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Stratonovich-Weyl correspondence via Berezin quantization

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Abstract. *Let G be a quasi-Hermitian Lie group and let K be a maximal compactly embedded subgroup of G. Let π be a unitary representation of G which is holomorphically induced from a unitary representation ρ of K. We introduce and study a notion of complex-valued Berezin symbol for an operator acting on the space of π and the corresponding notion of Stratonovich-Weyl correspondence. This generalizes some results already obtained in the case when ρ is a unitary character, see [19]. As an example, we treat in detail the case of the Heisenberg motion groups.*

Keywords: Stratonovich-Weyl correspondence, Berezin quantization, Berezin transform, quasi-Hermitian Lie group, coadjoint orbit, unitary representation, holomorphic representation, reproducing kernel Hilbert space, Heisenberg motion group.
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1. Introduction

There are different ways to extend the usual Weyl correspondence between functions on $\mathbb{R}^{2n}$ and operators on $L^2(\mathbb{R}^n)$ to the general setting of a Lie group acting on a homogeneous space [1, 13, 29]. In this paper, we focuse on Stratonovich-Weyl correspondences. The notion of Stratonovich-Weyl correspondence was introduced in [42] and its systematic study began with the work of J.M. Gracia-Bondìa, J.C. Vàrilly and their co-workers (see [11, 23, 25, 27, 28]). The following definition is taken from [27], see also [28].

Definition 1.1. *Let G be a Lie group and π a unitary representation of G on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G-space and let μ be a (suitably normalized) G-invariant measure on M. Then a Stratonovich-Weyl correspondence for the triple (G, π, M) is an isomorphism W from a vector space of operators on $\mathcal{H}$ to a space of (generalized) functions on M satisfying the following properties:*

1. *W maps the identity operator of $\mathcal{H}$ to the constant function* 1*;*

2. *the function $W(A^*)$ is the complex-conjugate of $W(A)$;*
3. *Covariance: we have $W(\pi(g)\,A\,\pi(g)^{-1})(x) = W(A)(g^{-1}\cdot x)$;*
4. *Traciality: we have*

$$\int_M W(A)(x)W(B)(x)\,d\mu(x) = \mathrm{Tr}(AB).$$

A basic example is the case when G is the $(2n+1)$-dimensional Heisenberg group H_n acting on $\mathbb{R}^{2n}$ by translations and π is a Schrödinger representation of H_n on $L^2(\mathbb{R}^n)$. In this case, the usual Weyl correspondence (see [26]) provides a Stratonovich-Weyl correspondence for the triple $(H_n, \pi, \mathbb{R}^{2n})$ [6, 40, 44].

Stratonovich-Weyl correspondences were constructed for various Lie group representations, in particular for the massive representations of the Poincaré group [23, 27].

In [19], we constructed and studied a Stratonovich-Weyl correspondence for a quasi-Hermitian Lie group G and a unitary representation π of G which is holomorphically induced from a unitary character of a compactly embedded subgroup K of G (see also [15] and [16]). In this case, M is taken to be a coadjoint orbit of G which is associated with π by the Kirillov-Kostant method of orbits [33, 34] and we can consider the Berezin calculus on M [9, 10]. Recall that the Berezin map S is an isomorphism from the Hilbert space of all Hilbert-Schmidt operators on $\mathcal{H}$ (endowed with the Hilbert-Schmidt norm) onto a space of square-integrable functions on a homogeneous complex domain [43]. In this situation, we can apply an idea of [25] (see also [3] and [4]) and construct a Stratonovich-Weyl correspondence for (G, π, M) by taking the isometric part W in the polar decomposition of S, that is, $W := (SS^*)^{-1/2}S$. Note that $B := SS^*$ is the so-called Berezin transform which have been intensively studied by many authors, see in particular [24, 38, 39, 43, 46].

In [19], we also showed that if the Lie algebra $\mathfrak{g}$ of G is reductive then W can be extended to a class of functions which contains $S(d\pi(X))$ for each $X \in \mathfrak{g}$ and that, for each simple ideal $\mathfrak{s}$ in $\mathfrak{g}$, there exists a constant $c \geq 0$ such that $W(d\pi(X)) = cS(d\pi(X))$ for each $X \in \mathfrak{s}$. Similar results have been obtained for different examples of non-reductive Lie groups, see in particular [21].

On the other hand, in [17] and [18] we also obtained a Stratonovich-Weyl correspondence for a non-scalar holomorphic discrete series representation of a semi-simple Lie group by introducing a generalized Berezin map.

In the present paper, we adapt the method and the arguments of [17] and [18] in order to generalize the results of [19] to the case when π is holomorphically induced from a unitary representation ρ of K (in a finite-dimensional vector space V) which is not necessarily a character. More precisely, we prove that the coadjoint orbit $\mathcal{O}$ of G associated with π is diffeomorphic to the product $\mathcal{D}\times o$ where $\mathcal{D}$ is a complex domain and o is the coadjoint orbit of K associated with ρ. Then, following [17], we introduce a Berezin calculus for $\mathrm{End}(V)$-valued functions on $\mathcal{D}$. By combining this calculus with the usual Berezin calculus s on o, we obtain a Berezin calculus S on $\mathcal{O}$ which is G-equivariant with respect to π. Thus, we get a Stratonovich-Weyl correspondence for the triple $(G, \pi, \mathcal{O})$ by taking the isometric part of S.

As an illustration, we consider the case when G is a Heisenberg motion group, that is, the semi-direct product of the $(2n+1)$-Heisenberg group H_n with a compact subgroup of the unitary group $U(n)$. Note that Heisenberg motion groups play an important role in the theory of Gelfand pairs, since the study of a Gelfand pair of the form (K_0, N) where K_0 is a compact Lie group acting by automorphisms on a nilpotent Lie group N can be reduced to that of the form (K_0, H_n) [7, 8].

In this case, the space $\mathcal{H}$ of π can be decomposed as $\mathcal{H}_0 \otimes V$ where $\mathcal{H}_0$ is the Fock space and we show that for each operator A on $\mathcal{H}$ of the form $A_1 \otimes A_2$ we have the decomposition formula $S(A)(Z, \varphi) = S_0(A_1)(Z)s(A_2)(\varphi)$ where S_0 denotes the Berezin calculus on $\mathcal{H}_0$. Moreover, we verify that the Berezin transform takes a simple form and then can be extended to the functions of the form $S(d\pi(X_1X_2\cdots X_p))$ for $X_1, X_2, \ldots, X_p \in \mathfrak{g}$ and we compute explicitely $W(d\pi(X))$ for $X \in \mathfrak{g}$.

2. Preliminaries

All the material of this section is taken from the excellent book of K.-H. Neeb, [37, Chapters VIII and XII], (see also [41, Chapter II] and, for the Hermitian case, [30, Chapter VIII] and [31, Chapter 6]).

Let $\mathfrak{g}$ be a real quasi-Hermitian Lie algebra, that is, a real Lie algebra for which the centralizer in $\mathfrak{g}$ of the center $\mathcal{Z}(\mathfrak{k})$ of a maximal compactly embedded subalgebra $\mathfrak{k}$ coincides with $\mathfrak{k}$ [37, p. 241]. We assume that $\mathfrak{g}$ is not compact. Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$ and $Z = X + iY \to Z^* = -X + iY$ the corresponding involution. We fix a compactly embedded Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$, [37, p. 241], and we denote by $\mathfrak{h}^c$ the corresponding Cartan subalgebra of $\mathfrak{g}^c$. We write $\Delta := \Delta(\mathfrak{g}^c, \mathfrak{h}^c)$ for the set of roots of $\mathfrak{g}^c$ relative to $\mathfrak{h}^c$ and $\mathfrak{g}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ for the root space decomposition of $\mathfrak{g}^c$. Note that $\alpha(\mathfrak{h}) \in i\mathbb{R}$ for each $\alpha \in \Delta$ [37, p. 233]. Recall that a root $\alpha \in \Delta$ is called compact if $\alpha([Z, Z^*]) > 0$ holds for some element $Z \in \mathfrak{g}_\alpha$. All other roots are called non-compact [37, p. 235]. We write Δ_k, respectively Δ_p, for the set of compact, respectively non-compact, roots. Note that $\mathfrak{k}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta_k} \mathfrak{g}_\alpha$ [37, p. 235]. Recall also that a subset $\Delta^+ \subset \Delta$ is called a positive system if there exists an element $X_0 \in i\mathfrak{h}$ such that $\Delta^+ = \{\alpha \in \Delta : \alpha(X_0) > 0\}$ and $\alpha(X_0) \neq 0$ for all $\alpha \in \Delta$. A positive system is then said to be adapted if for $\alpha \in \Delta_k$ and $\beta \in \Delta^+ \cap \Delta_p$ we have $\beta(X_0) > \alpha(X_0)$, [37, p. 236]. Here we fix a positive adapted system Δ^+ and we set $\Delta_p^+ := \Delta^+ \cap \Delta_p$ and $\Delta_k^+ := \Delta^+ \cap \Delta_k$, see [37,

p. 241].

Let G^c be a simply connected complex Lie group with Lie algebra $\mathfrak{g}^c$ and $G \subset G^c$, respectively, $K \subset G^c$, the analytic subgroup corresponding to $\mathfrak{g}$, respectively, $\mathfrak{k}$. We also set $K^c = \exp(\mathfrak{k}^c) \subset G^c$ as in [37, p. 506].

Let $\mathfrak{p}^+ = \sum_{\alpha \in \Delta_p^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \sum_{\alpha \in \Delta_p^+} \mathfrak{g}_{-\alpha}$. We denote by P^+ and P^- the analytic subgroups of G^c with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$. Then G is a group of the Harish-Chandra type [37, p. 507], that is, the following properties are satisfied:

1. $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$ is a direct sum of vector spaces, $(\mathfrak{p}^+)^* = \mathfrak{p}^-$ and $[\mathfrak{k}^+, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$;
2. The multiplication map $P^+K^cP^- \to G^c$, $(z, k, y) \to zky$ is a biholomorphic diffeomorphism onto its open image;
3. $G \subset P^+K^cP^-$ and $G \cap K^cP^- = K$.

Moreover, there exists an open connected K-invariant subset $\mathcal{D} \subset \mathfrak{p}^+$ such that one has $GK^cP^- = \exp(\mathcal{D})K^cP^-$, [37, p. 497]. We denote by $\zeta : P^+K^cP^- \to P^+$, $\kappa : P^+K^cP^- \to K^c$ and $\eta : P^+K^cP^- \to P^-$ the projections onto P^+-, K^c- and P^--component. For $Z \in \mathfrak{p}^+$ and $g \in G^c$ with $g \exp Z \in P^+K^cP^-$, we define the element $g \cdot Z$ of $\mathfrak{p}^+$ by $g \cdot Z := \log \zeta(g \exp Z)$. Note that we have $\mathcal{D} = G \cdot 0$.

We also denote by $g \to g^*$ the involutive anti-automorphism of G^c which is obtained by exponentiating $X \to X^*$. We denote by $p_{\mathfrak{p}^+}$, $p_{\mathfrak{k}^c}$ and $p_{\mathfrak{p}^-}$ the projections of $\mathfrak{g}^c$ onto $\mathfrak{p}^+$, $\mathfrak{k}^c$ and $\mathfrak{p}^-$ associated with the direct decomposition $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$.

The G-invariant measure on $\mathcal{D}$ is $d\mu(Z) := \chi_0(\kappa(\exp Z^* \exp Z))\, d\mu_L(Z)$ where χ_0 is the character on K^c defined by $\chi_0(k) = \mathrm{Det}_{\mathfrak{p}^+}(\mathrm{Ad}\, k)$ and $d\mu_L(Z)$ is a Lebesgue measure on $\mathcal{D}$ [37, p. 538].

Now, we construct a section of the action of G on $\mathcal{D}$, that is, a map $Z \to g_Z$ from $\mathcal{D}$ to G such that $g_Z \cdot 0 = Z$ for each $Z \in \mathcal{D}$. Such a section will be needed later. In [20], we proved the following proposition.

Proposition 2.1. *Let $Z \in \mathcal{D}$. There exists a unique element k_Z in K^c such that $k_Z^* = k_Z$ and $k_Z^2 = \kappa(\exp Z^* \exp Z)^{-1}$. Each $g \in G$ such that $g \cdot 0 = Z$ is then of the form $g = \exp(-Z^*)\, \zeta(\exp Z^* \exp Z) k_Z^{-1} h$ where $h \in K$. Consequently, the map $Z \to g_Z := \exp(-Z^*)\, \zeta(\exp Z^* \exp Z) k_Z^{-1}$ is a section for the action of G on $\mathcal{D}$.*

Note that we have

$$\begin{aligned} g_Z &= \exp(-Z^*)\, \zeta(\exp Z^* \exp Z) k_Z^{-1} \\ &= \exp Z \eta(\exp Z^* \exp Z)^{-1} \kappa(\exp Z^* \exp Z)^{-1} k_Z^{-1} \\ &= \exp Z \eta(\exp Z^* \exp Z)^{-1} k_Z \end{aligned}$$

and then $\kappa(g_Z) = k_Z$.

3. Representations

Let (ρ, V) be a (finite-dimensional) unitary irreducible representation of K with highest weight λ (relative to Δ_c^+). We also denote by ρ the extension of ρ to K^c and by $\tilde{\rho}$ the extension of ρ to K^cP^- which is trivial on P^-. First, we verify that the representation π of G which is associated with ρ as in [37, Proposition XII.2.1], can be obtained by holomorphic induction from ρ.

Let us introduce the Hilbert G-bundle $L := G \times_\rho V$ over G/K. Recall that an element of L is an equivalence class

$$[g, v] = \{(gk, \rho(k)^{-1}v) \,:\, k \in K\}$$

where $g \in G$, $v \in V$ and that G acts on L by left translations: $g\,[g', v] := [gg', v]$.

The projection map $[g, v] \to gK$ is then G-equivariant. The G-invariant Hermitian structure on L is given by

$$\langle [g, v], [g, v'] \rangle = \langle v, v' \rangle_V$$

where $g \in G$ and v, $v' \in V$.

The space G/K being endowed with the complex structure defined in Section 2, let $\mathcal{H}^0$ be the space of all holomorphic sections s of L which are square-integrable with respect to the invariant measure μ_0 on G/K, that is,

$$\|s\|^2_{\mathcal{H}^0} = \int_{G/K} \langle s(p)\,,\, s(p) \rangle \, d\mu_0(p) < +\infty.$$

We can consider the action π_0 of G on $\mathcal{H}^0$ defined by

$$(\pi_0(g)\, s)(p) = g\, s(g^{-1}p).$$

Recall also that the map $gK \to \log \zeta(g)$ is a diffeomorphism from G/K onto $\mathcal{D}$ (see Section 2) whose inverse is the diffeomorphism σ from $\mathcal{D}$ onto G/K defined by $\sigma(Z) = g_Z K$. We can verify that σ intertwines the natural action of G on G/K and the action of G on $\mathcal{D}$ introduced in Section 2, that is, we have $\sigma(g \cdot Z) = g\sigma(Z)$ for each $Z \in \mathcal{D}$ and each $g \in G$. Then we have $\mu_0 = (\sigma^{-1})^*(\mu)$.

Now, we will introduce a realization of π_0 on a space of functions on $\mathcal{D}$. To this aim, we associate with any $s \in \mathcal{H}^0$ the function $f_s : \mathcal{D} \to V$ defined by $s(\sigma(Z)) = [g_Z, \tilde{\rho}(g_Z^{-1} \exp Z) f_s(Z)]$. Then, for each s and s' in $\mathcal{H}^0$, we have

$$\begin{aligned}\langle s(\sigma(Z)), s'(\sigma(Z)) \rangle &= \langle \tilde{\rho}(g_Z^{-1} \exp Z) f_s(Z)\,,\, \tilde{\rho}(g_Z^{-1} \exp Z) f_{s'}(Z) \rangle_V \\ &= \langle \tilde{\rho}(g_Z^{-1} \exp Z)^* \tilde{\rho}(g_Z^{-1} \exp Z) f_s(Z), f_{s'}(Z) \rangle_V \\ &= \langle \tilde{\rho}(\kappa(\exp Z^* \exp Z)) f_s(Z), f_{s'}(Z) \rangle_V\end{aligned}$$

since $g_Z^*g_Z = e$ (the unit element of G).

This implies that

$$\langle s\,,\,s'\rangle_{\mathcal{H}^0} = \int_{\mathcal{D}} \langle \rho(\kappa(\exp Z^* \exp Z)) f_s(Z)\,,\, f_{s'}(Z)\rangle_V \, d\mu(Z).$$

This leads us to introduce the Hilbert space $\mathcal{H}$ of all holomorphic functions $f : \mathcal{D} \to V$ such that

$$\|f\|^2_{\mathcal{H}} := \int_{\mathcal{D}} \langle \rho(\kappa(\exp Z^* \exp Z)) f(Z)\,,\, f(Z)\rangle_V \, d\mu(Z) < +\infty.$$

On the other hand, for each $s \in \mathcal{H}^0$, $g \in G$ and $Z \in \mathcal{D}$, we have

$$\begin{aligned}(\pi_0(g)\, s)(\sigma(Z)) &= gs(g^{-1}\sigma(Z)) \\ &= g\,[g_{g^{-1}\cdot Z}, \tilde{\rho}(g^{-1}_{g^{-1}\cdot Z} \exp(g^{-1}\cdot Z)) f_s(g^{-1}\cdot Z)] \\ &= [g_Z, \tilde{\rho}(g_Z^{-1} g \exp(g^{-1}\cdot Z)) f_s(g^{-1}\cdot Z)].\end{aligned}$$

Then we get

$$\begin{aligned} f_{\pi_0(g)\, s}(Z) &= \tilde{\rho}(g_Z^{-1}\exp Z)^* \tilde{\rho}(g_Z^{-1} g \exp(g^{-1}\cdot Z)) f_s(g^{-1}\cdot Z) \\ &= \tilde{\rho}(\exp(-Z) g \exp(g^{-1}\cdot Z)) f_s(g^{-1}\cdot Z).\end{aligned}$$

Now, noting that

$$g^{-1}\exp Z = \exp(g^{-1}\cdot Z)\kappa(g^{-1}\exp Z)\eta(g^{-1}\exp Z),$$

we obtain

$$f_{\pi_0(g)\, s}(Z) = \rho(\kappa(g^{-1}\exp Z))^{-1} f_s(g^{-1}\cdot Z).$$

Let $J(g,Z) := \rho(\kappa(g \exp Z))$ for $g \in G$ and $Z \in \mathcal{D}$. Hence we can conclude that the equality

$$(\pi(g)f)(Z) = J(g^{-1}, Z)^{-1} f(g^{-1}\cdot Z)$$

defines a unitary representation π of G on $\mathcal{H}$ which is unitarily equivalent to π_0. This is precisely the representation of G introduced in [37, Proposition XII.2.1]. Note also that π is irreducible since ρ is irreducible, [37, p. 515].

We denote $K(Z,W) := \rho(\kappa(\exp W^* \exp Z))^{-1}$ for $Z, W \in \mathcal{D}$. The evaluation maps $K_Z : \mathcal{H} \to V, f \to f(Z)$ are continuous [37, p. 539]. The generalized coherent states of $\mathcal{H}$ are the maps $E_Z = K_Z^* : V \to \mathcal{H}$ defined by $\langle f(Z), v\rangle_V = \langle f, E_Z v\rangle$ for $f \in \mathcal{H}$ and $v \in V$.

We have the following result, see [37, p. 540].

Proposition 3.1. *(1) There exists a constant $c_\rho > 0$ such that $E_Z^* E_W = c_\rho K(Z,W)$ for each Z, $W \in \mathcal{D}$.*
(2) For $g \in G$ and $Z \in \mathcal{D}$, we have $E_{g\cdot Z} = \pi(g) E_Z J(g,Z)^*$.

In the rest of this section, we give an explicit expression for the derived representation $d\pi$. We use the following notation. If L is a Lie group and X is an element of the Lie algebra of L then we denote by X^+ the right invariant vector field on L generated by X, that is, $X^+(h) = \frac{d}{dt}(\exp tX)h|_{t=0}$ for $h \in L$. Then, by differentiating the multiplication map from $P^+ \times K^c \times P^-$ onto $P^+K^cP^-$, we can easily prove the following result.

Lemma 3.2. *Let $X \in \mathfrak{g}^c$ and $g = z\,k\,y$ where $z \in P^+$, $k \in K^c$ and $y \in P^-$. We have*

1. $d\zeta_g(X^+(g)) = (\mathrm{Ad}(z)\, p_{\mathfrak{p}^+}(\mathrm{Ad}(z^{-1})\, X))^+(z)$.
2. $d\kappa_g(X^+(g)) = (p_{\mathfrak{k}^c}(\mathrm{Ad}(z^{-1})\, X))^+(k)$.
3. $d\eta_g(X^+(g)) = (\mathrm{Ad}(k^{-1})\, p_{\mathfrak{p}^-}(\mathrm{Ad}(z^{-1})\, X))^+(y)$.

From this lemma, we deduce the following proposition (see also [37, p. 515]).

Proposition 3.3. *For $X \in \mathfrak{g}^c$ and $f \in \mathcal{H}$, we have*

$$(d\pi(X)f)(Z) = d\rho(p_{\mathfrak{k}^c}(e^{-\operatorname{ad} Z}\, X))\, f(Z) - (df)_Z\left(\frac{\operatorname{ad} Z}{1-e^{-\operatorname{ad} Z}} p_{\mathfrak{p}^+}(e^{-\operatorname{ad} Z}\, X)\right).$$

4. Berezin calculus

Here, we first introduce the Berezin calculus associated with ρ, see [5, 14, 45]. Let $\lambda \in (\mathfrak{h}^c)^*$ denote the highest weight of ρ relative to Δ_c^+. Let $\varphi_0 := -i\lambda \in (\mathfrak{h}^c)^*$. We also denote by φ_0 the restriction to $\mathfrak{k}$ of the trivial extension of φ_0 to $\mathfrak{k}^c$. Then the orbit $o(\varphi_0)$ of φ_0 under the coadjoint action of K is said to be associated with ρ [13, 45].

Note that a complex structure on $o(\varphi_0)$ is then defined by the diffeomorphism $o(\varphi_0) \simeq K/H \simeq K^c/H^cN^-$ where N^- is the analytic subgroup of K^c with Lie algebra $\sum_{\alpha\in\Delta_c^+} \mathfrak{g}_{-\alpha}$.

Without loss of generality, we can assume that V is a space of holomorphic functions on $o(\varphi_0)$ as in [14]. For each $\varphi \in o(\varphi_0)$ there exists a unique function $e_\varphi \in V$ (called a coherent state) such that $a(\varphi) = \langle a, e_\varphi\rangle_V$ for each $a \in V$. The Berezin calculus on $o(\varphi_0)$ associates with each operator B on V the complex-valued function $s(B)$ on $o(\varphi_0)$ defined by

$$s(B)(\varphi) = \frac{\langle Be_\varphi, e_\varphi\rangle_V}{\langle e_\varphi, e_\varphi\rangle_V}$$

which is called the symbol of B. In the following proposition, we recall some basic properties of the Berezin calculus, see for instance [5, 14, 22].

PROPOSITION 4.1. *1. The map $B \to s(B)$ is injective.*

2. For each operator B on V, we have $s(B^*) = \overline{s(B)}$.

3. For each $\varphi \in o(\varphi_0)$, $k \in K$ and $B \in \mathrm{End}(V)$, we have

$$s(B)(\mathrm{Ad}(k)\varphi) = s(\rho(k)^{-1}B\rho(k))(\varphi).$$

4. For each $U \in \mathfrak{k}$ and $\varphi \in o(\varphi_0)$, we have $s(d\rho(U))(\varphi) = i\beta(\varphi, U)$.

In order to define the Berezin symbol $S(A)$ of an operator A on $\mathcal{H}$, we first define the pre-symbol $S_0(A)$ of A as a $\mathrm{End}(V)$-valued function on $\mathcal{D}$, following [2, 17, 32].

Let $\mathcal{H}^s$ be the subspace of $\mathcal{H}$ generated by the functions $E_Z v$ for $Z \in \mathcal{D}$ and $v \in V$. Clearly, $\mathcal{H}^s$ is a dense subspace of $\mathcal{H}$. Let $\mathcal{C}$ be the space consisting of all operators A on $\mathcal{H}$ such that the domain of A contains $\mathcal{H}^s$ and the domain of A^* also contains $\mathcal{H}^s$. We define the pre-symbol $S_0(A)$ of $A \in \mathcal{C}$ by

$$S_0(A)(Z) = c_\rho^{-1}\,\rho(k_Z^{-1})E_Z^* A E_Z \rho(k_Z^{-1})^*$$

and then the Berezin symbol $S(A)$ of A is defined as the complex-valued function on $\mathcal{D} \times o(\varphi_0)$ given by

$$S(A)(Z, \varphi) = s(S_0(A)(Z))(\varphi).$$

In order to establish that S_0 hence S are G-equivariant with respect to π, we need the following lemma.

LEMMA 4.2. *For $g \in G$ and $Z \in \mathcal{D}$, let $k(g, Z) := k_Z^{-1}\kappa(g \exp Z)^{-1}k_{g\cdot Z}$. Then we have $k(g, Z) = g_Z^{-1}g^{-1}g_{g\cdot Z}$. In particular, $k(g, Z)$ is an element of K.*

Proof. Let $g \in G$ and $Z \in \mathcal{D}$. We can write $g_Z = \exp Z k_Z y$ where $y \in P^-$. Then, on the one hand, we have

$$g g_Z = g \exp Z k_Z y = \exp(g \cdot Z)\kappa(g \exp Z)\eta(g \exp Z)k_Z y.$$

On the other hand, we can also write $g_{g\cdot Z} = \exp(g\cdot Z)k_{g\cdot Z}y'$ where $y' \in P^-$. Since $(g g_Z)\cdot 0 = g \cdot Z = g_{g\cdot Z}\cdot 0$, we see that $k := (g g_Z)^{-1}g_{g\cdot Z}$ is an element of K. Then, by replacing $g g_Z$ and $g_{g\cdot Z}$ by the above expressions we get

$$k = y^{-1}k_Z^{-1}\eta(g \exp Z)^{-1}\kappa(g \exp Z)^{-1}k_{g\cdot Z}y'.$$

Thus, applying κ, we obtain $k = k(g, Z)$ hence the result. □

PROPOSITION 4.3. *1. Each $A \in \mathcal{C}$ is determined by $S_0(A)$.*

2. For each $A \in \mathcal{C}$ and each $Z \in \mathcal{D}$, we have $S_0(A^*)(Z) = S_0(A)(Z)^*$.

3. *For each $Z \in \mathcal{D}$, we have $S_0(I)(Z) = I_V$. Here I denotes the identity operator of $\mathcal{H}$ and I_V the identity operator of V.*

4. *For each $A \in \mathcal{C}$, $g \in G$ and $Z \in \mathcal{D}$, we have*

$$S_0(A)(g \cdot Z) = \rho(k(g,Z))^{-1} S_0(\pi(g)^{-1} A \pi(g))(Z) \rho(k(g,Z)).$$

Proof. The proof is similar to that of [17, Proposition 4.1]. Following [37, p. 15], we associate with any operator $A \in \mathcal{C}$ the function $K_A(Z,W) := E_Z^* A E_W$.

1. Let $A \in \mathcal{C}$. Since we have

$$\begin{aligned}\langle (Af)(Z), v\rangle_V &= \langle Af, E_Z v\rangle = \langle f, A^* E_Z v\rangle \\ &= \int_{\mathcal{D}} \langle K(W,W)^{-1} f(W), (A^* E_Z v)(W)\rangle_V \, d\mu(W) \\ &= \int_{\mathcal{D}} \langle K(W,W)^{-1} f(W), K_A(Z,W)^* v\rangle_V \, d\mu(W)\end{aligned}$$

we see that A is determined by K_A. Moreover, since $K_A(Z,W)$ is clearly holomorphic in the variable Z and anti-holomorphic in the variable W, we also see that K_A hence A is determined by $K_A(Z,Z)$ or, equivalently, by $S_0(A)(Z)$.

2. Clearly, for each $A \in \mathcal{C}$, Z, $W \in \mathcal{D}$, we have $K_{A^*}(Z,W) = K_A(W,Z)^*$. The result follows.

3. Let $Z \in \mathcal{D}$. We have

$$E_Z^* E_Z = c_\rho K(Z,Z) = c_\rho \rho(\kappa(\exp Z^* \exp Z))^{-1} = c_\rho \rho(k_Z k_Z^*).$$

The result therefore follows.

4. Let $A \in \mathcal{C}$, $g \in G$ and $Z \in \mathcal{D}$. We have

$$\begin{aligned}S_0(A)(g \cdot Z) &= \frac{1}{c_\rho}\, \rho(k_{g\cdot Z}^{-1})\, E_{g\cdot Z}^* A E_{g\cdot Z}\, \rho(k_{g\cdot Z}^{-1})^* \\ &= \frac{1}{c_\rho}\, \rho(k_{g\cdot Z}^{-1})\, \rho(\kappa(g \exp Z)) E_Z^* \pi(g)^{-1} A \pi(g) E_Z \rho(\kappa(g \exp Z))^* \rho(k_{g\cdot Z}^{-1})^* \\ &= \frac{1}{c_\rho}\, \rho(k(g,Z))^{-1} \rho(k_Z^{-1}) E_Z^* \pi(g)^{-1} A \pi(g) E_Z \rho(k_Z^{-1})^* \rho(k(g,Z)) \\ &= \rho(k(g,Z))^{-1} S_0(\pi(g)^{-1} A \pi(g))(Z) \rho(k(g,Z)).\end{aligned}$$

□

From this proposition and Proposition 4.1 we immediately deduce the following proposition.

PROPOSITION 4.4. *1. Each $A \in \mathcal{C}$ is determined by $S(A)$.*

2. For each $A \in \mathcal{C}$, we have $S(A^*) = \overline{S(A)}$.

3. *We have* $S(I)=1$.

4. *For each* $A\in\mathcal{C}$, $g\in G$, $Z\in\mathcal{D}$ *and* $\varphi\in o(\varphi_0)$, *we have*

$$S(A)(g\cdot Z,\varphi)=S(\pi(g)^{-1}A\pi(g))(Z,\mathrm{Ad}(k(g,Z))\varphi).$$

5. Berezin symbols of representation operators

In this section, we give some simple formulas for the Berezin pre-symbol of $\pi(g)$ for $g\in G$ and for the Berezin symbol of $d\pi(X)$ for $X\in\mathfrak{g}^c$.

PROPOSITION 5.1. *For* $g\in G$ *and* $Z\in\mathcal{D}$, *we have*

$$S_0(\pi(g))(Z)=\rho\left(k_Z^{-1}\kappa(\exp Z^*g^{-1}\exp Z)^{-1}(k_Z^{-1})^*\right).$$

Proof. For each $g\in G$, we have

$$\begin{aligned}S_0(\pi(g))(0)&=c_\rho^{-1}E_0^*\pi(g)E_0=c_\rho^{-1}E_0^*E_{g\cdot 0}J(g,0)^{*-1}\\&=K(0,g\cdot 0)J(g,0)^{*-1}=\rho(\kappa(g))^{*-1}=\rho(\kappa(g^{-1}))^{-1}\end{aligned}$$

by Proposition 3.1.

Now, by using G-equivariance of S_0 (see Proposition 4.3), we get

$$S_0(\pi(g))(Z)=S_0(\pi(g_Z^{-1}gg_Z))(0)=\rho(\kappa(g_Z^{-1}g^{-1}g_Z))^{-1}.$$

But writing $g_Z=\exp Zk_Zy$ with $y\in P^-$ we see that

$$g_Z^{-1}g^{-1}g_Z=g_Z^*g^{-1}g_Z=y^*k_Z^*\exp Z^*g^{-1}\exp Zk_Zy$$

hence $\kappa(g_Z^{-1}g^{-1}g_Z)=k_Z^*\kappa(\exp Z^*g^{-1}\exp Z)k_Z$. This gives the result. □

Now, we aim to compute $S_0(d\pi(X))$ and $S(d\pi(X))$ for $X\in\mathfrak{g}^c$. For $\varphi\in\mathfrak{k}^*$, we also denote by φ the restriction to $\mathfrak{g}$ of the extension of φ to $\mathfrak{g}^c$ which vanishes on $\mathfrak{p}^\pm$. Then we have the following result.

PROPOSITION 5.2. 1. *For each* $g\in G$ *and* $Z\in\mathcal{D}$, *we have*

$$S_0(d\pi(X))(Z)=(d\rho\circ p_{\mathfrak{k}^c})(\mathrm{Ad}(g_Z^{-1})X).$$

2. *For each* $g\in G$, $Z\in\mathcal{D}$ *and* $\varphi\in o(\varphi_0)$, *we have*

$$S(d\pi(X))(Z,\varphi)=i\langle\mathrm{Ad}^*(g_Z)\varphi,X\rangle.$$

Proof. We can deduce the first statement from the preceding proposition. Indeed, by using Lemma 3.2 we get

$$\frac{d}{dt}\,\rho(\kappa(\exp Z^*\exp(-tX)\exp Z)^{-1})|_{t=0}$$
$$=\rho(\kappa(\exp Z^*\exp Z)^{-1})(d\rho\circ p_{\mathfrak{k}^c})(\mathrm{Ad}(\zeta(\exp Z^*\exp Z)^{-1}\exp Z^*)X).$$

Recall that we have $g_Z=\exp(-Z^*)\,\zeta(\exp Z^*\exp Z)k_Z^{-1}$. Then we obtain

$$\begin{aligned}S_0(d\pi(X))(Z)=&(d\rho\circ p_{\mathfrak{k}^c})(\mathrm{Ad}(k_Z\zeta(\exp Z^*\exp Z)^{-1}\exp Z^*)X)\\=&(d\rho\circ p_{\mathfrak{k}^c})(\mathrm{Ad}(g_Z^{-1})X).\end{aligned}$$

The second statement follows from the first and 4 of Proposition 4.1. □

We are then lead to consider the map $\Psi:\mathcal{D}\times o(\varphi_0)\to\mathfrak{g}^*$ defined by $\Psi(Z,\varphi)=\mathrm{Ad}^*(g_Z)\varphi$. Note that by 4 of Proposition 4.4 and 2 of Proposition 5.2 we have

$$\Psi(g\cdot Z,\varphi)=\mathrm{Ad}^*(g)\,\Psi(Z,\mathrm{Ad}^*(k(g,Z))\varphi) \tag{1}$$

for each $g\in G$, $Z\in\mathcal{D}$ and $\varphi\in o(\varphi_0)$.

We say that $\xi_0\in\mathfrak{g}^*$ is *regular* if the stabilizer $G(\xi_0)$ of ξ_0 for the coadjoint action is connected and if the Hermitian form $(Z,W)\to\langle\xi_0,[Z,W^*]\rangle$ is not isotropic. Recall that we have denoted by $\varphi_0\in\mathfrak{g}^*$ the restriction to $\mathfrak{g}$ of the trivial extension to $\mathfrak{g}^c$ of $-i\lambda\in\mathfrak{h}^*$ where λ is the highest weight of ρ. Let $\mathcal{O}(\varphi_0)$ be the orbit of $\varphi_0\in\mathfrak{g}^*$ for the coadjoint action of G and let $K(\varphi_0)$ be the stabilizer of φ_0 for the coadjoint action of K. We assume that φ_0 is regular. Then we have the following result.

Lemma 5.3. *We have $G(\varphi_0)=K(\varphi_0)$.*

Proof. Let us denote by $\mathfrak{g}(\varphi_0)$ and $\mathfrak{k}(\varphi_0)$ the Lie algebras of $G(\varphi_0)$ and $K(\varphi_0)$. We first show that $\mathfrak{g}(\varphi_0)=\mathfrak{k}(\varphi_0)$.

Let $X\in\mathfrak{g}(\varphi_0)$. Then we have $\langle\varphi_0,[X,X']\rangle=0$ for each $X'\in\mathfrak{g}^c$. Now, we can write $X=Z+H+Y$ where $Z\in\mathfrak{p}^+$, $H\in\mathfrak{k}^c$ and $Y\in\mathfrak{p}^-$. Take $X'=Z$ in the preceding equation and recall that we have $\varphi_0|_{\mathfrak{p}^\pm}=0$ and $[\mathfrak{k}^c,\mathfrak{p}^\pm]\subset\mathfrak{p}^\pm$. Thus we get $\langle\varphi_0,[Z,Z^*]\rangle=0$ hence $Z=0$. Similarly, we obtain $Y=0$. This gives $X=H\in\mathfrak{k}(\varphi_0)$. This shows that $\mathfrak{g}(\varphi_0)=\mathfrak{k}(\varphi_0)$.

Now, $G(\varphi_0)$ is connected by hypothesis and $K(\varphi_0)$ is also connected by [35, Lemma 5]. Since $K(\varphi_0)\subset G(\varphi_0)$, we can conclude that $G(\varphi_0)=K(\varphi_0)$. □

We are now in position to establish the following proposition.

Proposition 5.4. *The map Ψ is a diffeomorphism form $\mathcal{D}\times o(\varphi_0)$ onto $\mathcal{O}(\varphi_0)$.*

Proof. For each $g \in G$, one has

$$\mathrm{Ad}^*(g)\varphi_0 = \mathrm{Ad}^*(g)\Psi(0,\varphi_0) = \Psi(g\cdot 0, \mathrm{Ad}^*(k(g,0))\varphi_0).$$

This implies that Ψ takes values in $\mathcal{O}(\varphi_0)$ and that Ψ is surjective.

Now, let (Z,φ) and (Z',φ') in $\mathcal{D}\times o(\varphi_0)$ such that $\Psi(Z,\varphi) = \Psi(Z',\varphi')$. Then we have $\mathrm{Ad}^*(g_Z)\varphi = \mathrm{Ad}^*(g_{Z'})\varphi'$. Write $\varphi = \mathrm{Ad}^*(k)\varphi_0$ and $\varphi' = \mathrm{Ad}^*(k')\varphi_0$ where k, $k'\in K$. Thus we get $\mathrm{Ad}^*(g_Z k)\varphi_0 = \mathrm{Ad}^*(g_{Z'}k')\varphi_0$ and, by Lemma 5.3, there exists $k_0 \in K(\varphi_0)$ such that $g_{Z'}k' = g_Z k k_0$. Consequently, we have $Z' = (g_{Z'}k')\cdot 0 = (g_Z k k_0)\cdot 0 = Z$ hence $k' = kk_0$ and, finally, we obtain $\varphi' = \mathrm{Ad}^*(k')\varphi_0 = \mathrm{Ad}^*(kk_0)\varphi_0 = \mathrm{Ad}^*(k)\varphi_0 = \varphi$. This shows that Ψ is injective.

Now we have to show that Ψ is regular. By using Equation 1, it is sufficient to prove that Ψ is regular at $(0,\varphi)$ for $\varphi\in o(\varphi_0)$. Recall that we have

$$\Psi(Z,\varphi) = \mathrm{Ad}^*\big(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z)k_Z^{-1}\big)\,\varphi.$$

Then, differentiating Ψ by using Lemma 3.2, we easily get

$$(d\Psi)(0,\varphi)(W, U^+(\varphi)) = \mathrm{ad}^*(W - W^* + U)\varphi$$

for each $W\in\mathfrak{p}^+$ and $U\in\mathfrak{k}^c$. Thus, for each $X\in\mathfrak{g}^c$, we have

$$\langle \varphi, [W - W^* + U, X]\rangle = 0.$$

Taking in particular $X = W^*$, we get $\langle\varphi,[W,W^*]\rangle = 0$. Since φ_0 hence φ is regular, we obtain $W = 0$ and, consequently, $\mathrm{ad}^*(U)\varphi = U^+(\varphi) = 0$. This finishes the proof. □

Note that we have also the following result.

PROPOSITION 5.5. *Assume that we have* $[\mathfrak{p}^+,\mathfrak{p}^-]\subset\mathfrak{k}^c$ *(this is the case, in particular, when* $\mathfrak{g}$ *is reductive). Let* $\varphi^0\in\mathfrak{h}^*$. *As usual, we denote also by* φ^0 *the restriction to* $\mathfrak{g}$ *of the trivial extension of* φ^0 *to* $\mathfrak{g}^*$. *Then* φ^0 *is regular if and only if the Hermitian form* $(Z,W)\to\langle\varphi^0,[Z,W^*]\rangle$ *is not isotropic. In that case, we also have* $G(\varphi_0) = K(\varphi_0)$.

Proof. Assume that the Hermitian form $(Z,W)\to\langle\varphi^0,[Z,W^*]\rangle$ is not isotropic. Let $g\in G(\varphi^0)$. Write $g = (\exp Z)ky$ where $Z\in\mathfrak{p}^+$, $k\in K^c$ and $Y\in\mathfrak{p}^-$. Then we have $\mathrm{Ad}^*(k\exp Y)\varphi^0 = \mathrm{Ad}^*(\exp Z)\varphi^0$ and, for each $X\in\mathfrak{g}^c$,

$$\langle\varphi^0, \mathrm{Ad}(\exp Z)^{-1}X\rangle = \langle\varphi^0, \mathrm{Ad}(k\exp Y)^{-1}X\rangle.$$

Taking $X = Z^*$, we find $\langle\varphi^0,[Z,Z^*]\rangle = 0$ hence $Z = 0$. Similarly, we verify that $Y = 0$. This gives $g = k\in K^c\cap G(\varphi^0) = K(\varphi^0)$. Consequently, $G(\varphi^0)$ is connected and φ^0 is regular. □

Moreover, by adapting the arguments of the proof of [19, Lemma 3.1], we also obtain the following proposition.

PROPOSITION 5.6. *Assume that* $\mathcal{H} \neq (0)$*. Then the Hermitian form* $(Z, W) \to \langle \varphi^0, [Z, W^*] \rangle$ *is not isotropic.*

6. Berezin transform and Stratonovich-Weyl correspondence

In this section, we introduce the Berezin transform and we review some of its properties. As an application, we construct a Stratonovich-Weyl correspondence for $(G, \pi, \mathcal{O}(\varphi_0))$.

Let us fix a K-invariant measure ν on $o(\varphi_0)$ normalized as in [14, Section 2]. Then the measure $\tilde{\mu} := \mu \otimes \nu$ on $\mathcal{D} \times o(\varphi_0)$ is invariant under the action of G on $\mathcal{D} \times o(\varphi_0)$ given by $g \cdot (Z, \varphi) := (g \cdot Z, \mathrm{Ad}(k(g, Z))^{-1}\varphi)$ and the measure $\mu_{\mathcal{O}(\varphi_0)} := (\Psi^{-1})^*(\tilde{\mu})$ is a G-invariant measure on $\mathcal{O}(\varphi_0)$.

We denote by $\mathcal{L}_2(\mathcal{H})$ (respectively $\mathcal{L}_2(V)$) the space of Hilbert-Schmidt operators on $\mathcal{H}$ (respectively V) endowed with the Hilbert-Schmidt norm. Since V is finite-dimensional, we have $\mathcal{L}_2(V) = \mathrm{End}(V)$. We denote by $L^2(\mathcal{D} \times o(\varphi_0))$ (respectively $L^2(\mathcal{D}), L^2(o(\varphi_0))$) the space of functions on $\mathcal{D} \times o(\varphi_0)$ (resp. $\mathcal{D}$, $o(\varphi_0)$) which are square-integrable with respect to the measure $\tilde{\mu}$ (respectively μ, ν). The following result is well-known, see for instance [15].

PROPOSITION 6.1. *For each* $\varphi \in o(\varphi_0)$*, let* p_φ *denote the orthogonal projection of* V *on the line generated by* e_φ*. Then the adjoint* s^* *of the operator* $s : \mathcal{L}_2(V) \to L^2(o(\varphi_0))$ *is given by*

$$s^*(a) = \int_{o(\varphi_0)} a(\varphi) p_\varphi \, d\nu(\varphi)$$

and the Berezin transform $b := ss^*$ *is given by*

$$b(a)(\psi) = \int_{o(\varphi_0)} a(\varphi) \frac{|\langle e_\psi, e_\varphi \rangle_V|^2}{\langle e_\varphi, e_\varphi \rangle_V \langle e_\psi, e_\psi \rangle_V} \, d\nu(\varphi)$$

for each $a \in L^2(o(\varphi_0))$

Following [18], we can easily obtain the following analogous results for S, see also [43].

PROPOSITION 6.2. *The map* S *is a bounded operator from* $\mathcal{L}_2(\mathcal{H})$ *to* $L^2(\mathcal{D} \times o(\varphi_0))$*. Moreover,* S^* *is given by*

$$S^*(f) = \int_{\mathcal{D} \times o(\varphi_0)} P_{Z,\varphi} f(Z, \varphi) d\mu(Z) d\nu(\varphi)$$

where $P_{Z,\varphi} := c_\rho^{-1} E_Z \rho(h_Z^{-1})^* p_\varphi \rho(h_Z^{-1}) E_Z^*$ *is the orthogonal projection of* $\mathcal{H}$ *on the line generated by* $E_Z\rho(h_Z^{-1})^* e_\varphi$.

From this result we easily deduce that the following proposition.

Proposition 6.3. *The Berezin transform* $B := SS^*$ *is a bounded operator of* $L^2(\mathcal{D} \times o(\varphi_0))$ *and, for each* $f \in L^2(\mathcal{D} \times o(\varphi_0))$, *we have the following integral formula*

$$B(f)(Z,\psi) = \int_{\mathcal{D}\times o(\varphi_0)} k(Z,W,\psi,\varphi)\, f(W,\varphi)\, d\mu(W) d\nu(\varphi)$$

where

$$k(Z,W,\psi,\varphi) := \frac{|\langle \rho(\kappa(g_Z^{-1} g_W))^{-1} e_\psi, e_\varphi\rangle_V|^2}{\langle e_\varphi, e_\varphi\rangle_V \langle e_\psi, e_\psi\rangle_V}.$$

Let us introduce the left-regular representation τ of G on $L^2(\mathcal{D} \times o(\varphi_0))$ defined by $(\tau(g)(f))(Z,\varphi) = f(g^{-1}\cdot(Z,\varphi))$. Clearly, τ is unitary. Moreover, since S is G-equivariant, we immediately verify that for each $f \in L^2(\mathcal{D}\times o(\varphi_0))$ and each $g \in G$, we have $B(\tau(g)f) = \tau(g)(B(f))$.

Now, we consider the polar decomposition of $S : \mathcal{L}_2(\mathcal{H}) \to L^2(\mathcal{D} \times o(\varphi_0))$. We can write $S = (SS^*)^{1/2} W = B^{1/2} W$ where $W := B^{-1/2} S$ is a unitary operator from $\mathcal{L}_2(\mathcal{H})$ to $L^2(\mathcal{D}\times o(\varphi_0))$. Then we have the following proposition.

Proposition 6.4.

1. *The map* $W : \mathcal{L}_2(\mathcal{H}) \to L^2(\mathcal{D} \times o(\varphi_0))$ *is a Stratonovich-Weyl correspondence for the triple* $(G, \pi, \mathcal{D} \times o(\varphi_0))$.
2. *The map* $\mathcal{W}$ *from* $\mathcal{L}_2(\mathcal{H})$ *to* $L^2(\mathcal{O}(\varphi_0), \mu_{\mathcal{O}(\varphi_0)})$ *defined by* $\mathcal{W}(f) = W(f \circ \Psi)$ *is a Stratonovich-Weyl correspondence for the triple* $(G, \pi, \mathcal{O}(\varphi_0))$.

7. Generalies on Heisenberg motion groups

We first introduce the Heisenberg group. For $z, w \in \mathbb{C}^n$, we denote $zw := \sum_{k=1}^n z_k w_k$. Consider the symplectic form ω on $\mathbb{C}^{2n}$ defined by

$$\omega((z,w),(z',w')) = \frac{i}{2}(zw' - z'w).$$

for $z, w, z', w' \in \mathbb{C}^n$. The $(2n+1)$-dimensional real Heisenberg group is $H_n := \{((z,\bar z), c) : z \in \mathbb{C}^n, c \in \mathbb{R}\}$ endowed with the multiplication

$$((z,\bar z),c)\cdot((z',\bar z'),c') = ((z+z', \bar z + \bar z'), c + c' + \tfrac{1}{2}\omega((z,\bar z),(z',\bar z'))). \tag{2}$$

Then the complexification H_n^c of H_n is $H_n^c := \{((z,w),c) : z, w \in \mathbb{C}^n, c \in \mathbb{C}\}$ and the multiplication of H_n^c is obtained by replacing $(z,\bar z)$ by (z,w) and $(z',\bar z')$

by (z', w') in Equation 2. We denote by $\mathfrak{h}_n$ and $\mathfrak{h}_n^c$ the Lie algebras of H_n and H_n^c.

Let K_0 be a closed subgroup of $U(n)$. Then K_0 acts on H_n by $k\cdot((z,\bar{z}),c) = ((kz,\bar{kz}),c)$ and we can form the semi-direct product $G := H_n \rtimes K_0$ which is called a Heisenberg motion group. The elements of G can be written as $((z,\bar{z}),c,h)$ where $z \in \mathbb{C}^n$, $c \in \mathbb{R}$ and $h \in K_0$. The multiplication of G is then given by

$$((z,\bar{z}),c,h)\cdot((z',\bar{z}'),c',h') \\ = ((z,\bar{z})+(hz',\bar{hz'}),c+c'+\tfrac{1}{2}\omega((z,\bar{z}),(hz',\bar{hz'})),hh').$$

We denote by K_0^c the complexification of K_0. In order to describe the complexification G^c of G, it is convenient to introduce the action of K_0^c on $\mathbb{C}^n\times\mathbb{C}^n$ given by $k\cdot(z,w) = (kz,(k^t)^{-1}w)$ (here, the subscript t denotes transposition). The group G^c is then the semi-direct product $G^c = H_n^c \rtimes K_0^c$. The elements of G^c can be written as $((z,w),c,h)$ where z, $w \in \mathbb{C}^n$, $c \in \mathbb{C}$ and $h \in K_0^c$ and the multiplication law of G^c is given by

$$((z,w),c,h)\cdot((z',w'),c',h') \\ = ((z,w)+h\cdot(z',w'),c+c'+\tfrac{1}{2}\omega((z,w),h\cdot(z',w')),hh').$$

We denote by $\mathfrak{k}_0$, $\mathfrak{k}_0^c$, $\mathfrak{g}$ and $\mathfrak{g}^c$ the Lie algebras of K_0, K_0^c, G and G^c. The derived action $\mathfrak{k}_0^c$ on $\mathbb{C}^n\times\mathbb{C}^n$ is $A\cdot(z,w) := (Az,-A^tw)$ and the Lie brackets of $\mathfrak{g}^c$ are given by

$$[((z,w),c,A),((z',w'),c',A')] \\ = (A\cdot(z',w')-A'\cdot(z,w),\omega((z,w),(z',w')),[A,A']).$$

Recall that, for each $X \in \mathfrak{g}^c$, we have $X^* = -\theta(X)$ where θ denotes conjugation over $\mathfrak{g}$. We can easily verify that if $X = ((z,w),c,A) \in \mathfrak{g}^c$ then $X^* = ((-\bar{w},-\bar{z}),c,\bar{A}^t)$.

Here we take $K = \{((0,0),c,h) : c \in \mathbb{R},\ h \in K_0\}$ for the maximal compactly embedded subgroup of G. Also, let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$. Then we take $\mathfrak{h} := \{((0,0),c,A) : c \in \mathbb{R},\ A \in \mathfrak{h}_0\}$ for the compactly embedded Cartan subalgebra of $\mathfrak{g}$, see Section 2. Moreover, we can choose the positive non-compact roots in such a way that $P^+ = \{((z,0),0,I_n) : z \in \mathbb{C}^n\}$ and $P^- = \{((0,w),0,I_n) : w \in \mathbb{C}^n\}$. The $P^+K^cP^-$-decomposition of $g = ((z_0,w_0),c_0,h) \in G^c$ is given by

$$g = ((z_0,0),0,I_n)\cdot((0,0),c,h)\cdot((0,w_0),0,I_n)$$

where $c = c_0 - \frac{i}{4}z_0w_0$. From this, we deduce that the action of the element $g = ((z_0,w_0),c_0,h)$ of G on $Z = ((z,0),0,0) \in \mathfrak{p}^+$ is given by $g\cdot Z = \log\zeta(g\exp Z) = ((z_0+hz,0),0,0)$. Then we have $\mathcal{D} = \mathfrak{p}^+ \simeq \mathbb{C}^n$.

We can also easily compute the section $Z \to g_Z$. We find that if $Z = ((z,0),0,0) \in \mathcal{D}$ then $g_Z = ((z,\bar{z}),0,I_n)$ and $k_Z = \kappa(g_Z) = ((0,0),-\frac{i}{4}|z|^2, I_n)$.

Now we compute the adjoint action of G^c. Let $g = (v_0, c_0, h_0) \in G^c$ where $v_0 \in \mathbb{C}^{2n}$, $c_0 \in \mathbb{C}$, $h_0 \in K_0^c$ and $X = (w, c, A) \in \mathfrak{g}^c$ where $w \in \mathbb{C}^{2n}$, $c \in \mathbb{C}$ and $A \in \mathfrak{k}_0^c$. We set $\exp(tX) = (w(t), c(t), \exp(tA))$. Then, since the derivatives of $w(t)$ and $c(t)$ at $t = 0$ are w and c, we find that

$$\begin{aligned} &\mathrm{Ad}(g)X = \frac{d}{dt}(g\exp(tX)g^{-1})|_{t=0} \\ &= \big(h_0 w - (\mathrm{Ad}(h_0)A)\cdot v_0, c + \omega(v_0, h_0 w) - \tfrac{1}{2}\omega(v_0, (\mathrm{Ad}(h_0)A)\cdot v_0), \mathrm{Ad}(h_0)A\big). \end{aligned}$$

From this, we deduce the coadjoint action of G^c. Let us denote by $\xi = (u, d, \phi)$, where $u \in \mathbb{C}^{2n}$, $d \in \mathbb{C}$ and $\phi \in (\mathfrak{k}_0^c)^*$, the element of $(\mathfrak{g}^c)^*$ defined by

$$\langle \xi, (w, c, A)\rangle = \omega(u, w) + dc + \langle \phi, A\rangle.$$

Also, for $u, v \in \mathbb{C}^{2n}$, we denote by $v \times u$ the element of $(\mathfrak{k}_0^c)^*$ defined by $\langle v \times u, A\rangle := \omega(u, A\cdot v)$ for $A \in \mathfrak{k}_0^c$.

Now, let $\xi = (u, d, \phi) \in (\mathfrak{g}^c)^*$ and $g = (v_0, c_0, h_0) \in G^c$. Recall that we have $\langle \mathrm{Ad}^*(g)\xi, X\rangle = \langle \xi, \mathrm{Ad}(g^{-1})X\rangle$ for each $X \in \mathfrak{g}^c$. Then we obtain

$$\mathrm{Ad}^*(g)\xi = \big(h_0 u - d v_0, d, \mathrm{Ad}^*(h_0)\phi + v_0 \times (h_0 u - \tfrac{d}{2} v_0)\big)$$

By restriction, we also get the analogous formula for the coadjoint action of G. From this, we deduce that if a coadjoint orbit of G contains a point (u, d, ϕ) with $d \neq 0$ then it also contains a point of the form $(0, d, \phi_0)$. Such an orbit is called *generic*.

8. Representations of Heisenberg motion groups

We retain the notation of the previous section and introduce some additional notation. Let ρ_0 be a unitary irreducible representation of K_0 on a (finite-dimensional) Hilbert space V and let $\gamma \in \mathbb{R}$. Then we take ρ to be the representation of K on V defined by $\rho((0,0), c, h) = e^{i\gamma c}\rho_0(h)$ for each $c \in \mathbb{R}$ and $h \in K_0$. Thus, for each $Z = ((z,0),0,0)$, $W = ((w,0),0,0) \in \mathcal{D}$, we have $K(Z,W) = \rho(\kappa(\exp W^* \exp Z))^{-1} = e^{\gamma z\bar{w}/2} I_V$. Hence the Hilbert product on $\mathcal{H}$ is given by

$$\langle f, g\rangle = \int_{\mathcal{D}} \langle f(Z), g(Z)\rangle_V \, e^{-\gamma|z|^2/2}\, d\mu(Z)$$

where μ is the G-invariant measure on $\mathcal{D} \simeq \mathbb{C}^n$ defined by $d\mu(Z) := (\frac{\gamma}{2\pi})^n\, dx\, dy$. Here $Z = ((z,0),0,0)$ and $z = x + iy$ with x and y in $\mathbb{R}^n$. Note that we have $c_\rho = 1$. Moreover, for each $v \in V$, $Z = ((z,0),0,0)$, $W = ((w,0),0,0) \in \mathcal{D}$, we have $(E_W v)(Z) = K(Z,W)v = e^{\frac{\gamma}{2} z\bar{w}} v$.

On the other hand, we easily verify that, for each $g=((z_0,\bar z_0),c_0,h)\in G$ and $Z=((z,0),0,0),\in\mathcal{D}$, we have

$$J(g,Z)=\rho\left(\kappa(g\exp Z)\right)=\exp\left(i\gamma c_0+\tfrac{\gamma}{2}\bar z_0(hz)+\tfrac{\gamma}{4}|z_0|^2\right)\,\rho_0(h)$$

and consequently, we get the following formula for π:

$$(\pi(g)f)(Z)=\exp\left(i\gamma c_0+\tfrac{\gamma}{2}\bar z_0 z-\tfrac{\gamma}{4}|z_0|^2\right)\,\rho_0(h)\,f(h^{-1}(z-z_0),0,0)$$

where $g=((z_0,\bar z_0),c_0,h)\in G$ and $Z=((z,0),0,0),\in\mathcal{D}$.

Let $\phi_0\in\mathfrak{k}_0^*$. Assume that the orbit $o(\phi_0)$ of ϕ_0 for the coadjoint action of K_0 is associated with ρ_0 as in Section 4. Then, in the notation of Section 4, the coadjoint orbit of $\varphi_0:=((0,0),\gamma,\phi_0)$ for the coadjoint action of G is then associated with π. Note that the orbit $o(\varphi_0)$ of $\varphi_0:=((0,0),\gamma,\phi_0)$ for the coadjoint action of K can be identify to $o(\phi_0)$ via $\phi\to((0,0),\gamma,\phi)$.

In the present situation, Proposition 3.3 can be reformulated as follows.

Proposition 8.1. *Let $X=((a,b),c,A)\in\mathfrak{g}^c$. Then, for each $f\in\mathcal{H}$ and each $Z=((z,0),0,0),\in\mathcal{D}$, we have*

$$(d\pi(X)f)(Z)=d\rho_0(A)f(Z)+\gamma(ic-\tfrac{1}{2}bz)f(Z)-df_Z((a+Az,0),0,0).$$

Now consider the Hilbert space $\mathcal{H}_0$ of all holomorphic functions $f_0:\mathcal{D}\to\mathbb{C}$ such that

$$\|f\|_0^2=\int_{\mathcal{D}}|f(Z)|^2\,e^{-\gamma|z|^2/2}\,d\mu(Z)<+\infty.$$

Then for each $Z\in\mathcal{D}$ there exists a coherent state $e_Z^0\in\mathcal{H}_0$ such that $f(Z)=\langle f,e_Z^0\rangle_0$ for each $f\in\mathcal{H}_0$. More precisely, for each $Z=((z,0),0,0)$, $W=((w,0),0,0)\in\mathcal{D}$, we have $e_Z^0(W)=e^{\gamma\bar z w/2}$.

Clearly, one has $\mathcal{H}=\mathcal{H}_0\otimes V$. For $f_0\in\mathcal{H}_0$ and $v\in V$, we denote by $f_0\otimes v$ the function $Z\to f_0(Z)v$. Moreover, if A_0 is an operator of $\mathcal{H}_0$ and A_1 is an operator of V then we denote by $A_0\otimes A_1$ the operator of $\mathcal{H}$ defined by $(A_0\otimes A_1)(f_0\otimes v)=A_0f_0\otimes A_1v$.

Let π_0 be the unitary irreducible representation of H_n on $\mathcal{H}_0$ defined by

$$(\pi_0((z_0,\bar z_0),c_0)f_0)(Z)=\exp\left(i\gamma c_0+\tfrac{\gamma}{2}\bar z_0 z-\tfrac{\gamma}{2}|z_0|^2\right)\,f_0((z-z_0,0),0,0)$$

for each $Z=((z,0),0,0)\in\mathcal{D}$ and let σ_0 be the left-regular representation of K_0 on $\mathcal{H}_0$, that is, $(\sigma_0(h)f_0)(Z)=f_0(h^{-1}\cdot Z)$. Then we have

$$\pi((z_0,\bar z_0),c_0,h)=\pi_0((z_0,\bar z_0),c_0)\circ\sigma_0(h)\otimes\rho_0(h)$$

for each $z_0\in\mathbb{C}^n$, $c_0\in\mathbb{R}$ and $h\in K_0$. This is precisely Formula (3.18) in [7].

9. Berezin and Stratonovich-Weyl symbols for Heisenberg motion groups

In this section, we first establish a decomposition formula for the Berezin symbol of an operator on $\mathcal{H}$ of the form $A_0 \otimes A_1$ where A_0 is an operator of $\mathcal{H}_0$ and A_1 is an operator of V. As an application, we compute explicitely the Berezin and the Stratonovich-Weyl symbols of the representation operators.

We also need here the Berezin calculus for operators on $\mathcal{H}_0$. Recall that the Berezin symbol $S^0(A_0)$ of an operator A_0 on $\mathcal{H}_0$ is the function on $\mathcal{D}$ defined by

$$S^0(A_0)(Z) := \frac{\langle A_0\, e^0_Z\, ,\, e^0_Z\rangle}{\langle e^0_Z\, ,\, e^0_Z\rangle} = e^{-\gamma|z|^2/2}\,(A_0 e^0_Z)(Z),$$

see, for instance, [12]. In particular, S^0 is H_n-equivariant with respect to π_0. Let $B^0 := S^0(S^0)^*$ be the corresponding Berezin transform.

On the other hand, recall that $\varphi_0 = ((0,0),\gamma,\phi_0)$ and that we have identified the coadjoint orbit $o(\varphi_0)$ of K with the coadjoint orbit $o(\phi_0)$ of K_0. Then, for $\varphi = ((0,0),\gamma,\phi)$, we can identify the coherent state e_φ on $o(\varphi_0)$ with the coherent state e_ϕ on $o(\phi_0)$. Hence, the corresponding Berezin calculus can be also identified.

Let f_0 be a complex-valued function on $\mathcal{D}$ and f_1 be a complex-valued function on $o(\phi_0)$. Then we denote by $f_0 \otimes f_1$ the function on $\mathcal{D} \times o(\phi_0)$ defined by $f_0 \otimes f_1(Z,\phi) = f_0(Z) f_1(\phi)$.

Proposition 9.1. *Let A_0 be an operator on $\mathcal{H}_0$ and let A_1 be an operator on V. Let $A := A_0 \otimes A_1$. Then*

1. *For each $Z \in \mathcal{D}$, we have $S_0(A)(Z) = S^0(A_0)(Z) A_1$.*
2. *For each $Z \in \mathcal{D}$ and each $\phi \in o(\phi_0)$, we have $S(A)(Z,\phi) = S^0(A_0)(Z) s(A_1)(\phi)$, that is, $S(A) = S^0(A_0) \otimes s(A_1)$.*

Proof. Let $Z = ((z,0),0,0) \in \mathcal{D}$ and $v \in V$. We have

$$S_0(A)(Z)v = e^{-\gamma|z|^2/2} E_Z^* A E_Z v = e^{-\gamma|z|^2/2} A(E_Z v)(Z).$$

Now, recall that $E_Z v = e^0_Z \otimes v$. Then we get $A(E_Z v) = A_0 e^0_Z \otimes A_1 v$ and, consequently,

$$S_0(A)(Z)v = e^{-\gamma|z|^2/2}(A_0 e^0_Z)(Z) A_1 v = S^0(A_0)(Z) A_1.$$

This proves 1. Assertion 2 immediately follows from 1. □

The preceding proposition is useful to compute the Berezin symbol of an operator on $\mathcal{H}$ which is a sum of operators of the form $A_0 \otimes A_1$. This is precisely the case of the representation operators $\pi(g)$, $g \in G$ and $d\pi(X)$, $X \in \mathfrak{g}^c$ and then we have the following propositions.

Proposition 9.2. *Let $g = ((z_0, \bar{z}_0), c_0, h) \in G$. For each $Z = ((z, 0), 0, 0) \in \mathcal{D}$ and each $\phi \in o(\phi_0)$, we have*

$$S(\pi(g))(Z, \phi) \\ = \exp \gamma \left(ic_0 + \tfrac{1}{2}\bar{z}_0 z - \tfrac{1}{4}|z_0|^2 - \tfrac{1}{2}|z|^2 + \tfrac{1}{2}\bar{z}h^{-1}(z - z_0) \right) s(\rho_0(h))(\phi).$$

Proof. Recall that, for each $g = ((z_0, \bar{z}_0), c_0, h) \in G$, we have

$$\pi(g) = \pi_0((z_0, \bar{z}_0), c_0) \circ \sigma(h) \otimes \rho_0(h).$$

Then the result follows from Proposition 9.1. □

Proposition 9.3. *1. For each $X = ((a, b), c, A) \in \mathfrak{g}^c$, $Z = ((z, 0), 0, 0) \in \mathcal{D}$ and $\phi \in o(\phi_0)$, we have*

$$S(d\pi(X))(Z, \phi) = i\gamma c - \frac{\gamma}{2}\left(a\bar{z} + bz + \bar{z}(Az)\right) + s(d\rho_0(A))(\phi).$$

2. For each $X = ((a, b), c, A) \in \mathfrak{g}^c$ and $Z = ((z, 0), 0, 0) \in \mathcal{D}$ and $\phi \in o(\phi_0)$, we have $S(d\pi(X))(Z, \phi) = i\langle \Psi(Z, \phi), X\rangle$ where the diffeomorphism $\Psi : \mathcal{D} \times o(\phi_0) \to \mathcal{O}(\varphi_0)$ is defined by

$$\Psi(Z, \phi) = \left(-\gamma(z, \bar{z}), \gamma, \phi - \tfrac{\gamma}{2}(z, \bar{z}) \;\times (z, \bar{z})\right).$$

Proof. Assertion 1 follows from Proposition 3.3 and Proposition 9.1 and Assertion 2 follows from the equality $\Psi(Z, \phi) = \mathrm{Ad}^*(g_Z)\varphi_0$. □

By adapting Proposition 6.3 to the present situation, we get the following decomposition of the Berezin transform $B = SS^*$.

Proposition 9.4. *For each $f \in L^2(\mathcal{D} \times o(\phi_0))$, we have*

$$B(f)(Z, \psi) = \int_{\mathcal{D}\times o(\phi_0)} k(Z, W, \psi, \phi)\, f(W, \phi)\, d\mu(W) d\nu(\phi)$$

where

$$k(Z, W, \psi, \phi) = e^{-\gamma|z-w|^2/2}\, \frac{|\langle e_\psi, e_\phi\rangle_V|^2}{\langle e_\phi, e_\phi\rangle_V \langle e_\psi, e_\psi\rangle_V}.$$

In particular, for each $f_0 \in L^2(\mathcal{D})$ and $f_1 \in L^2(o(\phi_0))$, we have $B(f_0 \otimes f_1) = B_0(f_0) \otimes b(f_1)$.

Proof. We can compute $k(Z, W, \psi, \phi)$ (see Proposition 6.3) as follows. Let $Z = ((z, 0), 0, 0)$ and $W = ((w, 0), 0, 0) \in \mathcal{D}$. Then we have

$$g_Z^{-1} g_W = \left((-z + w, -\bar{z} + \bar{w}), -\frac{i}{4}(z\bar{w} - \bar{z}w), I_n \right).$$

Thus

$$\kappa(g_Z^{-1}g_W) = \left((0,0), -\frac{i}{4}(z\bar{z} + w\bar{w} - 2\bar{z}w), I_n\right).$$

Consequently, we get

$$\rho(\kappa(g_Z^{-1}g_W))^{-1} = e^{-\gamma(|z|^2+|w|^2-2\bar{z}w)/4}\, I_V.$$

Since we have

$$|e^{-\gamma(|z|^2+|w|^2-2\bar{z}w)/4}|^2 = e^{-\gamma|z-w|^2/2},$$

the first assertion follows from Proposition 6.3. The second assertion is an immediate consequence of the first one. □

In the following proposition, we study the form of the function

$$S(d\pi(X_1X_2\cdots X_q))$$

for $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$.

Proposition 9.5. *Let $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$. Then*

1. *The function $S(d\pi(X_1X_2\cdots X_q))(Z,\phi)$ is a sum of terms of the form*

$$P(Z)Q(\bar{Z})s(d\rho_0(Y_1Y_2\cdots Y_r))(\phi)$$

where P, Q are polynomials of degree $\leq q$, $r \leq q$ and $Y_1, Y_2, \ldots, Y_r \in \mathfrak{k}_0^c$.

2. *We have $S(d\pi(X_1X_2\cdots X_q)) \in L^2(\mathcal{D} \times o(\phi_0))$.*

Proof. 1. By using Proposition 3.3, we can verify by induction on q that, for each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, $d\pi(X_1X_2\cdots X_q)$ is a sum of terms of the form

$$P(Z)d\rho_0(Y_1Y_2\cdots Y_r)\partial_{i_1}\partial_{i_2}\cdots\partial_{i_s}$$

where P is a polynomial of degree $\leq q$, r, $s \leq q$ and $Y_1, Y_2, \ldots, Y_r \in \mathfrak{k}_0^c$. Here we write as usual $Z = ((z,0),0,0)$ with $z \in \mathbb{C}^n$ and ∂_i stands for the derivative with respect to z_i.

Taking Proposition 9.1 into account, this implies that $S(d\pi(X_1X_2\cdots X_q))$ is a sum of terms of the form

$$P(Z)S^0(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_s})(Z)\, s(d\rho_0(Y_1Y_2\cdots Y_r))(\phi).$$

But recall that $e_Z^0(W) = e^{\gamma\bar{z}w/2}$. Then we have

$$(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_s}e_Z^0)(W) = \bar{w}_{i_1}\bar{w}_{i_1}\cdots\bar{w}_{i_s}e_Z^0(W).$$

Thus we see that

$$S^0(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_s})(Z) = e_Z^0(Z)^{-1}(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_s}e_Z^0)(Z) = \bar{w}_{i_1}\bar{w}_{i_1}\cdots\bar{w}_{i_s}.$$

The result follows.

2. This assertion is a consequence of 1. Indeed, the function $P(Z)Q(\bar{Z})$ with P, Q polynomials is clearly square-integrable with respect to μ_0. On the other hand, recall that V is finite-dimensional, that $o(\phi_0)$ is compact and that we have the property $|s(A_0)| \leq \|A_0\|_{\mathrm{op}}$ for each operator A_0 on V. Then we see that $s(d\rho_0(Y_1 Y_2 \cdots Y_s))$ is bounded hence square-integrable on $o(\phi_0)$. □

In the general case, by contrast to the preceding proposition, the function $S(d\pi(X_1 X_2 \cdots X_q))$ is not usually square-integrable. However, when $\mathfrak{g}$ is reductive, we have proved that B can be extended to a class of fonctions which contains $S(d\pi(X_1 X_2 \cdots X_q))$ for $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$ and $q \leq q_\pi$ where q_π only depends on π, see [18, 19].

Finally, we compute $W(d\pi(X))$, $X \in \mathfrak{g}^c$ which is well-defined thanks to the preceding proposition. Consider the Stratonovich-Weyl correspondences $W := B^{-1/2}S$, $W_0 := B_0^{-1/2}S^0$ and $w := b^{-1/2}s$ on $\mathcal{D} \times o(\phi_0)$, $\mathcal{D}$ and $o(\phi_0)$, respectively. Clearly, for any A_0 operator on $\mathcal{H}_0$ and any A_1 operator on V, we have $W(A_0 \otimes A_1) = W_0(A_0) \otimes w(A_1)$ by Proposition 9.1 and Proposition 9.4.

Proposition 9.6. *For each $X = ((a,b),c,A) \in \mathfrak{g}^c$, $Z = ((z,0),0,0) \in \mathcal{D}$ and $\phi \in o(\phi_0)$, we have*

$$W(d\pi(X))(Z,\phi) = ic\gamma + w(d\rho_0(A))(\phi) + \frac{1}{2}\,\mathrm{Tr}(A) - \frac{\gamma}{2}\left(a\bar{z} + bz + \bar{z}(Az)\right).$$

Proof. Let $\Delta := 4\sum_{k=1}^n (\partial_{z_k}\partial_{\bar{z}_k})$ be the Laplace operator. Then it is well-known that we have $B_0 = \exp(\Delta/2\gamma)$, see [36]. Thus we get

$$W_0 = \exp(-\Delta/4\gamma)S^0$$

and, by applying Proposition 9.3 and Proposition 9.4, we find that

$$\begin{aligned} W(d\pi(X))(Z,\phi) &= ic\gamma + w(d\rho_0(A))(\phi) - \frac{\gamma}{2}\exp(-\Delta/4\gamma)\left(a\bar{z} + bz + \bar{z}(Az)\right) \\ &= ic\gamma + w(d\rho_0(A))(\phi) + \frac{1}{2}\,\mathrm{Tr}(A) - \frac{\gamma}{2}\left(a\bar{z} + bz + \bar{z}(Az)\right). \end{aligned}$$

□

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