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On Grothendieck's counterexample to the Generalized Hodge Conjecture

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Abstract. *For a smooth complex projective variety X, let N^p and F^p denote respectively the coniveau filtration on $H^i(X,\mathbb{Q})$ and the Hodge filtration on $H^i(X,\mathbb{C})$. Hodge proved that $N^pH^i(X,\mathbb{Q}) \subset F^pH^i(X,\mathbb{C}) \cap H^i(X,\mathbb{Q})$, and conjectured that equality holds. Grothendieck exhibited a threefold X for which the dimensions of $N^1H^3(X,\mathbb{Q})$ and $F^1H^3(X,\mathbb{C}) \cap H^3(X,\mathbb{Q})$ differ by one. Recently the point of view of Hodge was somewhat refined (Portelli, 2014), and we aimed to use this refinement to revisit Grothendieck's example. We explicitly compute the classes in this second space which are not in $N^1H^3(X,\mathbb{Q})$. We also get a complete clarification that the representation of the homology customarily used for complex tori does not allow to apply the methods of (Portelli, 2014) to give a different proof of $N^1H^3(X,\mathbb{Q}) \subsetneq F^1H^3(X,\mathbb{C}) \cap H^3(X,\mathbb{Q})$.*

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1. Introduction

First of all, let us quickly recall the Generalized Hodge Conjecture.

Let X be a projective n-dimensional variety over $\mathbb{C}$, smooth and connected. To understand the algebraic geometry of X it is certainly of great interest the knowledge of the cohomology classes $\xi \in H^i(X,\mathbb{Q})$ for which there exists an algebraic subvariety $Y \subset X$ such that the image of ξ in the map $H^i(X,\mathbb{Q}) \to H^i(X-Y,\mathbb{Q})$ induced by the inclusion $X - Y \subset X$, is zero. We will say that ξ is *supported* by Y, or that Y is a *support* for ξ.

For any fixed integer $p \geq 0$ we can then consider the subspace of $H^i(X,\mathbb{Q})$

$$N^pH^i(X,\mathbb{Q}) := \sum_{\substack{Y \subset X \ Zariski\ closed \\ codim\, Y \geq p}} Ker\big(H^i(X,\mathbb{Q}) \to H^i(X-Y,\mathbb{Q})\big) .$$

When p varies the $N^pH^i(X,\mathbb{Q})$ form a decreasing filtration of $H^i(X,\mathbb{Q})$, the so called coniveau filtration. Working with homology instead of cohomology, this filtration was introduced by Hodge ([3, p. 213]). He always deal with single classes instead of spaces [1].

In particular, Hodge gave the following necessary condition for a class $\xi \in H^i(X,\mathbb{Q})$ to be supported by an algebraic subvariety $Y \subset X$, of codimension $\geq p$ (for details and the proof, see § 2, Proposition 2.1). Assume that a singular $(2n-i)$-cycle Γ is homologous to a $(2n-i)$-cycle, contained into a subvariety Y as above. Let α be any *closed* $(2n-i)$-form on X, which contains in any of its local expressions at least $n-p+1$ dz's. Then $\int_\Gamma \alpha = 0$ (under suitable smoothness conditions on Γ). The translation between homology and cohomology is made by means of the Poincaré Duality, and it is implicit in the above statement that $[\Gamma]$ and ξ are Poincaré duals each other.

As we will see, in modern terms this amounts to say that $N^pH^i(X,\mathbb{Q})$ is contained into $F^pH^i(X,\mathbb{C})$, where $F^pH^i(X,\mathbb{C}) = \oplus_{a\geq p} H^{a\,,\,i-a}(X)$ is a space of the Hodge filtration on $H^i(X,\mathbb{C})$. Therefore we can conclude

$$N^pH^i(X,\mathbb{Q}) \subset F^pH^i(X,\mathbb{C}) \cap H^i(X,\mathbb{Q}) \; . \tag{1}$$

After this, Hodge raised a "problem" ([3, p. 214]; see also [4], where these contents of [3] have been presented to a large audience) of whether the above inclusion is, actually, an equality. Over the years this problem has become known as the Generalized Hodge Conjecture (from now on GHC, for short). If $i = 2p$ in (1), then the conjectured equality is the ordinary Hodge Conjecture.

Twenty eight years after [3], Grothendieck exhibited in [2] a particular abelian threefold X for which (1) is a strict inclusion, thus answering Hodge's question in the negative.

However, Grothendieck also showed that it is possible to correct the GHC simply by asking whether $N^pH^i(X,\mathbb{Q})$ (instead of being equal to $F^pH^i(X,\mathbb{C}) \cap H^i(X,\mathbb{Q})$) is the *maximal* rational sub-Hodge structure of $H^i(X,\mathbb{Q})$ which is contained into $F^pH^i(X,\mathbb{C})$. Of course, the abelian threefold X satisfies this amended GHC.

Let us remark here that, although Grothendieck also gives a variant of his amended GHC valid for a single class [2], he puts the major emphasis in a direct comparison between the *spaces* at the left and right hand sides of (1).

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[1] However, it seems to us that his explicit concern to allow Y reducible (loc. cit.) is an indication that he was aware of the fact that these classes could be added together to obtain still classes of the same type.

[2] He wrote in [2, p. 300], "... *, an element of* $H^i(X,\mathbb{C})$ *should belong to* $N^pH^i(X,\mathbb{Q})$ (he certainly meant here: to the complexification of this space, N.d.A.) *if and only if all its bihomogeneous components* (i.e. the components with respect to the Hodge decomposition of $H^i(X,\mathbb{C})$, N.d.A.) *belong to the* $\mathbb{C}$*-vector space spanned by the right hand side of* (1)."

The ordinary Hodge Conjecture does not require to be corrected.

Finally, to put the present paper in the right perspective we have to spend a few words about the content of [6], where we assumed a point of view which is close to that of Hodge. The starting point of that paper was the remark that, if we assume from the beginning that Γ is contained into the subvariety Y, then $\int_\Gamma \alpha = 0$ holds true for *any* r-form α on X, *not necessarily closed*, which contains in its local expressions at least $n-p+1$ dz's.

It is not clear to us whether Hodge was aware or not of this fact. On the one hand, he considered (limits of) integrals of a given form on suitably "small" sets, see e.g. [3, pp. 113–115], to express punctual properties of the form. On the other hand, he was looking for a handy criterion to check whether a homology class on X is algebraic or not. And the model for such a criterion was, undoubtely, Lefschetz theorem on $(1,1)$-classes.

However, starting from the above remark, in [6] we proved that, if Γ is a suitable r-cycle such that $\int_\Gamma \alpha = 0$ for *every* r-form containing at least $n-p+1$ dz's, then Γ is contained in an algebraic subvariety Y of X, of codimension $\geq p$.

To prove this, the main ingredients in the proof are the following. First of all, our complex projective variety X can be thought as a smooth, compact *real* algebraic variety, of real dimension $2n$. It is rather well known that these varieties can be triangulated into simplexes which are real-analytic and semi-algebraic. Moreover, it is necessary to use only certain peculiar systems of local holomorphic coordinates, of essentially (complex) algebraic-geometric nature.

All this may give rise to the feeling that, perhaps, it is rather difficult to apply the results of [6] to deal with some concrete case of the GHC.

This paper is a first attempt to make such an application. More precisely, we analized if it is possible to check that the GHC fails for the Grothendieck's example, by an argument exclusively based on homology, in the spirit of [6]. It turns out that the customary representation of the homology classes in the case of complex tori, which we used throughout in the paper, is completely inadequate for this purpose. In a certain sense, this confirms the above feeling.

The content of the paper is as follows.

In the first section we examine in detail the two main steps which lead to the Generalized Hodge Conjecture as amended by Grothendieck, namely Hodge's necessary condition for a cohomology class to belong to $N^pH^i(X,\mathbb{Q})$ and the translation from homology to cohomology of the whole stuff. Grothendieck's example X is briefly introduced at the end of the section. In the next three sections we undertake a thorough analysis of this example. More precisely, in § 2, the homology and cohomology of X are quickly recalled for the reader convenience, ant to fix notations. In § 3, we determine a basis for the vector space $H^2(X,\mathbb{Q}) \cap H^{1,1}(X)$ over $\mathbb{Q}$. Moreover, for every element of such a basis, we determine a smooth, integral surface in X representing such class. In this way we obtain the surfaces $S_1, S_2, S_3, T_1, T_2, T_3$ of X. The contribution of all these surface to $N^1H^3(X,\mathbb{Q})$ is computed in § 4. In § 5 we compute $F^1H^3(X,\mathbb{C}) \cap H^3(X,\mathbb{Q})$, thus completing the examination of Grothendieck's counterexample to the GHC. In the last section we compare the Poincaré duals of two classes inside $F^1H^3(X,\mathbb{C}) \cap H^3(X,\mathbb{Q})$, only one of which belongs to $N^1H^3(X,\mathbb{Q})$, while the other does not. The representation of the homology classes used in the paper reveals to be completely inefficient to detect the difference between the two.

2. The Generalized Hodge Conjecture, from Hodge to Grothendieck

From now on we will set $r = 2n - i$.

Let us start with Hodge's result, which is the following:

PROPOSITION 2.1. *Let $Y \subset X$ be an algebraic subvariety, of codimension $\geq p$. Consider a class $[\Gamma] \in H_r(Y,\mathbb{Q})$. We can assume without loss of generality that the singular r-cycle Γ is a linear combination of $\mathscr{C}^\infty$ singular r-simplexes. Let α be any closed r-form on X such that every term of the expression of α in any system of local holomorphic coordinates contains at least $n-p+1$ dz's. Then*

$$\int_\Gamma \alpha = 0 . \tag{2}$$

Proof. First of all, notice that the image of $[\Gamma]$ in the canonical map $H_r(Y,\mathbb{Q}) \to H_r(X,\mathbb{Q})$ induced by the inclusion $Y \subset X$ can be represented by the same cycle Γ.

Notice moreover that the form which is actually integrated here is the pull-back of α to the various singular simplexes of Γ. An intermediate step in this pull-back procedure is the pull-back of α to Y_{sm}. But α contains too many dz's to be supported by Y_{sm}, hence

$$\alpha_{|_{Y_{sm}}} \equiv 0$$

and (2) is proved.

A possible doubt here is that some simplex S of Γ can be contained, actually, into the singular locus of Y. But the codimension of $Sing(Y)$ is $\geq p+1$, hence we can use the above argument with $Sing(Y)$ at the place of Y, to conclude that $\int_S \alpha = 0$. □

Remark 2.2. *The argument given above to prove (2) does not use the assumption that α is a closed form. Only the dimension of Y and the type of the form α are relevant.*

To deal with the Generalized Hodge Conjecture, Grothendieck had the idea to translate everything from homology to cohomology. The device for this is the Poincaré duality isomorphism (recall that $i = 2n - r$)

$$PD : H_r(X, \mathbb{C}) \to H^i(X, \mathbb{C}) \ ,$$

which works as follows. Fix $[\Gamma] \in H_r(X, \mathbb{C})$, and let $j : \Gamma \to X$ denote the inclusion (*cum grano salis*, because Γ is a cycle). Then we have a well defined $\mathbb{C}$-linear map

$$\lambda_{[\Gamma]} : H^r(X, \mathbb{C}) \to \mathbb{C} \qquad \text{given by} \qquad [\omega] \mapsto \int_\Gamma j^*\omega \ .$$

Therefore, thanks to the canonical perfect pairing

$$\Psi : H^r(X, \mathbb{C}) \times H^i(X, \mathbb{C}) \to \mathbb{C} \ , \qquad ([\omega], [\omega']) \mapsto \int_X \omega \wedge \omega' \ , \tag{3}$$

there is one and only one $[\xi] \in H^i(X, \mathbb{C})$ such that $\lambda_{[\Gamma]} = \Psi(\_, [\xi])$. In more down-to-earth terms

$$\int_\Gamma j^*\omega = \int_X \omega \wedge \xi \tag{4}$$

for any closed r-form ω.

The class $[\xi] \in H^i(X, \mathbb{C})$ is called the Poincaré dual of $[\Gamma] \in H_r(X, \mathbb{Q})$.

The attentive reader had certainly noticed that, to introduce Poincaré duality as above, one represents cohomology classes à la de Rham, i.e. by mean of closed forms. This forces us to use cohomology with *complex* coefficients. How to deal with *rational* cohomology classes in this set-up?

Recall that, if η is a closed differential s-form on X, then a *period* of η is any complex number of the form

$$\int_\Gamma \eta \ ,$$

where Γ is a s-cycle with *integral* coefficients. Then, η represents a class in $H^s(X, \mathbb{Q})$ ($\subset H^s(X, \mathbb{C})$) if and only if all its periods are in $\mathbb{Q}$ ([8, pp. 34–35]).

With these last preparations, we have at hand everything we need to translate in cohomological terms Hodge's necessary condition.

Given a proper map $f : Z \to X$, where Z is a smooth complex projective variety, equidimensional of dimension $t \le n-p$, Poincaré duality allows us to define the Gysin map $f_* : H^{2t-r}(Z,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ as the composition

$$H^{2t-r}(Z,\mathbb{Q}) \xrightarrow{PD} H_r(Z,\mathbb{Q}) \xrightarrow{f_*=can.} H_r(X,\mathbb{Q}) \xrightarrow{PD} H^i(X,\mathbb{Q}) \ .$$

For this Gysin map we have the *exact* sequence ([1, Coroll. (8.2.8)])

$$H^{2t-r}(Z,\mathbb{Q}) \xrightarrow{f_*} H^i(X,\mathbb{Q}) \longrightarrow H^i(X-f(Z),\mathbb{Q}) \ .$$

Let us remark that $Y := f(Z)$ is Zariski closed inside X because f is proper. Moreover, the map f can be thought of as the composition of a resolution of the singularities $Z \to f(Z)$ of $f(Z) =: Y$ with the inclusion $Y \subset X$. All this shows that the coniveau filtration on $H^i(X,\mathbb{Q})$ is also given by

$$N^pH^i(X,\mathbb{Q}) \ = \sum_{f \ as \ above} Im(f_*) \ .$$

Finally, a simple weight argument shows that (for more details the reader is referred to [6])

$$Im\big(f_* : H^{2t-r}(Z,\mathbb{Q}) \to H^i(X,\mathbb{Q})\big) = PD\Big(Im\big(H_r(Y,\mathbb{Q}) \to H_r(X,\mathbb{Q})\,\big)\Big) \ . \quad (5)$$

Of course, Hodge's was concerned to characterize the classes in $H_r(X,\mathbb{Q})$ which were in the image of some map $H_r(Y,\mathbb{Q}) \to H_r(X,\mathbb{Q})$.

The assumption on α in the statement of Proposition 2.1 can be simply rephrased saying that $[\alpha] \in F^{n-p+1}H^r(X,\mathbb{C})$. The relation (2) then becomes, thanks to (4) ,

$$\int_X \alpha \wedge \xi \ = \ 0$$

for every $[\alpha] \in F^{n-p+1}H^r(X,\mathbb{C})$, i.e.

$$PD([\Gamma]) \ = \ [\xi] \ \in \ \big(F^{n-p+1}H^r(X,\mathbb{C})\,\big)^{\perp} \ ,$$

where the orthogonal subspace is taken with respect to the canonical perfect pairing (3). But it is easily computed that

$$\big(F^{n-p+1}H^r(X,\mathbb{C})\,\big)^{\perp} \ = \ F^pH^i(X,\mathbb{C}) \ . \quad (6)$$

In fact, if $[\omega] \in H^{\,a\,,\,i-a}(X)$ with $a \ge p$, then $n-p+1+a > n$ and therefore we have trivially $[\omega] \wedge F^{n-p+1}H^r(X,\mathbb{C}) \ = \ 0$. On the other hand, if $[\omega] \in$

$H^{a,i-a}(X)$ with $a < p$, then $*\overline{[\omega]} \in F^{n-p+1}H^r(X,\mathbb{C})$ and it is well known that

$$\int_X [\omega] \wedge *\overline{[\omega]} > 0 .$$

Therefore the complete translation into cohomology of Proposition 2.1 amounts to the inclusion (1).

An advantage of the coomological translation is that the Gysin maps f_* *are maps of rational Hodge structures* ([7, 7.3.2]). Hence, for any proper map $f : Z \to X$, where Z is a smooth complex projective variety, equidimensional of dimension $t \leq n-p$, we have that $Im(f_*)$ is a rational sub-Hodge structure of $H^i(X,\mathbb{Q})$. Then, by general facts on the category of (pure) rational Hodge structures, the space $N^pH^i(X,\mathbb{Q})$ is also a rational sub-Hodge structure of $H^i(X,\mathbb{Q})$.

But then, from $N^1 H^3(X,\mathbb{Q}) \subset F^1 H^3(X,\mathbb{C})$ it follows that the Hodge decomposition of $N^1 H^3(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ has necessarily the form

$$N^1 H^3(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = U^{1,2} \oplus U^{2,1} ,$$

for suitable complex subspaces $U^{1,2}$ and $U^{2,1}$ of $H^3(X,\mathbb{C})$. Finally, since we have $\overline{U^{1,2}} = U^{2,1}$, we conclude that *the dimension of* $N^1 H^3(X,\mathbb{Q})$ *is even.*

On the other hand, in Grothendieck's example the right hand side of (1) is odd-dimensional. The example is constructed as follows.

Let E be an elliptic curve over the field of complex numbers. The projective manifolds we are interested in are the abelian threefolds

$$X := E \times E \times E = E^3 .$$

More precisely, X can be defined as follows. Let e_1, e_2, e_3 denote the standard basis of $\mathbb{C}^3$, and let z_1, z_2, z_3 denote the corresponding complex coordinates. Fix a complex number $\tau = u + iv$, where $u, v \in \mathbb{R}$, with $v > 0$. Then $e_1, e_2, e_3, \tau e_1, \tau e_2, \tau e_3$ is an (ordered) basis of a lattice $\Lambda \simeq \mathbb{Z}^6$ contained into $\mathbb{C}^3$, and we set

$$X := \mathbb{C}^3/\Lambda .$$

So everything depends on the choice of τ. To avoid some minor difficulties, we will assume from now on that $[\mathbb{Q}(\tau) : \mathbb{Q}] \geq 3$. Of course, we do not exclude that τ may be a transcendental number, but if τ is algebraic, then its degree is not 2.

3. The topology of X

The homology and cohomology of complex tori is a completely standard topic. Hence this section is just for the reader convenience, and to fix the notations.

We will denote by $u_1, u_2, u_3, u_4, u_5, u_6$ the real coordinates in $\mathbb{C}^3$ with respect to the basis of Λ fixed above, namely

$$z_h \;=\; u_h \;+\; \tau\, u_{h+3} \;, \qquad\qquad h = 1, 2, 3 \;. \tag{7}$$

Concerning the topology of X, let us consider integral homology first.

Let $I = [0,1] \subset \mathbb{R}$, and define maps $\gamma_i : I \to \mathbb{C}^3$ by setting

$$\gamma_i(t) \;=\; t\, e_i \quad \text{for} \quad i = 1,2,3 \qquad \text{and} \qquad \gamma_i(t) = t\,\tau\, e_{i-3} \quad \text{for} \quad i = 4,5,6 \;.$$

If we compose these γ_i with the canonical map $\pi : \mathbb{C}^3 \to X$ we get six singular 1-cycles of X, whose classes are a basis for the free abelian group $H_1(X,\mathbb{Z})$. So inside X there are six copies of S^1, the images of the $\pi \circ \gamma_i$; we will denote them by $C_1, \ldots, C_6$. It is well known that a basis for $H_r(X,\mathbb{Z})$ is then given by the classes of all the r-cycles

$$C_{i_1} \times C_{i_2} \times \ldots \times C_{i_r} \qquad\qquad \text{where} \quad 1 \le i_1 < i_2 < \ldots < i_r \le 6 \;\;. \tag{8}$$

Now we turn to the cohomology with complex coefficients of X.

A basis for $H^r(X,\mathbb{C})$ is given by the classes of the closed r-forms

$$d_{H} \;=\; du_{h_1 h_2 \ldots h_r} \;=\; du_{h_1} \wedge du_{h_2} \wedge \ldots \wedge du_{h_r} \;, \tag{9}$$

where $H = (\, h_1 h_2 \ldots h_r\,)$ is a multi-index, and $1 \le h_1 < h_2 < \ldots < h_r \le 6$. A straightforward computation then shows that

$$\int_{C_{i_1} \times C_{i_2} \times \ldots \times C_{i_r}} du_{h_1} \wedge du_{h_2} \wedge \ldots \wedge du_{h_r} \;=\; \delta^{h_1}_{i_1}\, \delta^{h_2}_{i_2} \,\ldots\, \delta^{h_r}_{i_r} \;, \tag{10}$$

where the δ's are Kronecker's. As remarked in the previous section, then *the classes of the forms* (9) *are also a basis for* $H^r(X,\mathbb{Q})$ *over* $\mathbb{Q}$.

Example 3.1. *For future use, let us show how goes the computation (10), at least in a particular case, for* $r = 3$. *Parametrize* $C_2 \times C_4 \times C_5$ *by first defining*

$$\varphi : [0,1]^3 \to \mathbb{C}^3 \qquad\quad \varphi : (t_1, t_2, t_3) \mapsto t_1\, e_2 \;+\; t_2\, \tau\, e_1 \;+\; t_3\, \tau\, e_2$$

and then composing with the canonical map $\pi : \mathbb{C}^3 \to X$. *Namely we have*

$$u_1 \;=\; 0 \qquad u_2 \;=\; t_1 \qquad u_3 \;=\; 0 \qquad u_4 \;=\; t_2 \qquad u_5 \;=\; t_3 \qquad u_6 \;=\; 0 \;.$$

Actually, we can consider φ *defined in an open neighborhood of* $[0,1]^3$ *inside* $\mathbb{R}^3$, *and therefore*

$$\varphi^* \left(du_2 \wedge du_4 \wedge du_5 \right) \;=\; dt_1 \wedge dt_2 \wedge dt_3 \;,$$

which yields the result.

We turn now to the Hodge decomposition of the spaces $H^r(X,\mathbb{C})$, and to their relations with $H^r(X,\mathbb{Q})$.

For our purposes we have to use also the basis $dz_1, \dots, d\bar{z}_3$ of $H^1(X,\mathbb{C})$. Because of (7), the simple relations between the dz_h, $d\bar{z}_k$ and the du_j are

$$dz_h = du_h + \tau\, du_{h+3} \qquad d\bar{z}_h = du_h + \bar{\tau}\, du_{h+3} \tag{11}$$

and

$$\begin{aligned} du_h &= \left(\tfrac{1}{2} + i\,\tfrac{u}{2v}\right) dz_h + \left(\tfrac{1}{2} - i\,\tfrac{u}{2v}\right) d\bar{z}_h \ , \\ du_{h+3} &= -\tfrac{i}{2v}\, dz_h + \tfrac{i}{2v}\, d\bar{z}_h \end{aligned} \tag{12}$$

for any $h = 1,2,3$.

Finally, we can compute the various classes $PD([C_i \times C_j \times C_k])$ with respect to the basis (9) by means of formula (4). To be precise, assume that $\{i,j,k,l,m,n\} = \{1,2,3,4,5,6\}$, that $1 \le i < j < k \le 6$ and that $1 \le l < m < n \le 6$. Moreover, denote by σ the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ i & j & k & l & m & n \end{pmatrix} .$$

Then it is easily checked that

$$PD([C_i \times C_j \times C_k]) = (-1)^{sign(\sigma)+1}\, du_{_{lmn}} . \tag{13}$$

4. Divisors on X

To test Hodge's and Grothendieck's guesses on X we have to produce elements of $N^1 H^3(X,\mathbb{Q})$. This requires a rather detailed knowledge of the surfaces on X.

PROPOSITION 4.1. *A basis of the $\mathbb{Q}$-module $H^2(X,\mathbb{Q}) \cap H^{1,1}(X)$ is given by the classes*

$$\frac{i}{2v}\, dz_h \wedge d\bar{z}_h = du_h \wedge du_{h+3} \ , \qquad h = 1,2,3 \tag{14}$$

and

$$\frac{i}{2v}\left(dz_h \wedge d\bar{z}_k + dz_k \wedge d\bar{z}_h\right) = du_h \wedge du_{k+3} + du_k \wedge du_{h+3} \ , \qquad h = 1,2,3 . \tag{15}$$

Hence

$$dim_{_\mathbb{Q}}\left(H^2(X,\mathbb{Q}) \cap H^{1,1}(X)\right) = 6 .$$

Proof. Consider the closed 2-form

$$F \;=\; \sum_{1\le h,k\le 3} a_{hk}\, dz_h \wedge d\bar z_k \;=\; \sum_{1\le s<t\le 6} b_{st}\, du_s \wedge du_t \ , \tag{16}$$

where the a_{hk} and b_{st} are all in $\mathbb{C}$. By (10), we have that $[F] \in H^2(X,\mathbb{Q})$ if and only if all the b_{st} are in $\mathbb{Q}$. Murasaki's idea in [5] is to write

$$F \;=\; F_1 + F_2 + F_3 + F_{12} + F_{13} + F_{23} \ , \tag{17}$$

where, for every $h = 1, 2, 3$,

$$F_h \;:=\; a_{hh}\, dz_h \wedge d\bar z_h$$

and for every $1 \le h < k \le 3$,

$$F_{hk} \;:=\; a_{hk}\, dz_h \wedge d\bar z_k + a_{kh}\, dz_k \wedge d\bar z_h \ .$$

LEMMA 4.2. *F represents a rational cohomology class if and only if all the F_h and the F_{hk} represent rational cohomology classes.*

Proof. One direction is obvious, so assume that F represents a rational cohomology class. From (11) we get the relations

$$F_h \;=\; -\,2iv\, a_{hh}\ du_h \wedge du_{h+3}$$

$$\text{and} \tag{18}$$

$$\begin{aligned} F_{hk} \;=\;& (\,a_{hk} - a_{kh}\,)\, du_h \wedge du_k \;+\; (\,a_{hk}\,\bar\tau - a_{kh}\,\tau\,)\, du_h \wedge du_{k+3} \,+ \\ &+ (\,a_{kh}\,\bar\tau - a_{hk}\,\tau\,)\, du_k \wedge du_{h+3} \;+\; (\,a_{hk} - a_{kh}\,)\,\tau\,\bar\tau\, du_{h+3} \wedge du_{k+3} \ . \end{aligned}$$

This shows that each of the six terms in (17) involves different elements of the basis $du_i \wedge du_j$ of $H^2(X,\mathbb{C})$, hence the lemma is completely proved. □

Now, the first equation of (18) yields by (16) that

$$-\,2iv\, a_{hh} \;=\; b_{h,h+3} \ .$$

Therefore $b_{h,h+3} \in \mathbb{Q}$ if and only if

$$a_{hh} \;=\; \frac{i}{2\,v}\, r \qquad \text{where} \quad r \in \mathbb{Q} \ .$$

In other words, all the classes (14) are in $H^2(X,\mathbb{Q}) \cap H^{1,1}(X)$, and they are independent over $\mathbb{Q}$.

Concerning the class F_{hk}, for any fixed $1 \le h < k \le 3$, the second relation in (18) implies that all the following numbers are rational:

$$a_{hk} - a_{kh} \quad , \qquad (a_{hk} - a_{kh})\, \tau\, \bar\tau \quad , \qquad a_{hk}\, \bar\tau - a_{kh}\, \tau \quad , \qquad a_{kh}\, \bar\tau - a_{hk}\, \tau \ .$$

From this we get, in particular

$$(a_{hk} - a_{kh})\,(\tau + \bar\tau) \ \in \ \mathbb{Q} \ .$$

Therefore, *if* $a_{hk} - a_{kh} \neq 0$, *then necessarily* τ *is an algebraic number over* $\mathbb{Q}$, *of degree* $[\mathbb{Q}(\tau) : \mathbb{Q}] \le 2$. We ruled out this possibility at the end of § 2.

Hence if F is rational, then necessarily $a_{hk} = a_{kh} = a$ and

$$F_{hk} \ = \ a\,(\bar\tau - \tau)\,\big(du_h \wedge du_{k+3} + du_k \wedge du_{h+3}\big) \ =$$

$$= \ -2\, i\, a\, v\,\big(du_h \wedge du_{k+3} + du_k \wedge du_{h+3}\big) \ .$$

We get in this way the classes (15) of $H^2(X, \mathbb{Q}) \cap H^{1,1}(X)$, which are linearly independent over $\mathbb{Q}$, and are also independent of the classes (14). □

Let us consider the divisors now, i.e. the surfaces on X.

First of all, the abelian surface $E \times E$ can be embedded in X in a trivial way by setting, for an arbitrary $P \in E$:

$$S_3 \ := \ E \times E \times P \,.$$

The family $\{E \times E \times P\}_{P \in E}$ is a fibration of X. Moreover, if $P, Q \in E$, then $E \times E \times P$ and $E \times E \times Q$ are algebraically equivalent, hence they are homologically equivalent.

To determine the cohomology class of S_3, we remark that as a singular 4-cycle inside X we have $S_3 = C_1 \times C_2 \times C_4 \times C_5$. Then, for any closed 4-form

$$\alpha \ = \sum_{\#I = 4} b_{_I}\, du_{_I} \quad , \qquad b_{_I} \in \mathbb{Q} \quad \text{ for any } \quad I \ ,$$

the relation (10) implies

$$\int_{S_3} \alpha \ = \ \int_{C_1 \times C_2 \times C_4 \times C_5} \alpha \ = \ b_{(1,2,4,5)} \ .$$

Therefore, (4) will be satisfied for every form α if we take

$$\omega_3 \ := \ du_3 \wedge du_6 \ = \ \frac{i}{2\,v}\, dz_3 \wedge d\bar z_3 \ . \tag{19}$$

In other words

$$PD(\,[S_3]\,) \;=\; \frac{i}{2\,v}\,dz_3 \wedge d\bar{z}_3 \;. \tag{20}$$

On X we have also two other obvious families of surfaces, given respectively by

$$S_1 \;:=\; P \times E \times E \quad , \qquad\qquad S_2 \;:=\; E \times P \times E \;.$$

The corresponding cohomology classes, computed as above, are

$$PD(\,[S_1]\,) \;=\; \omega_1 \;:=\; \frac{i}{2\,v}\,dz_1 \wedge d\bar{z}_1 \quad , \qquad PD(\,[S_2]\,) \;=\; \omega_2 \;:=\; \frac{i}{2\,v}\,dz_2 \wedge d\bar{z}_2 \;.$$

To produce divisors not equivalent to the S_i's, we have to embed $E \times E$ inside X by using the diagonal map $\Delta : E \to E \times E$ like in

$$f : E \times E \xhookrightarrow{\Delta \times id_E} E \times E \times E \quad \cdot \tag{21}$$

We will denote by T_3 the image of $E \times E$ in the proper map f. Two other surfaces T_1 and T_2 can be defined inside X as the images of the (proper) map

$$E \times E \xhookrightarrow{id_E \times \Delta} E \times E \times E \quad ,$$

and similarly for T_2 .

We will determine now the cohomology class of the divisor T_3 .

To compute the pull-back of forms it is better to describe the map f in (21) as follows. If $\varepsilon_1, \varepsilon_2$ is the standard basis of $\mathbb{C}^2$, we have the real basis $\varepsilon_1, \varepsilon_2, \tau\,\varepsilon_1, \tau\,\varepsilon_2$ of this space. It generates over the integers a lattice $L \subset \mathbb{C}^2$, and of course $E \times E = \mathbb{C}^2/L$. Moreover, let v_1, v_2, v_3, v_4 denote real coordinates in $\mathbb{C}^2$ with respect to $\varepsilon_1, \varepsilon_2, \tau\,\varepsilon_1, \tau\,\varepsilon_2$. Then the map f is induced by the map $\mathbb{C}^2 \to \mathbb{C}^3$ given in real coordinates by

$$u_1 \,=\, v_1 \ , \quad u_2 \,=\, v_1 \ , \quad u_3 \,=\, v_2 \ , \quad u_4 \,=\, v_3 \ , \quad u_5 \,=\, v_3 \ , \quad u_6 \,=\, v_4 \,. \tag{22}$$

First consequences of these relations are

$$f^*\,(\,du_1 \wedge du_2\,) \;=\; 0 \quad , \qquad\qquad f^*\,(\,du_4 \wedge du_5\,) \;=\; 0 \;.$$

Hence, given any rational closed 4-form $\alpha = b_{1234}\,du_1 \wedge du_2 \wedge du_3 \wedge du_4 + \ldots$ on X, we have

$$f^*\,\alpha \;=\; -\,(\,b_{1346} + b_{1356} + b_{2346} + b_{2356}\,)\,dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4 \;.$$

If we set

$$\eta_3 \;:=\; du_{15} + du_{24} - du_{14} - du_{25} \ , \tag{23}$$

it is easily checked that

$$\eta_3 \wedge \alpha \;=\; -\,(\,b_{1346} + b_{1356} + b_{2346} + b_{2356}\,)\,du_1 \wedge du_2 \wedge \ldots \wedge du_6 \ ,$$

and we can conclude that

$$\int_{T_3} f^*\,\alpha \;=\; \int_X \eta_3 \wedge \alpha$$

for any rational closed 4-form α on X. Namely, we have that the Poincaré dual σ_3 of T_3 is represented by the closed form η_3 , which can also be written as

$$\eta_3 \;=\; \frac{i}{2\,v}\left(dz_1 \wedge d\bar{z}_2 \,+\, dz_2 \wedge d\bar{z}_1 \,-\, dz_1 \wedge d\bar{z}_1 \,-\, dz_2 \wedge d\bar{z}_2 \right) , \tag{24}$$

thanks to (14) and (15).

5. Classes of $N^1H^3(X,\mathbb{Q})$

The purpose of this section is to compute the contribution to $N^1H^3(X,\mathbb{Q})$ of the surfaces $S_1, S_2, S_3, T_1, T_2, T_3$ on X, introduced in the previous section. More concretely, we will prove the

PROPOSITION 5.1. *Consider the abelian threefold* $X = E \times E \times E$, *where* E *is the elliptic curve* $\mathbb{C}/\mathbb{Z} + \tau\,\mathbb{Z}$. *Here we assume that* τ *is a complex number with* $\Im(\tau) > 0$, *and such that* $[\mathbb{Q}(\tau) : \mathbb{Q}] \geq 3$. *Then, for every such threefold* $N^1H^3(X,\mathbb{Q})$ *contains a rational sub-Hodge structure* M *of* $H^3(X,\mathbb{Q})$, *with* $dim_{\mathbb{Q}} M = 16$.

Proof. Let i denote the inclusion $S_1 \subset X$. We start by computing the image of the Gysin map $i_* : H^1(S_1,\mathbb{Q}) \to H^3(X,\mathbb{Q})$. A basis for $H_3(S_1,\mathbb{Q})$ is given by the classes

$$[C_2 \times C_3 \times C_5] \ \ , \quad [C_2 \times C_3 \times C_6] \ \ , \quad [C_2 \times C_5 \times C_6] \ \ , \quad [C_3 \times C_5 \times C_6] \ .$$

They are sent by i_* into the same classes, viewed as elements of $H_3(X,\mathbb{Q})$. Finally, by (13) we conclude that $i_*(\,H^1(S_1,\mathbb{Q})\,)$ is generated by

$$du_{146} \ , \ du_{145} \ , \ du_{134} \ , \ du_{124} \ .$$

Similarly, bases for the images of the Gysin maps for the surfaces S_2 and S_3 are given respectively by

$$du_{256} \ , \ du_{245} \ , \ du_{235} \ , \ du_{125} \qquad \text{and} \qquad du_{356} \ , \ du_{346} \ , \ du_{236} \ , \ du_{136} \ .$$

These twelve 3-forms are distinct elements of the basis (9) of $H^3(X,\mathbb{Q})$; let M_1 denote the subspace they generate.

The contribution of the surfaces T_1, T_2 and T_3 to $N^1H^3(X,\mathbb{Q})$ is not easily determined in homology, so we switch directly to cohomology.

Consider the map f defined in (21) (or (22), in coordinates). The image of f was denoted by T_3; for simplicity, we will still denote by f the inclusion of T_3 into X.

LEMMA 5.2. *Recall that the closed form η_3 given in (23) represents the class $\sigma_3 = PD([T_3])$. The following diagram is then commutative*

$$\begin{tikzcd} H^1(X,\mathbb{Q}) \arrow[rr, "\sigma_3 \cup \_"] \arrow[dr, "f^*"'] & & H^3(X,\mathbb{Q}) \\ & H^1(T_3,\mathbb{Q}) \arrow[ur, "f_*"'] & \end{tikzcd} \quad . \tag{25}$$

Proof. of Lemma 5.2.

In fact, consider the following commutative diagram, where $[X] \in H_6(X,\mathbb{Q})$ and $[T_3] \in H_4(T_3,\mathbb{Q})$ are the fundamental classes of X and T_3 respectively

$$\begin{tikzcd} H^1(X,\mathbb{Q}) \arrow[r, "f^*"] & H^1(T_3,\mathbb{Q}) \arrow[r, "f_*"] \arrow[d, "\_ \cap [T_3]"', "\wr"] & H^3(X,\mathbb{Q}) \arrow[d, "\wr"', "\_ \cap [X]"] \\ & H_3(T_3,\mathbb{Q}) \arrow[r, "f_*"'] & H_3(X,\mathbb{Q}) \end{tikzcd} \; .$$

The classes σ_3 and $[T_3]$ are related by Poincaré duality in the following way

$$f_*[T_3] \;=\; \sigma_3 \,\cap\, [X] \; .$$

Then, for every $x \in H^1(X,\mathbb{Q})$ we have by the "projection formula" and the above relation

$$(\, f_* \, f^* \, x\,) \cap [X] \;=\; f_* (\, f^* \, x \cap [T_3]\,) \;=\; x \cap f_* \, [T_3] \;=$$
$$x \cap (\, \sigma_3 \cap [X]\,) \;=\; (\, x \cup \sigma_3\,) \cap [X] \;=\; (\, \sigma_3 \cup x\,) \cap [X] \; ,$$

where the last equality is true because the degree of σ_3 is 2. Since the Poincaré duality map is an isomorphism, the commutativity of (25) is completely proved. $\square$

Now, the space $Im(\,\sigma_3 \,\cup\, \_\,)$ is generated by the classes of

$$\begin{array}{rcrcl} \eta_3 \wedge du_1 & = & \eta_3 \wedge du_2 & = & du_{124} - du_{125} \in M_1 \\ & & \eta_3 \wedge du_3 & = & \underbrace{du_{134} + du_{235}}_{\in M_1} - du_{135} - du_{234} \; , \end{array}$$

$$\begin{array}{rcrcl} \eta_3 \wedge du_4 & = & \eta_3 \wedge du_5 & = & du_{245} - du_{145} \in M_1 \\ & & \eta_3 \wedge du_6 & = & \underbrace{-du_{256} - du_{146}}_{\in M_1} + du_{246} + du_{156} \; . \end{array}$$

These relations shows that $Im(\sigma_3 \cup \_)$ has dimension four. But this forces f^* to be onto because $dim_{\mathbb{Q}} H^1(T_3, \mathbb{Q}) = 4$, hence $Im(f_*) = Im(\sigma_3 \cup \_)$.

To summarize, the contribution of $Im(f_*)$ to the generation of the space $N^1H^3(X, \mathbb{Q})$ is given by the classes

$$du_{135} + du_{234} \quad , \qquad du_{246} + du_{156} \ .$$

Similar computations can be performed for the surfaces T_1 and T_2, which contribute to the generation of $N^1H^3(X, \mathbb{Q})$ respectively with the classes

$$du_{135}+du_{126} \ , \quad du_{246}+du_{345} \qquad \text{and} \qquad du_{126}-du_{234} \ , \quad du_{156}-du_{345} \ .$$

Finally, denote by M_2 the subspace of $H^3(X, \mathbb{Q})$ generated by the six classes above. It is easily seen that a basis for M_2 is given by

$$du_{126} - du_{234} \ , \qquad du_{156} - du_{345} \ , \qquad du_{246} + du_{345} \ , \qquad du_{135} + du_{234} \tag{26}$$

and that $M_1 \cap M_2 = 0$. Then $M := M_1 \oplus M_2$ is a rational sub-Hodge structure of $H^3(X, \mathbb{Q})$, contained into $N^1H^3(X, \mathbb{Q})$. Since $dim_{\mathbb{Q}} M = 16$, the proof of Proposition 5.1 is complete. □

6. Computation of $F^1H^3(X, \mathbb{C}) \cap H^3(X, \mathbb{Q})$

For this computation we will exploit (6) and the fact that $H^{3,0}(X)$ is isomorphic to $\mathbb{C}$, generated by the class of the following closed form α (which we give also in terms of the base (9))

$$\begin{aligned} dz_1 \wedge dz_2 \wedge dz_3 \quad = \quad & du_{123} + \tau\,(du_{126} - du_{135} + du_{234}) + \\ & + \tau^2\,(du_{156} - du_{246} + du_{345}) + \tau^3\, du_{456} \ . \end{aligned} \tag{27}$$

Then, for an arbitrary $\omega = \sum_{1 \le i < j < k \le 6} r_{ijk}\, du_{ijk}$ where $r_{ijk} \in \mathbb{Q}$ for any i, j, k, we have that

$$\omega \wedge \alpha = \big(r_{123}\, \tau^3 - (\, r_{234} - r_{135} + r_{126}\,)\, \tau^2 + (\, r_{345} - r_{246} + r_{156}\,)\, \tau - r_{456} \big)\, du_{123456} \ .$$

Hence $[\omega]$ is orthogonal to $[\alpha]$ with respect to (3) if and only if

$$r_{123}\, \tau^3 - (\, r_{234} - r_{135} + r_{126}\,)\, \tau^2 + (\, r_{345} - r_{246} + r_{156}\,)\, \tau - r_{456} \ = \ 0 \ . \tag{28}$$

At the end of § 2 we made the assumption $[\mathbb{Q}(\tau) : \mathbb{Q}] \ge 3$. Therefore, if τ is not algebraic over $\mathbb{Q}$, of degree 3, the above relation is satisfied only if all the coefficients in it vanish. Since the linear system

$$\left\{ \begin{array}{rcl} r_{123} & = & 0 \\ r_{234} - r_{135} + r_{126} & = & 0 \\ r_{345} - r_{246} + r_{156} & = & 0 \\ r_{456} & = & 0 \end{array} \right.$$

has rank four, we conclude $dim_{\mathbb{Q}}\left(H^3(X,\mathbb{Q}) \cap F^1 H^3(X,\mathbb{C})\right) = 16$, and we have the following straightforward consequence of Proposition 5.1:

Proposition 6.1. *If* $[\mathbb{Q}(\tau):\mathbb{Q}] > 3$ *(in particular, if* τ *is transcendental over* $\mathbb{Q}$*), then*

$$N^1H^3(X,\mathbb{Q}) = F^1H^3(X,\mathbb{C}) \cap H^3(X,\mathbb{Q}) ,$$

i.e. the Generalized Hodge conjecture is true in its original form for X.

On the other hand, Grothendieck considered the case when τ is algebraic over $\mathbb{Q}$, of degree 3. Let $f := X^3 + \mu_1 X^2 + \mu_2 X + \mu_3$ be the minimal polynomial of τ over $\mathbb{Q}$. Then relation (28) can be rewritten as

$$(r_{234}-r_{135}+r_{126}+\mu_1 r_{123})\,\tau^2 - (r_{345}-r_{246}+r_{156}-\mu_2 r_{123})\,\tau + r_{456}+\mu_3 r_{123} = 0 .$$

Since $[\mathbb{Q}(\tau):\mathbb{Q}] = 3$, we have necessarily

$$\left\{\begin{array}{rcl} r_{234} - r_{135} + r_{126} + \mu_1 r_{123} & = & 0 \\ r_{345} - r_{246} + r_{156} - \mu_2 r_{123} & = & 0 \\ r_{456} + \mu_3 r_{123} & = & 0 \end{array}\right. .$$

This linear system has rank 3, hence

$$dim_{\mathbb{Q}}\left(H^3(X,\mathbb{Q}) \cap F^1 H^3(X,\mathbb{C})\right) = 17 , \qquad (29)$$

and *the Generalized Hodge Conjecture fails in its original form.*

But, since we know a priori that the dimension of $N^1H^3(X,\mathbb{Q})$ is even, (29) forces $N^1H^3(X,\mathbb{Q}) = M$, the space introduced in Proposition 5.1, and this is also the maximal rational sub-Hodge structure of $F^1H^3(X,\mathbb{C})$. Hence, the Generalized Hodge Conjecture as amended by Grothendieck is true for such threefolds X.

7. Final remarks

Let us set $\varphi := d_{123} - \mu_1 d_{234} + \mu_2 d_{345} - \mu_3 d_{456}$. Then the detailed computations performed in the last two sections show that

$$F^1H^3(X,\mathbb{C}) \cap H^3(X,\mathbb{Q}) = \mathbb{Q}[\varphi] \oplus N^1H^3(X,\mathbb{Q}) .$$

We would like to be able to conclude that $[\varphi] \notin N^1H^3(X,\mathbb{Q})$ *by a direct examination of* $[\varphi]$, or, in a spirit close to Hodge's, by a direct examination of the Poincaré dual of $[\varphi]$. If we set

$$\Gamma_1 := C_4{\times}C_5{\times}C_6 \ , \ \Gamma_2 := C_1{\times}C_5{\times}C_6 \ , \ \Gamma_3 := C_1{\times}C_2{\times}C_6 \ , \ \Gamma_4 := C_1{\times}C_2{\times}C_3$$

then $PD([\varphi])$ is represented by the cycle $\Gamma := \Gamma_1 + \mu_1\,\Gamma_2 + \mu_2\,\Gamma_3 + \mu_3\,\Gamma_4$, where the rational numbers μ_i are the coefficients of the minimal polynomial of τ. Now, (10) and (27) together imply

$$\int_{\Gamma_1} \alpha \;=\; \tau^3 \quad , \qquad \int_{\Gamma_2} \alpha \;=\; \tau^2 \quad , \qquad \int_{\Gamma_3} \alpha \;=\; \tau \quad , \qquad \int_{\Gamma_4} \alpha \;=\; 1 \;. \tag{30}$$

Therefore

$$\int_{\Gamma} \alpha \;=\; \tau^3 + \mu_1\tau^2 + \mu_2\tau + \mu_3 \;=\; 0 \;,$$

which simply means that $[\varphi] \in F^1H^3(X,\mathbb{C})$, as we already know.

The integrals (30) imply, in particular, that no one of the 3-cycles $\Gamma_1, \dots, \Gamma_4$ can be contained into an algebraic surface $Y \subset X$.

Consider now the cycles $\Gamma_5 := C_2 \times C_3 \times C_4$ and $\Gamma_6 := C_1 \times C_3 \times C_5$. Then, it is easily checked that

$$\int_{\Gamma_5+\Gamma_6} \alpha \;=\; \int_{\Gamma_5} \alpha \;+\; \int_{\Gamma_6} \alpha \;=\; \tau - \tau \;=\; 0 \;.$$

In particular, the computation shows also that neither of the two cycles Γ_5 and Γ_6 can be contained into an effective divisor on X. But

$$PD(\,[\Gamma_5] + [\Gamma_6]\,) \;=\; [d_{156} + d_{246}] \;\in\; N^1H^3(X,\mathbb{Q}) \;,$$

as we have seen.

Let us add another remark on the above cycles, in the spirit of [6]. Let Γ denote any of the cycles $\Gamma_1, \dots, \Gamma_4, \Gamma_5, \Gamma_6$. Since Γ is smooth, for any point $P \in \Gamma$ we have $T_P\Gamma \subset T_PX$, where X denotes here the differentiable manifold underlying the projective, smooth variety. Moreover, if $J : T_PX \to T_PX$ denotes the complex structure on the *real* vector space T_PX, then the non vanishing of the integral of the form α on Γ implies that

$$T_P\Gamma \;+\; J(T_P\Gamma) \;=\; T_PX \;.$$

This relation can also be easily checked by an easy, direct computation.

To summarize, thanks to the detailed computations performed in §§ 5 and 6, we know that $[d_{156} + d_{246}]$ is supported by an algebraic surface $Y \subset X$, whereas $[\varphi]$ is not.

On the other hand, our aim would be to be able to directly determine this different nature of these two classes, by examining their respective Poincaré duals, to get a direct proof that $N^1H^3(X,\mathbb{Q}) \subsetneq F^1H^3(X,\mathbb{C}) \cap H^3(X,\mathbb{Q})$.

Throughout the paper we used the representation of the homology classes which is customary for complex tori. The above remarks show that this particular representation is completely inadequate to reach our goal.

All this seems to indicate that to apply the results of [6] to the computation of some concrete case of the Generalized Hodge Conjecture, it is necessary to use bases for the homology spaces, which are induced by some suitable real-analytic semi-algebraic triangulation of X.

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