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p-Buchsbaum rank 2 bundles on the projective space

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Abstract. *It has been proved by various authors that a normalized, 1-Buchsbaum rank 2 vector bundle on $\mathbb{P}^3$ is a nullcorrelation bundle, while a normalized, 2-Buchsbaum rank 2 vector bundle on $\mathbb{P}^3$ is an instanton bundle of charge 2. We find that the same is not true for 3-Buchsbaum rank 2 vector bundles on $\mathbb{P}^3$, and propose a conjecture regarding the classification of such objects.*

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Introduction

A coherent sheaf E on $\mathbb{P}^3$ is said to be *p-Buchsbaum* if p is the minimal power of the irrelevant ideal which annihilates $H^1_*(E)$. The complete list of p-Buchsbaum rank 2 bundles on $\mathbb{P}^3$ for $p \leq 2$ has been established by several authors, see for example [7, 9, 14, 15, 16]. More precisely, we have the following.

Theorem 1. *Let E be a normalized p-Buchsbaum rank 2 vector bundle on $\mathbb{P}^3$. Then*

- $p = 0$ *if and only if E is direct sum of line bundles;*
- $p = 1$ *if and only if E is a null correlation bundle, i.e. an instanton bundle of charge* 1*;*
- $p = 2$ *if and only if E is an instanton bundle of charge* 2.

After examining this list, two questions natually arise. First, is every rank 2 instanton bundle of charge k on $\mathbb{P}^3$ k-Buchsbaum? Second, since every bundle is p-Buchsbaum for some sufficiently high p, for which values of p can we find a p-Buchsbaum rank 2 bundle which is not instanton?

The goal of this paper is to provide partial answers to these questions. In particular, we show that every rank 2 instanton bundle of charge 3 is 3-Buchsbaum. However, this is false for instantons of higher charge. On the other hand, we show that the generic instanton of charge 4 or 5 is also 3-Buchsbaum. In addition, we provide an explicit example of a 3-Buchsbaum bundle of rank 2 which is not an instanton, and conjecture that every 3-Buchsbaum rank 2 bundle on $\mathbb{P}^3$ is one of these.

1. Preliminaries

In this section we will fix the notation and recall the basic definitions used throughout this paper.

1.1. Buchsbaum sheaves

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let us denote by $S = \mathbb{K}[x_0, x_1, x_2, x_3]$ the ring of polynomials in four variables, so that $\mathbb{P}^3 := \mathrm{Proj}(S)$, and let $\mathfrak{m} = (x_0, x_1, x_2, x_3)$ denote the irrelevant ideal.

Let V be a $\mathbb{K}$-vector space of dimension $m+1$, with V^* denoting its dual. The projective space $\mathbb{P}(V) = \mathbb{P}^m$ is understood as the set of equivalence classes of m-dimensional subspaces of V, or, equivalently, the equivalence classes of the lines of V^*.

Given a coherent sheaf E on $\mathbb{P}^3$, consider the following graded S-module:

$$H^1_*(E) = \bigoplus_{n\in\mathbb{Z}} H^1(E(n)).$$

Definition 1.1. *A coherent sheaf E on $\mathbb{P}^3$ is said to be p-Buchsbaum if and only if p is the minimal power of the irrelevant ideal which annihilates the S-module $H^1_*(E)$, i.e.*

$$p = \min\left\{ t \mid \mathfrak{m}^t H^1_*(E) = 0 \right\}.$$

In this work, we will only consider locally free sheaves on $\mathbb{P}^3$.

1.2. Monads and regularity

Recall that a *monad* on a projective variety X of dimension n is a complex of locally free sheaves on X of the form

$$M_\bullet : A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

such that the map α is injective and the map β is surjective. It follows that $E := \ker\beta/\operatorname{Im}\alpha$ is the only nontrivial cohomology of the complex $M_\bullet$. The coherent sheaf E is called the *cohomology* of $M_\bullet$; it is locally free if and only if the map α is injective in every fiber.

The monad $M_\bullet$ is called a *Horrocks monad* if, in addition:

i) A and C are direct sum of invertible sheaves,

ii) $H^1_*(B) = H^{n-1}_*(B) = 0$.

Furthermore, the monad is also called *minimal* if it satisfies

iii) no direct sum of A is isomorphic to a direct sum of B,

iv) no direct sum of C is the image of a line subbundle of B.

Let us recall the following result on minimal Horrocks monads, cf. [12, Theorem 2.3].

THEOREM 1.2. *Let X be an arithmetically Cohen–Macaulay variety of dimension $n \geq 3$, and let E be a locally free sheaf on X. Then there is a 1-1 correspondence between collections*

$$\{n_1, \dots, n_r, m_1, \dots, m_s\} \text{ with } n_i \in H^1(E^\vee \otimes \omega_X(k_i)) \text{ and } m_j \in H^1(E(-l_j))$$

for integers k_i's and l_j's, and equivalence classes of Horrocks monads of the form

$$M_\bullet : \bigoplus_{i=1}^{r} \omega_X(k_i) \xrightarrow{\alpha} F \xrightarrow{\beta} \bigoplus_{j=1}^{s} \mathcal{O}_X(l_j),$$

whose cohomology is isomorphic to E.

Moreover, the correspondence is such that $M_\bullet$ is minimal if and only if the elements m_j generate $H^1_*(E)$ and the elements n_i generate $H^1_*(E^\vee \otimes \omega_X)$ as modules.

Recall that a coherent sheaf E on $\mathbb{P}^n$ is said to be m-regular in the sense of Castelnuovo–Mumford if $H^i(\mathbb{P}^n, E(m-i)) = 0$ for $i > 0$. Costa and Miró-Roig studied in [3] the Castelnuovo–Mumford regularity of the cohomology of a certain class of monads which include monads of the following form:

$$\mathcal{O}_{\mathbb{P}^3}(-l)^{\oplus k} \xrightarrow{\alpha} \bigoplus_{j=1}^{2+2k} \mathcal{O}_{\mathbb{P}^3}(b_j) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(d)^{\oplus k}, \tag{1}$$

where $l, k, c \geq 1$ and $-l < b_1 \leq \cdots \leq b_{2+2k} < d$. Specializing [3, Theorem 3.2] to monads of the form (1), one obtains the following result.

PROPOSITION 1.3. *If E is the cohomology of a monad of the form (1), then E is m-regular for any integer m such that*

$$m \geq \max\{(k+2)d - (b_1 + \cdots + b_{k+3}) - 2, l\}.$$

1.3. Cohomology of generic instanton bundles

Recall that a bundle E of rank 2 on $\mathbb{P}^3$ is called an *instanton bundle* if it is isomorphic to the cohomology of a monad of the following form:

$$\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus k} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2+2k} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus k} \tag{2}$$

The integer k is called the *charge* of E; notice that $c_1(E) = 0$ and $c_2(E) = k$. Note also that nullcorrelation bundles are precisely instanton bundles of charge 1.

Alternatively, an instanton bundle can also be defined as a bundle E on $\mathbb{P}^3$ with $c_1(E) = 0$ and satisfying the following cohomological conditions:

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$

The Hilbert polynomial of an instanton bundle is given by

$$\begin{aligned} P_E(t) &= 2(k+1)\chi(\mathcal{O}_{\mathbb{P}^3}(t)) - k\chi(\mathcal{O}_{\mathbb{P}^3}(t-1)) - k\chi(\mathcal{O}_{\mathbb{P}^3}(t+1)) \\ &= \frac{1}{3}(t+2)((t+3)(t+1) - 3k) \\ &= \frac{1}{3}(t+2)(t+2+\sqrt{3k+1})(t+2-\sqrt{3k+1}). \end{aligned} \tag{3}$$

Note also that $P_E(t) = h^0(E(t)) - h^1(E(t))$ for $t \geq -2$.

On another direction, recall that a coherent sheaf F on $\mathbb{P}^3$ is said to have *natural cohomology* if for each $t \in \mathbb{Z}$, at most one of the cohomology groups $H^p(F(t))$, where $p = 0, \dots, 3$, is nonzero; every torsion free coherent sheaf with natural cohomology is in fact locally free [10, Lemma 1.1]. In addition, every rank 2 locally free sheaf with $c_1 = 0$, $c_2 > 0$ and natural cohomology is an instanton bundle [10, p. 365].

Hartshorne and Hirschowitz have shown in [10] that the generic instanton bundle has natural cohomology. More precisely, let $\mathcal{I}(k)$ denote the moduli space of rank 2 locally free instanton sheaves of charge k; this is known to be an affine [4], irreducible [18, 19], nonsingular variety of dimension $8k-3$ [13]. Let $\mathcal{N}(k)$ denote the subset of $\mathcal{I}(k)$ consisting of instanton bundles with natural cohomology; it is easy to see that $\mathcal{N}(k)$ is open within $\mathcal{I}(k)$, and [10, Theorem 0.1 (a)] tells us that it is nonempty.

More recently, Eisenbud and Schreyer have introduced the notion of *supernatural bundles*, see [6, p. 862]: a locally free sheaf on $\mathbb{P}^3$ is called supernatural if it has natural cohomology and its Hilbert polynomial has distinct integral roots. Therefore we see that there exists a rank 2 supernatural bundle with $c_1 = 0$ and $c_2 = k > 0$ if and only if $3k+1$ is a perfect square; the first three possible values for k are $k = 1, 5, 8$.

2. Instanton vs Buchsbaum

We start by introducing the following function on the positive integers

$$m(k) = \left\lfloor \sqrt{3k+1} - 2 \right\rfloor ,$$

where $\lfloor\cdot\rfloor$ denotes the largest positive integer which is smaller than or equal to the argument.

Proposition 2.1. *A rank* 2 *instanton bundle* E *is* p*-Buchsbaum if and only if* $h^1(E(p-2)) \neq 0$ *and* $h^1(E(p-1)) = 0$*. In addition, every rank* 2 *instanton bundle of charge* k *is* p*-Buchsbaum for some* $m(k)+2 \leq p \leq k$*.*

Proof. By Theorem 1.2, we get that $H^1_*(E)$ is generated in $H^1(E(-1))$. Thus if $h^1(E(p-2)) \neq 0$ and $h^1(E(p-1)) = 0$ (and hence $h^1(E(t)) = 0$ for every $t \geq p-1$), then $H^1_*(E)$ must be p-Buchsbaum. Conversely, if E is p-Buchsbaum, then $h^1(E(p-2)) \neq 0$ (otherwise, $H^1_*(E)$ would be annihilated by the $(p-1)$-th power of the irrelevant ideal) and $h^1(E(p-1)) = 0$.

By Proposition 1.3, we have that E is k-regular (cf. also [3, Corollary 3.3]). Hence $H^1(E(k-1)) = 0$, and it follows that every rank 2 instanton bundle is at most k-Buchsbaum.

Finally, note from (3) that for $-1 \leq t \leq m(k)$ we have $\chi(E(t)) < 0$. Since $h^3(E(t)) = 0$ in this range, it follows that $h^1(E(t)) \neq 0$ for $t = m(k)$. Thus every rank 2 instanton bundle is at least $(m(k)+2)$-Buchsbaum. □

Since $m(3)+2 = 3$, the first immediate consequence of the previous Proposition is given by the following Corollary.

Corollary 2.2. *Every rank* 2 *instanton bundle of charge* 3 *is* 3*-Buchsbaum.*

However, it is not true that every rank 2 instanton bundle of charge 3 has natural cohomology, as observed in [10, Example 1.6.1]. Indeed, recall that an instanton bundle E is called a *'t Hooft instanton* if $h^0(E(1)) \neq 0$, cf. [1]; more formally, consider the set

$$\mathcal{H}(k) := \{ E \in \mathcal{I}(k) \mid h^0(E(1)) \neq 0 \} ,$$

which is known to be a locally closed subvariety of $\mathcal{I}(k)$ of dimension $5k+4$, irreducible and rational [1, Theorem 2.5]. On the other hand, let $\mathcal{U}(k) := \mathcal{I}(k) \setminus \mathcal{N}(k)$, the subvariety of "unnatural" instanton bundles.

Lemma 2.3. *For every* $k \geq 3$*, we have* $\mathcal{H}(k) \subset \mathcal{U}(k)$*, while* $\mathcal{H}(3) = \mathcal{U}(3)$*.*

Proof. If E is a rank 2 instanton bundle of charge $k \geq 3$, then $h^1(E(1)) \neq 0$ (because $\chi(E(-1)) < 0$). Hence if E is a 't Hooft instanton, then it does not have natural cohomology, showing that $\mathcal{H}(k) \subset \mathcal{U}(k)$.

Conversely, let now E be a rank 2 instanton bundle of charge 3 which does not have natural cohomology. We then know that

(i) $h^0(E(t)) = 0$ for $t \leq 0$;

(ii) $h^1(E(t)) = 0$ for $t \neq -1, 0, 1$;

(iii) $h^2(E(t)) = 0$ for $t \neq -5, -4, -3$;

(iv) $h^3(E(t)) = 0$ for $t \geq -4$.

The last two claims are obtained by Serre duality and the fact $E \simeq E^*$. Therefore the only way in which E may fail to have natural cohomology is if $h^0(E(1)) = h^3(E(-5)) \neq 0$. It follows that $\mathcal{U}(3) \subset \mathcal{H}(3)$. □

It would be interesting to determine properties of the $\mathcal{U}(k)$ for $k \geq 4$, particularly its dimension and number of irreducible components. The previous lemma tells us that $\dim \mathcal{U}(k) \geq 5k + 4$.

Another immediate consequence of Proposition 2.1 is the following.

COROLLARY 2.4. *The generic rank* 2 *instanton bundle of charge* k *is* $(m(k)+2)$*-Buchsbaum.*

In particular, since $m(4) + 2 = m(5) + 2 = 3$, the generic rank 2 instanton bundle of charges 4 and 5 are 3-Buchsbaum, while instanton bundles of charge $k \geq 6$ are at least 4-Buchsbaum.

3. A 3-Buchsbaum rank 2 bundle with $c_1 = -1$

Theorem 1 tells us, in particular, that the first Chern class of every 1- and 2-Buchsbaum rank 2 bundle on $\mathbb{P}^3$ is zero. In this section, we show that the same is not true for p-Buchsbaum bundles with $p \geq 3$, providing an example of a 3-Buchsbaum rank 2 bundle with $c_1 = -1$.

Indeed, consider the monad

$$\mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \quad , \tag{4}$$

which is the simplest example of a class of monads originally introduced by Ein in [5, eq. 3.1.A]. The existence of such monads can be easily established; consider for instance the following explicit maps

$$\alpha = \begin{pmatrix} -z^2 \\ -w^2 \\ x \\ y \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} x \\ y \\ z^2 \\ w^2 \end{pmatrix}$$

where $[x : y : z : w]$ are homogeneous coordinates on $\mathbb{P}^3$.

Let F denote the locally free cohomology of a monad of the form (4); it is a rank 2 bundle with $c_1(F) = -1$ and $c_2(F) = 2$. Ein claims in [5, p. 21], without proof, that F is μ-stable. For the sake of completeness, we include a proof below.

LEMMA 3.1. *Every locally free sheaf F obtained as the cohomology of a monad of the form (4) is μ-stable.*

Proof. First consider the kernel bundle $K := \ker\beta$ defined by the sequence

$$0 \to K \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1).$$

It follows from [2, Theorem 2.7] that K is μ-semistable (but not μ-stable). Therefore, since $\mu(K) = -1$, we must have $h^0(K) = 0$. Now, from the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} K \to F \to 0$$

we have that $h^0(F) = 0$, which implies that F is μ-stable. $\square$

We now show that the bundles considered in this Section are 3-Buchsbaum.

PROPOSITION 3.2. *Every locally free sheaf F obtained as cohomology of a monad of the form (4) is 3-Buchsbaum.*

Proof. By Theorem 1.2, we get that $H^1_*(F)$ is generated in $H^1(F(-1))$. On the other hand, Proposition 1.3 tells us that F is 3-regular, thus $h^1(F(2)) = 0$.

If we also had $h^1(F(1)) = 0$, F would be 2-Buchsbaum, which, by Theorem 1 cannot happen. Therefore F must be 3-Buchsbaum. $\square$

Note also that, since $h^0(F(1)) = h^1(F(1)) = 1$ [11, 2.2], such bundles do not have natural cohomology.

Based on the evidence here presented and also motivated by results due to Roggero and Valabrega in [17], specially Propositions 5 and 6 and Theorem 2 there, we propose the following classification of 3-Buchsbaum rank 2 bundles on $\mathbb{P}^3$.

CONJECTURE 3.3. *Every normalized, 3-Buchsbaum rank 2 bundle on $\mathbb{P}^3$ is either an instanton bundle of charge 3, 4 or 5, if $c_1 = 0$, or the cohomology of a monad of the form (4), if $c_1 = -1$.*

Finally, let us comment on p-Buchsbaum rank 2 bundles on $\mathbb{P}^3$ for $p \geq 4$. An interesting, possible source of examples of such bundles is provided by Ein's *generalized nullcorrelation bundles*, described in [5]. These are bundles obtained as cohomologies of monads of the following two types:

$$\mathcal{O}_{\mathbb{P}^3}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(d) \quad , \tag{5}$$

and

$$\mathcal{O}_{\mathbb{P}^3}(-d-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-b-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-a-1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(d) \ , \tag{6}$$

where $d > b \geq a \geq 0$. Let us denote the cohomology of such monads by $E_{a,b,d}$ and $F_{a,b,d}$, respectively.

Note that, by Theorem 1.2 and Proposition 1.3, $H^1_*(E_{a,b,d})$ is generated in degree $-d$, and that $E_{a,b,d}$ is $(3d-2)$-regular when $d \geq 1$. Therefore, such bundles are at most $(4d-3)$-Buchsbaum, being precisely $(4d-3)$-Buchsbaum provided $h^1(E_{a,b,d}(3d-4)) \neq 0$.

Similarly, note that $H^1_*(F_{a,b,d})$ is generated in degree $-d$, and that $F_{a,b,d}$ is $3d$-regular. Therefore, such bundles are at most $(4d-1)$ – Buchsbaum, being precisely $(4d-1)$ –Buchsbaum provided $h^1(F_{a,b,d}(3d-2)) \neq 0$.

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