \to F^{r\star}\Omega_H(p^r) \to \mathscr{I}_{E|R}(s) \to 0, \tag{3}$$

being $\mathscr{I}_{E|R} \subset \mathscr{O}_R$ the ideal sheaf of E. We want to prove that $d \leq s^2 - s + 1 + p^r$.

Note that $c_1(\mathscr{N}) = -p^r - s$ and

$$c_2(\mathscr{N}) = s^2 + p^r s + p^{2r} - \deg E \leq s^2 + p^r s + p^{2r} - \deg X. \tag{4}$$

As in the case of the lifting problem for curves in $\mathbb{P}^3$ the solution to the problem lies in the second Chern class of a sheaf, that in this case is just reflexive.

By [24, Proposition 1] and [13, Theorem 3.2] (see also Langer's remark in [13] after Corollary 6.3) the Bogomolov inequality holds also in positive characteristic for semistable reflexive sheaves in $\mathbb{P}^n$. So we see that if $\mathscr{N}$ is semistable, by the Bogomolov inequality and by the fact that $\deg E \geq \deg Y = \deg X$ we get the statement. So we can suppose that $\mathscr{N}$ is unstable. To get a contradiction we need to restrict the sequence (3) to a generic plane and so we need some further conditions on the generic plane section Z. The difference with the proof in the case of curves, that we saw in the previous section, is in the rank of the sheaf $\mathscr{N}$, which now is 3. So, while previously the proof followed quite easily, now we will see that we require a careful study of the scheme E in order to get the contradiction we are looking for.

Since the Hilbert function of X is of decreasing type, an easy computation shows that we can suppose that $\Delta H_Z(s+i) = s - i - 1$ for any $i \leq p^r$. Given $g \in H^0\mathscr{O}_K(s)$ such that $f|_K$ and g are generators of I_Z in degree s, by [15, Proposition 1.4] we see that $f|_K$ and g are the only generators of I_Z in degree $\leq s + p^r$. By this remark we will get a contradiction.

Restricting (3) to K we get:

$$0 \to \mathscr{N}|_K \to F^{r\star}\Omega_K(p^r) \oplus \mathscr{O}_K \to \mathscr{I}_{E\cap K|R\cap K}(s) \to 0, \tag{5}$$

where $\mathscr{I}_{E\cap K|R\cap K} \subset \mathscr{O}_{R\cap K}$ is the ideal sheaf of $E \cap K$ in $R \cap K$. Since $\mathscr{N}$ is unstable of rank 3, $F^{r\star}\Omega_H(p^r)$ is stable and $c_1(F^{r\star}\Omega_H(p^r)) = -p^r < 0$, the maximal destabilizing subsheaf $\mathscr{F}$ of $\mathscr{N}$ has rank at most 2 and $c_1(\mathscr{F}) < 0$. By [16, Theorem 3.1] we see that $\mathscr{F}|_K$ is still semistable and so it must be $h^0\mathscr{N}|_K = 0$. By (5) we see that $h^0\mathscr{I}_{E\cap K|R\cap K}(s) \geq 1$, which implies that $h^0\mathscr{I}_{E\cap K}(s) \geq 2$ and, since $E \cap K \supseteq Z$ and $h^0\mathscr{I}_Z(s) = 2$, we get that

$h^0\mathscr{I}_{E\cap K}(s)=2$. Since $R\cap K$ is integral of degree s and $R\cap K\supset E\cap K$, we see that $\deg(E\cap K)\le s^2$.

Recall that for any $i,j=0,\dots,4$:

$$x_i{}^{p^r}f_j-x_j{}^{p^r}f_i|_H\in H^0\mathscr{I}_E(s+p^r)\Rightarrow x_i{}^{p^r}f_j-x_j{}^{p^r}f_i|_K\in H^0\mathscr{I}_Z(s+p^r)$$

where $p^r\le m+1$. By the assumption that $f|_K$ and g generate I_Z in degree $\le s+p^r$ we can say that:

$$x_i{}^{p^r}f_j-x_j{}^{p^r}f_i|_K=h_{ij}f|_K+l_{ij}g,$$

for some $h_{ij},l_{ij}\in H^0\mathscr{O}_K(p^r)$. So:

$$E\cap K=V\left(f|_K,l_{ij}g\mid i,j=0,\dots,4\right). \tag{6}$$

So $E\cap K$ contains the complete intersection of two curves of degree s $V(f|_K,g)$, but we have seen that $\deg(E\cap K)\le s^2$. This implies that $E\cap K$ is the complete intersection $V(f|_K,g)$ and so $\mathscr{I}_{E\cap K|R\cap K}\cong\mathscr{O}_{R\cap K}(-s)$. So by (5) we have:

$$0\to\mathscr{N}|_K\to F^{r\star}\Omega_K(p^r)\oplus\mathscr{O}_K\to\mathscr{O}_{R\cap K}\to 0. \tag{7}$$

By the fact that $h^0\mathscr{N}|_K=0$, that $R\cap K$ is integral and by the commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & 0 \arrow[r] \arrow[d] & \mathscr{O}_K \arrow[r] \arrow[d] & \mathscr{O}_K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{N}|_K \arrow[r] \arrow[d] & F^{r\star}\Omega_K(p^r)\oplus\mathscr{O}_K \arrow[r] \arrow[d] & \mathscr{O}_{R\cap K} \arrow[r] & 0 \\
 & \mathscr{N}|_K \arrow[r] \arrow[d] & F^{r\star}\Omega_K(p^r) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

we get the exact sequence:

$$0\to\mathscr{O}_K(-s)\to\mathscr{N}|_K\to F^{r\star}\Omega_K(p^r)\to 0. \tag{8}$$

By the exact sequence:

$$0\to F^{r\star}\Omega_K(p^r)\to\mathscr{O}_K^{\oplus 3}\to\mathscr{O}_K(p^r)\to 0$$

and by the fact that $p^r \leq m+1 < \frac{s}{2}$ we see that $\mathrm{Ext}^1(F^{r\star}\Omega_K(p^r), \mathscr{O}_K(-s)) = 0$ and so $\mathscr{N}|_K \cong F^{r\star}\Omega_K(p^r) \oplus \mathscr{O}_K(-s)$. Since $F^{r\star}\Omega_K(p^r)$ is stable and:

$$\mu(F^{r\star}\Omega_K(p^r)) = -\frac{p^r}{2} > \mu(\mathscr{O}_K(-s)) = -s,$$

we see that the maximal destabilizing subsheaf of $\mathscr{N}|_K$ is $F^{r\star}\Omega_K(p^r)$. So, since $\mathscr{N}$ is unstable of rank 3, by [16, Theorem 3.1] the maximal destabilizing subsheaf of $\mathscr{N}$ must be a reflexive sheaf $\mathscr{F}$ of rank 2 such that:

$$\mathscr{F}|_K \cong F^{r\star}\Omega_K(p^r). \tag{9}$$

So, being $\mathscr{F}$ the maximal destabilizing sheaf of $\mathscr{N}$, we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{N} \arrow[r] \arrow[d] & F^{r\star}\Omega_H(p^r) \arrow[r] \arrow[d] & \mathscr{I}_{E|R}(s) \arrow[r] \arrow[d] & 0 \\
 & \mathscr{I}_T(-s) \arrow[d] & \mathscr{Q} \arrow[d] & \mathscr{I}_{E|R}(s) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $\mathscr{I}_T$ is the ideal sheaf in $\mathscr{O}_H$ of a zero-dimensional scheme T and $\mathscr{Q}$ is a rank 1 sheaf such that $c_1(\mathscr{Q}) = 0$. Since $Q|_K \cong \mathscr{O}_K$, $\mathscr{Q}$ must be torsion free and so $\mathscr{Q} = \mathscr{I}_W$ for some zero-dimensional scheme W. So we get:

$$0 \to \mathscr{I}_T(-s) \to \mathscr{I}_W \to \mathscr{I}_{E|R}(s) \to 0,$$

by which we get that $W \neq \emptyset$, because $h^0\mathscr{I}_Y(s) = 1$. Moreover:

$$h^1\mathscr{I}_E(n) = h^1\mathscr{I}_{E|R}(n) = \deg W - \deg T \tag{10}$$

for any $n < s$ and:

$$h^1\mathscr{I}_E(s) = h^1\mathscr{I}_{E|R}(s) = \deg W - \deg T - 1, \tag{11}$$

because $h^0\mathscr{I}_{E|R}(s) = 0$.

Let $F \subset E$ be the equidimensional component of dimension 1. Then there exists a sheaf $\mathscr{K}$ of finite length determining the following exact sequence:

$$0 \to \mathscr{I}_E \to \mathscr{I}_F \to \mathscr{K} \to 0.$$

Then we see that $h^1\mathscr{I}_E(n) = h^0\mathscr{K}$ for $n \ll 0$, so that by (10) we see that $h^0\mathscr{K} = \deg W - \deg T$. Moreover:

$$h^0\mathscr{I}_E(s) - h^0\mathscr{I}_F(s) + h^0\mathscr{K} - h^1\mathscr{I}_E(s) + h^1\mathscr{I}_F(s) = 0$$

and so, since $Y \subset F \subset E$, $h^0\mathscr{I}_E(s) = h^0\mathscr{I}_F(s) = 1$ and by (11) we get:

$$h^1\mathscr{I}_F(s) = h^1\mathscr{I}_E(s) - h^0\mathscr{K} = -1.$$

This is impossible and so we get a contradiction.

Corollary 4.6. *Let $h^0\mathscr{I}_Y(s) \neq 0$ and let $p < s$. If $\deg X > s^2$, then $h^0\mathscr{I}_X(s) \neq 0$.*

In the following theorem we see that for $p \geq s$ the bound for d is independent of the order of the sporadic zero α and coincides with the bound of the characteristic zero case (see [19] and [18]).

Theorem 4.7. *Let $h^0\mathscr{I}_Y(s) \neq 0$, $h^0\mathscr{I}_X(s) = 0$ and let $p \geq s$. Then $\deg X \leq s^2 - s + 2$.*

Proof. The proof works as in Theorem 4.4. We just need to remark that in the case $p \geq s$ it must be $r = 0$, which means $p^r = 1$. Indeed, again by [4, Lemma 4.1], we get an exact sequence:

$$0 \to \mathscr{I}_X(s - p^r) \to \mathscr{I}_X(s) \to \mathscr{I}_{X\cap H_r|H_r}(s) \to 0,$$

where $\mathscr{I}_{X\cap H_r|H_r} \subset \mathscr{O}_{H_r}$ is the ideal sheaf of $X \cap H_r$. Since $h^0\mathscr{I}_{X\cap H_r|H_r}(s) \neq 0$ and $h^0\mathscr{I}_X(s) = 0$, it must be $h^1\mathscr{I}_X(s - p^r) \neq 0$. By the fact that X is integral we see that it must be $p^r < s$ and so $r = 0$ and $p^r = 1$. □

Now, generalizing Example 4.2 we show that the bounds given in Theorem 4.4 and Theorem 4.7 are sharp.

Example 4.8. Let r, p, $s \in \mathbb{N}$ such that $s \geq 2p^r$. Let us consider $\mathscr{E} = \mathscr{O}_{\mathbb{P}^4}(p^r - 2s) \oplus \mathscr{O}_{\mathbb{P}^4}(-p^r - s)^{\oplus 2}$ and $\mathscr{F} = F^{r\star}\Omega_{\mathbb{P}^4}(p^r - s)$. Then, since $\mathscr{E}^\vee \otimes \mathscr{F}$ is generated by global sections, by [12] the dependency locus of a general momorphism $\varphi \in \mathrm{Hom}(\mathscr{E}, \mathscr{F})$ is a smooth surface $X \subset \mathbb{P}^4$ and it is determined by the sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^4}(p^r - 2s) \oplus \mathscr{O}_{\mathbb{P}^4}^{\oplus 2}(-p^r - s) \to F^{r\star}\Omega_{\mathbb{P}^4}(p^r - s) \to \mathscr{I}_X \to 0. \qquad (12)$$

Together with:
$$0 \to F^{r\star}\Omega_{\mathbb{P}^4}(p^r) \to \mathscr{O}_{\mathbb{P}^4}^5 \to \mathscr{O}_{\mathbb{P}^4}(p^r) \to 0 \tag{13}$$
this implies that $h^1\mathscr{I}_X = 0$, so that $h^0\mathscr{O}_X = 1$ and X is connected and, being smooth, X is integral. Moreover, $h^0\mathscr{I}_X(s) = 0$ and by a computation with Chern classes we see that $\deg X = s^2 - p^r s + 2p^{2r}$.

Let $H \subset \mathbb{P}^4$ be a general hyperplane and let $H_r \subset \mathbb{P}^4$ be the nonreduced hypersurface of degree p^r such that $H_r|_{\text{red}} = H$. Then, $(F^r)^{-1}(H) = H_r$. This shows that we have a commutative diagram:

$$\begin{tikzcd} H_r \arrow[r, "\pi"] \arrow[d, hook, "i"'] & H \arrow[d, hook, "j"] \\ \mathbb{P}^4 \arrow[r, "F^r"'] & \mathbb{P}^4 \end{tikzcd}$$

So we have:
$$i^\star(F^{r\star}\Omega_{\mathbb{P}^4}(p^r)) = i^\star(F^{r\star}(\Omega_{\mathbb{P}^4}(1))) = \pi^\star(j^\star(\Omega_{\mathbb{P}^4}(1))) \cong \pi^\star(\Omega_H(1)) \oplus \mathscr{O}_{H_r}.$$

This implies that $h^0(F^{r\star}\Omega_{\mathbb{P}^4}(p^r)|_{H_r}) \geq 1$. In particular, by (12) we see that $h^0\mathscr{I}_{X\cap H_r|H_r}(s) \neq 0$, so that $h^0\mathscr{I}_Y(s) \neq 0$. Moreover, by (12) and by (13) we see that $h^1\mathscr{I}_X(s-p^r-1) = 0$. This shows that X has a sporadic zero of degree s and order $m = p^r - 1$. So:

1. if $r = 0$ and $s \geq 2$, then $p^r = 1$, $m = 0$ and $\deg X = s^2 - s + 2$;
2. if $s = 2p^r + 1$, then $s = 2m + 3$ and $\deg X = s^2 - \frac{s-1}{2} = s^2 - s + p^r + 1$;
3. if $s = 2p^r$, then $s = 2m + 2$ and $\deg X = s^2$.

This shows that the bounds in Theorem 4.4 and Theorem 4.7 are sharp.

5. Lifting problems in higher dimensions: open problems

In characteristic zero the following results have been proved:

THEOREM 5.1 ([18, Theorem 4.10]). *Let $X \subset \mathbb{P}^5$ be a non degenerate integral variety of dimension* 3 *and degree d. Let s be the minimal degree of a degenerate hyper surface containing a general hyperplane section of X. If $d > s^2 - 2s + 4$ and $s > 5$, then X is contained in a hypersurface of degree s.*

Mezzetti proved this result using the method used for the proof of Theorem 4.1. In that paper she conjectured that, given an integral projective variety X of dimension r and degree d in $\mathbb{P}^{r+2}$ and given the minimal degree s of a hypersurface containing the general hyperplane section of X, X is contained in a hypersurface of degree s if $d > s^2 - (r-1)s + \binom{r}{2} + 1$.

The conjecture comes form Gruson and Peskine's proof of Theorem 2.1. Indeed, as we have seen, it is possible to follow the idea in order to get an exact sequence:

$$0 \to \mathscr{N} \to \Omega_H(1) \to \mathscr{I}_{E|\Gamma}(s) \to 0,$$

where H is the generic hyperplane, Γ is the hypersurface in H of degree s containing $X \cap H$, $\mathscr{N}$ is a reflexive sheaf of rank $r+1$ and $\mathscr{I}_{E|\Gamma} \subset \mathscr{O}_\Gamma$ is the ideal sheaf of a scheme E of dimension $r-1$ containing $X \cap H$. Moreover:

$$c_2(\mathscr{N}(1)) = s^2 - (r-1)s + \binom{r}{2} + 1 - \deg E \leq s^2 - (r-1)s + \binom{r}{2} + 1 - d.$$

The conjecture is proved if one proves that $c_2(\mathscr{N}(1)) \geq 0$.

We need to remark that it is possible to generalize the examples given previously in order to get an integral variety $X \subset \mathbb{P}^{r+2}$ of dimension r and degree $d = s^2 - (r-1)s + \binom{r}{2} + 1$ and such that $h^0 \mathscr{I}_X(s) = 0$, while its generic hyperplane section is contained in precisely one hypersurface of degree s (see [5]). Such a variety is determined by the sequence:

$$0 \to \mathscr{O}^r_{\mathbb{P}^{r+2}}(-1) \oplus \mathscr{O}_{\mathbb{P}^{r+2}}(r-1-s) \to \Omega_{\mathbb{P}^{r+2}}(1) \to \mathscr{I}_X(s) \to 0.$$

Roggero proved the conjecture for $n = 6$ in [20] and in the general case in [23] under some assumption on the generic *plane* section of X. We need to remark that Roggero's proof uses the generic initial ideal, making the proof strictly related to the fact that the characteristic considered is 0.

In positive characteristic, as we saw, following Gruson' and Peskine's idea we get a similar exact sequence:

$$0 \to \mathscr{N} \to F^n\Omega_H(p^n) \to \mathscr{I}_{E|\Gamma}(s) \to 0,$$

where F is the absolute Frobenius. In this case, one would want to prove that $c_2(\mathscr{N}(p^n)) \geq 0$, in order to get that:

$$d \leq s^2 - (r-1)p^n s + \binom{r}{2} p^{2n} + p^{2n}.$$

The problem is that, in general, $\mathscr{N}$ is a rank $r+1$ reflexive sheaf in $H \equiv \mathbb{P}^{r+1}$.

With this procedure, the lifting problem in codimension 2 becomes the problem of determining conditions so that the second Chern class of a rank m reflexive sheaf in $\mathbb{P}^m$ is non negative. Under this perspective the lifting problem in the general case does not seem to depend on the characteristic of the base field. Since unfortunately this has been obtained in the proof of Theorem 4.4 by looking at the general *plane* section Z of X and since Roggero proved the result with some additional conditions on Z, the questions arising are the following: is it possible to prove a result similar to Roggero's in positive characteristic (considering that if $p > s$ we expect the same bound as in characteristic 0)? Is it possible to get some equality on this second Chern class without making the restriction to the plane section? And, obviously, is Mezzetti's conjecture true? Or is it possible to determine a counterexample?

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Author's address:

Paola Bonacini
Dipartimento di Matematica e Informatica
Università di Catania
Viale A. Doria 6
95125, Catania, Italy
E-mail: `bonacini@dmi.unict.it`

Received October 2, 2014
Revised April 8, 2015